College Algebra

ELEVENTH EDITION

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To the memory of my friend, Charles Steib
E.J.H.

To Kurtis, Clayton, and Grady
C.J.D.

Contents

Preface xiii
Supplements Guide xvi

R Review of Basic Concepts 1

R.1 Sets 2
Basic Definitions ■ Operations on Sets

R.2 Real Numbers and Their Properties 8
Sets of Numbers and the Number Line ■ Exponents ■ Order of Operations ■ Properties of Real Numbers ■ Order on the Number Line ■ Absolute Value

R.3 Polynomials 21
Rules for Exponents ■ Polynomials ■ Addition and Subtraction ■ Multiplication ■ Division

R.4 Factoring Polynomials 32
Factoring Out the Greatest Common Factor ■ Factoring by Grouping ■ Factoring Trinomials ■ Factoring Binomials ■ Factoring by Substitution

R.5 Rational Expressions 41
Rational Expressions ■ Lowest Terms of a Rational Expression ■ Multiplication and Division ■ Addition and Subtraction ■ Complex Fractions

R.6 Rational Exponents 49
Negative Exponents and the Quotient Rule ■ Rational Exponents ■ Complex Fractions Revisited

R.7 Radical Expressions 59
Radical Notation ■ Simplified Radicals ■ Operations with Radicals ■ Rationalizing Denominators

Test Prep 70 ■ Review Exercises 73 ■ Test 77

1 Equations and Inequalities 79

1.1 Linear Equations 80
Basic Terminology of Equations ■ Solving Linear Equations ■ Identities, Conditional Equations, and Contradictions ■ Solving for a Specified Variable (Literal Equations)

1.2 Applications and Modeling with Linear Equations 86
Solving Applied Problems ■ Geometry Problems ■ Motion Problems ■ Mixture Problems ■ Modeling with Linear Equations

1.3 Complex Numbers 97
Basic Concepts of Complex Numbers ■ Operations on Complex Numbers

1.4 Quadratic Equations 104
Solving a Quadratic Equation ■ Completing the Square ■ The Quadratic Formula ■ Solving for a Specified Variable ■ The Discriminant

Chapter 1 Quiz (Sections 1.1–1.4) 113

1.5 Applications and Modeling with Quadratic Equations 114
Geometry Problems ■ Using the Pythagorean Theorem ■ Height of a Projected Object ■ Modeling with Quadratic Equations

1.6 Other Types of Equations and Applications 124
Rational Equations ■ Work Rate Problems ■ Equations with Radicals ■ Equations with Rational Exponents ■ Equations Quadratic in Form

Summary Exercises on Solving Equations 136

1.7 Inequalities 137
Linear Inequalities ■ Three-Part Inequalities ■ Quadratic Inequalities ■ Rational Inequalities

1.8 Absolute Value Equations and Inequalities 149
Basic Concepts ■ Absolute Value Equations ■ Absolute Value Inequalities ■ Special Cases ■ Absolute Value Models for Distance and Tolerance

Test Prep 156 ■ Review Exercises 160 ■ Test 166

2 Graphs and Functions 169

2.1 Rectangular Coordinates and Graphs 170
Ordered Pairs ■ The Rectangular Coordinate System ■ The Distance Formula ■ The Midpoint Formula ■ Graphing Equations in Two Variables

2.2 Circles 180
Center-Radius Form ■ General Form ■ An Application

2.3 Functions 188
Relations and Functions ■ Domain and Range ■ Determining Whether Relations Are Functions ■ Function Notation ■ Increasing, Decreasing, and Constant Functions

2.4 Linear Functions 203
Graphing Linear Functions ■ Standard Form $Ax + By = C$ ■ Slope ■ Average Rate of Change ■ Linear Models

Chapter 2 Quiz (Sections 2.1–2.4) 216

2.5 Equations of Lines and Linear Models 217
Point-Slope Form ■ Slope-Intercept Form ■ Vertical and Horizontal Lines ■ Parallel and Perpendicular Lines ■ Modeling Data ■ Solving Linear Equations in One Variable by Graphing

Summary Exercises on Graphs, Circles, Functions, and Equations 230

2.6 Graphs of Basic Functions 231

Continuity ■ The Identity, Squaring, and Cubing Functions ■ The Square Root and Cube Root Functions ■ The Absolute Value Function ■ Piecewise-Defined Functions ■ The Relation $x = y^2$

2.7 Graphing Techniques 242

Stretching and Shrinking ■ Reflecting ■ Symmetry ■ Even and Odd Functions ■ Translations

Chapter 2 Quiz (Sections 2.5–2.7) 257

2.8 Function Operations and Composition 258

Arithmetic Operations on Functions ■ The Difference Quotient ■ Composition of Functions and Domain

Test Prep 271 ■ Review Exercises 275 ■ Test 280

3 Polynomial and Rational Functions 283

3.1 Quadratic Functions and Models 284

Quadratic Functions ■ Graphing Techniques ■ Completing the Square ■ The Vertex Formula ■ Quadratic Models

3.2 Synthetic Division 300

Synthetic Division ■ Evaluating Polynomial Functions Using the Remainder Theorem ■ Testing Potential Zeros

3.3 Zeros of Polynomial Functions 306

Factor Theorem ■ Rational Zeros Theorem ■ Number of Zeros ■ Conjugate Zeros Theorem ■ Finding Zeros of a Polynomial Function ■ Descartes' Rule of Signs

3.4 Polynomial Functions: Graphs, Applications, and Models 318

Graphs of $f(x) = ax^n$ ■ Graphs of General Polynomial Functions ■ Behavior at Zeros ■ Turning Points and End Behavior ■ Graphing Techniques ■ Intermediate Value and Boundedness Theorems ■ Approximating Real Zeros ■ Polynomial Models

Summary Exercises on Polynomial Functions, Zeros, and Graphs 337

3.5 Rational Functions: Graphs, Applications, and Models 339

The Reciprocal Function $f(x) = \frac{1}{x}$ ■ The Function $f(x) = \frac{1}{x^2}$ ■ Asymptotes ■ Steps for Graphing Rational Functions ■ Rational Function Models

Chapter 3 Quiz (Sections 3.1–3.5) 359

Summary Exercises on Solving Equations and Inequalities 359

3.6 Variation 361

Direct Variation ■ Inverse Variation ■ Combined and Joint Variation

Test Prep 370 ■ Review Exercises 375 ■ Test 380

4 Inverse, Exponential, and Logarithmic Functions 383

4.1 Inverse Functions 384

One-to-One Functions ■ Inverse Functions ■ Equations of Inverses ■ An Application of Inverse Functions to Cryptography

4.2 Exponential Functions 397

Exponents and Properties ■ Exponential Functions ■ Exponential Equations ■ Compound Interest ■ The Number *e* and Continuous Compounding ■ Exponential Models

4.3 Logarithmic Functions 413

Logarithms ■ Logarithmic Equations ■ Logarithmic Functions ■ Properties of Logarithms

Summary Exercises on Inverse, Exponential, and Logarithmic Functions 426

4.4 Evaluating Logarithms and the Change-of-Base Theorem 427

Common Logarithms ■ Applications and Models with Common Logarithms ■ Natural Logarithms ■ Applications and Models with Natural Logarithms ■ Logarithms with Other Bases

Chapter 4 Quiz (Sections 4.1–4.4) 439

4.5 Exponential and Logarithmic Equations 439

Exponential Equations ■ Logarithmic Equations ■ Applications and Models

4.6 Applications and Models of Exponential Growth and Decay 451

The Exponential Growth or Decay Function ■ Growth Function Models ■ Decay Function Models

Summary Exercises on Functions: Domains and Defining Equations 462

Test Prep 465 ■ Review Exercises 468 ■ Test 472

5 Systems and Matrices 475

5.1 Systems of Linear Equations 476

Linear Systems ■ Substitution Method ■ Elimination Method ■ Special Systems ■ Applying Systems of Equations ■ Solving Linear Systems with Three Unknowns (Variables) ■ Using Systems of Equations to Model Data

5.2 Matrix Solution of Linear Systems 493

The Gauss-Jordan Method ■ Special Systems

5.3 Determinant Solution of Linear Systems 504

Determinants ■ Cofactors ■ Evaluating $n \times n$ Determinants ■ Determinant Theorems ■ Cramer's Rule

5.4 **Partial Fractions** 516

Decomposition of Rational Expressions ■ Distinct Linear Factors ■ Repeated Linear Factors ■ Distinct Linear and Quadratic Factors ■ Repeated Quadratic Factors

Chapter 5 Quiz (Sections 5.1–5.4) 522

5.5 **Nonlinear Systems of Equations** 523

Solving Nonlinear Systems with Real Solutions ■ Solving Nonlinear Systems with Nonreal Complex Solutions ■ Applying Nonlinear Systems

Summary Exercises on Systems of Equations 533

5.6 **Systems of Inequalities and Linear Programming** 534

Solving Linear Inequalities ■ Solving Systems of Inequalities ■ Linear Programming

5.7 **Properties of Matrices** 545

Basic Definitions ■ Adding Matrices ■ Special Matrices ■ Subtracting Matrices ■ Multiplying Matrices ■ Applying Matrix Algebra

5.8 **Matrix Inverses** 558

Identity Matrices ■ Multiplicative Inverses ■ Solving Systems Using Inverse Matrices

Test Prep 569 ■ Review Exercises 575 ■ Test 580

6 Analytic Geometry 583

6.1 **Parabolas** 584

Conic Sections ■ Horizontal Parabolas ■ Geometric Definition and Equations of Parabolas ■ An Application of Parabolas

6.2 **Ellipses** 593

Equations and Graphs of Ellipses ■ Translated Ellipses ■ Eccentricity ■ Applications of Ellipses

Chapter 6 Quiz (Sections 6.1–6.2) 604

6.3 **Hyperbolas** 604

Equations and Graphs of Hyperbolas ■ Translated Hyperbolas ■ Eccentricity

6.4 **Summary of the Conic Sections** 614

Characteristics ■ Identifying Conic Sections ■ Geometric Definition of Conic Sections

Test Prep 621 ■ Review Exercises 623 ■ Test 625

7 Further Topics in Algebra 627

7.1 Sequences and Series 628
Sequences ■ Series and Summation Notation ■ Summation Properties

7.2 Arithmetic Sequences and Series 638
Arithmetic Sequences ■ Arithmetic Series

7.3 Geometric Sequences and Series 647
Geometric Sequences ■ Geometric Series ■ Infinite Geometric Series ■ Annuities

Summary Exercises on Sequences and Series 657

7.4 The Binomial Theorem 658
A Binomial Expansion Pattern ■ Pascal's Triangle ■ *n*-Factorial ■ Binomial Coefficients ■ The Binomial Theorem ■ *k*th Term of a Binomial Expansion

7.5 Mathematical Induction 665
Proof by Mathematical Induction ■ Proving Statements ■ Generalized Principle of Mathematical Induction ■ Proof of the Binomial Theorem

Chapter 7 Quiz (Sections 7.1–7.5) 671

7.6 Counting Theory 672
Fundamental Principle of Counting ■ Permutations ■ Combinations ■ Distinguishing between Permutations and Combinations

7.7 Basics of Probability 682
Basic Concepts ■ Complements and Venn Diagrams ■ Odds ■ Union of Two Events ■ Binomial Probability

Test Prep 693 ■ Review Exercises 697 ■ Test 701

Glossary 703
Solutions to Selected Exercises S-1
Answers to Selected Exercises A-1
Photo Credits C-1
Index of Applications I-1
Index I-5

Preface

WELCOME TO THE 11TH EDITION

As authors, we have called upon our classroom experiences, use of MyMathLab, suggestions from users and reviewers, and many years of writing to provide tools that will support learning and teaching. This new edition of *College Algebra* continues our effort to provide a sound pedagogical approach through logical development of the subject matter. This approach forms the basis for all of the Lial team's instructional materials available from Pearson Education, in both print and technology forms.

Our goal is to produce a textbook that will be an integral component of the student's experience in learning college algebra. With this in mind, we have provided a textbook that students can read more easily, which is often a difficult task, given the nature of mathematical language. We have also improved page layouts for better flow, provided additional side comments, and updated many figures.

We realize that today's classroom experience is evolving and that technology-based teaching and learning aids have become essential to address the ever-changing needs of instructors and students. As a result, we've worked to provide support for all classroom types—traditional, hybrid, and online. In the 11th edition, text and online materials are more tightly integrated than ever before. This enhances flexibility and ease of use for instructors and increases success for students. See pages xvi–xviii for descriptions of these materials.

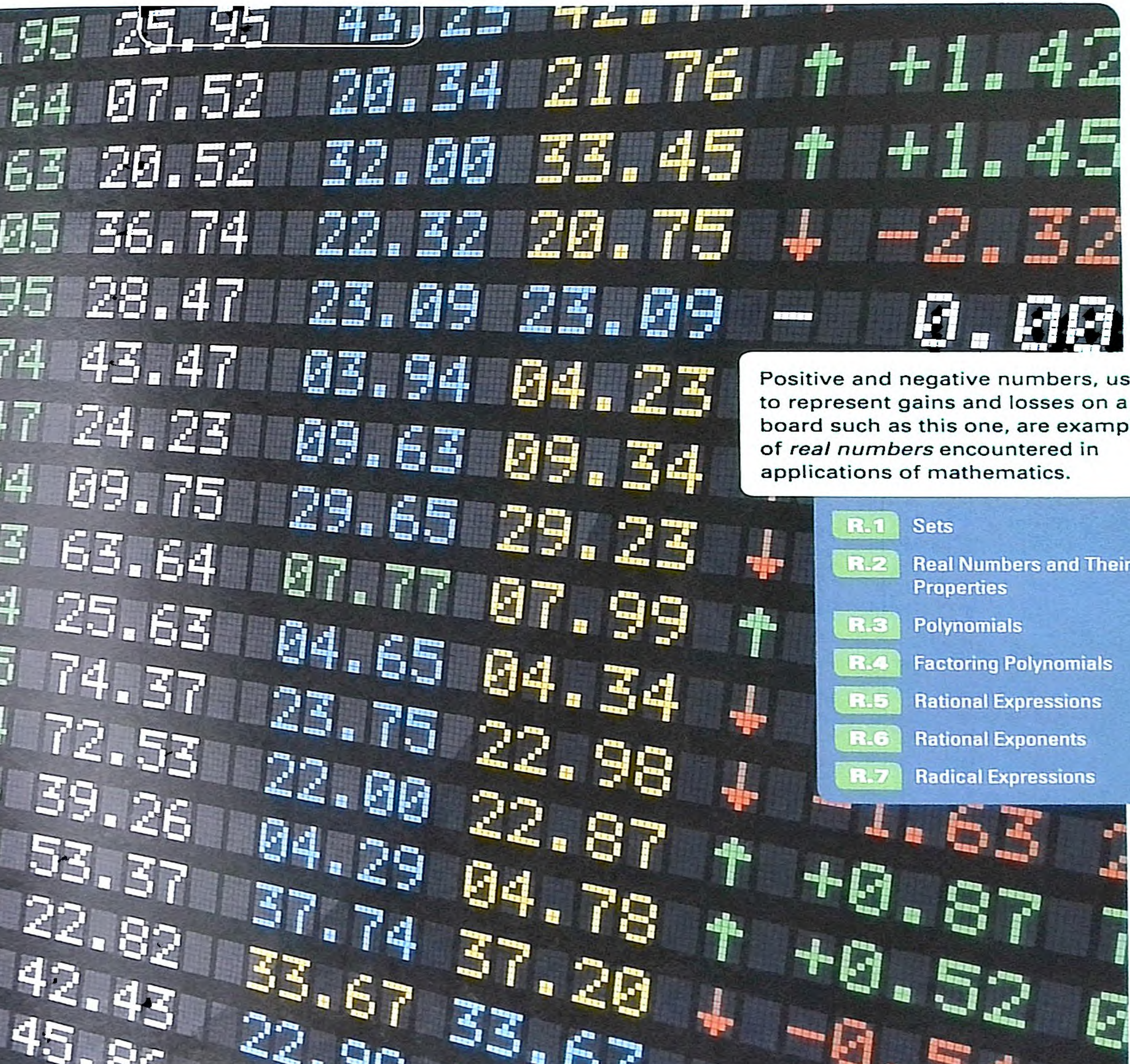
NEW TO THE 11TH EDITION

- In **Chapter 1** we have rewritten the introduction to the set of complex numbers and have included a new diagram illustrating the relationships among its subsets. We have also expanded the discussion of solving absolute value equations and inequalities.
- In **Chapter 2** we have prepared new art for shrinking and stretching graphs, added new exercises on translations, and expanded the exercises involving the difference quotient to include additional greater-degree functions.
- **Chapter 3** has undergone a particularly extensive revision. We have updated the opening discussion by providing a table of examples of polynomial functions (constant, linear, quadratic, cubic, quartic) and have identified the degree and leading coefficient. There is an increased emphasis on displaying all possibilities for positive, negative, and nonreal complex zeros in a table format, with graphical representations to illustrate them. We now have a new summary exercise set on solving both equations and inequalities (linear, quadratic, polynomial, rational, and miscellaneous).
- In **Chapter 4** we have added more work with translations of exponential and logarithmic functions, and we have included additional examples and exercises on solving equations that involve these functions.
- In **Chapter 5** we now present the determinant theorems within the exposition, and we have written new examples and exercises showing how they are used.
- **Chapter 6** now offers more discussion on graphing ellipses and hyperbolas centered away from the origin.
- The discussion of binomial probability in **Chapter 7** has been rewritten to provide more detail and greater clarity.

College Algebra

ELEVENTH EDITION

R Review of Basic Concepts



Positive and negative numbers, used to represent gains and losses on a board such as this one, are examples of *real numbers* encountered in applications of mathematics.

- **R.1** Sets
- **R.2** Real Numbers and Their Properties
- **R.3** Polynomials
- **R.4** Factoring Polynomials
- **R.5** Rational Expressions
- **R.6** Rational Exponents
- **R.7** Radical Expressions

The **set operations** are summarized below.

Set Operations

Let A and B be sets, with universal set U.

The **complement** of set A is the set A' of all elements in the universal set that do *not* belong to set A.

$$A' = \{x \mid x \in U, \quad x \notin A\}$$

The **intersection** of sets A and B, written $A \cap B$, is made up of all the elements belonging to both set A *and* set B.

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

The **union** of sets A and B, written $A \cup B$, is made up of all the elements belonging to set A *or* to set B.

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

R.1 Exercises

Identify each set as finite *or* infinite. *Then determine whether* 10 *is an element of the set. See Example 1.*

1. $\{4, 5, 6, \ldots, 15\}$

2. $\{1, 2, 3, 4, 5, \ldots, 75\}$

3. $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$

4. $\{4, 5, 6, \ldots\}$

5. $\{x \mid x$ is a natural number greater than $11\}$

6. $\{x \mid x$ is a natural number greater than or equal to $10\}$

7. $\{x \mid x$ is a fraction between 1 and $2\}$

8. $\{x \mid x$ is an even natural number$\}$

Use set notation, and list all the elements of each set. See Example 2.

9. $\{12, 13, 14, \ldots, 20\}$

10. $\{8, 9, 10, \ldots, 17\}$

11. $\{1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{32}\}$

12. $\{3, 9, 27, \ldots, 729\}$

13. $\{17, 22, 27, \ldots, 47\}$

14. $\{74, 68, 62, \ldots, 38\}$

15. $\{x \mid x$ is a natural number greater than 7 and less than $15\}$

16. $\{x \mid x$ is a natural number not greater than $4\}$

Insert $\in$ or $\notin$ in each blank to make the resulting statement true. See Examples 1 and 2.

17. 6 _____ $\{3, 4, 5, 6\}$

18. 9 _____ $\{3, 2, 5, 9, 8\}$

19. -4 _____ $\{4, 6, 8, 10\}$

20. -12 _____ $\{3, 5, 12, 14\}$

21. 0 _____ $\{2, 0, 3, 4\}$

22. 0 _____ $\{0, 5, 6, 7, 8, 10\}$

23. $\{3\}$ _____ $\{2, 3, 4, 5\}$

24. $\{5\}$ _____ $\{3, 4, 5, 6, 7\}$

25. $\{0\}$ _____ $\{0, 1, 2, 5\}$

26. $\{2\}$ _____ $\{2, 4, 6, 8\}$

27. 0 _____ $\emptyset$

28. $\emptyset$ _____ $\emptyset$

Determine whether each statement is true *or* false. ***See Examples 1–3.***

29. $3 \in \{2, 5, 6, 8\}$
30. $6 \in \{-2, 5, 8, 9\}$
31. $1 \in \{3, 4, 5, 11, 1\}$
32. $12 \in \{18, 17, 15, 13, 12\}$
33. $9 \notin \{2, 1, 5, 8\}$
34. $3 \notin \{7, 6, 5, 4\}$
35. $\{2, 5, 8, 9\} = \{2, 5, 9, 8\}$
36. $\{3, 0, 9, 6, 2\} = \{2, 9, 0, 3, 6\}$
37. $\{5, 8, 9\} = \{5, 8, 9, 0\}$
38. $\{3, 7, 12, 14\} = \{3, 7, 12, 14, 0\}$
39. $\{x \mid x \text{ is a natural number less than } 3\} = \{1, 2\}$
40. $\{x \mid x \text{ is a natural number greater than } 10\} = \{11, 12, 13, \ldots\}$

Let $A = \{2, 4, 6, 8, 10, 12\}$, $B = \{2, 4, 8, 10\}$, $C = \{4, 10, 12\}$, $D = \{2, 10\}$, *and* $U = \{2, 4, 6, 8, 10, 12, 14\}$.

Determine whether each statement is true *or* false. ***See Example 3.***

41. $A \subseteq U$
42. $C \subseteq U$
43. $D \subseteq B$
44. $D \subseteq A$
45. $A \subseteq B$
46. $B \subseteq C$
47. $\varnothing \subseteq A$
48. $\varnothing \subseteq \varnothing$
49. $\{4, 8, 10\} \subseteq B$
50. $\{0, 2\} \subseteq D$
51. $B \subseteq D$
52. $A \subseteq C$

Insert $\subseteq$ *or* $\nsubseteq$ *in each blank to make the resulting statement true.* ***See Example 3.***

53. $\{2, 4, 6\}$ ______ $\{3, 2, 5, 4, 6\}$
54. $\{1, 5\}$ ______ $\{0, -1, 2, 3, 1, 5\}$
55. $\{0, 1, 2\}$ ______ $\{1, 2, 3, 4, 5\}$
56. $\{5, 6, 7, 8\}$ ______ $\{1, 2, 3, 4, 5, 6, 7\}$
57. $\varnothing$ ______ $\{1, 4, 6, 8\}$
58. $\varnothing$ ______ $\varnothing$

Determine whether each statement is true *or* false. ***See Examples 4–6.***

59. $\{5, 7, 9, 19\} \cap \{7, 9, 11, 15\} = \{7, 9\}$
60. $\{8, 11, 15\} \cap \{8, 11, 19, 20\} = \{8, 11\}$
61. $\{2, 1, 7\} \cup \{1, 5, 9\} = \{1\}$
62. $\{6, 12, 14, 16\} \cup \{6, 14, 19\} = \{6, 14\}$
63. $\{3, 2, 5, 9\} \cap \{2, 7, 8, 10\} = \{2\}$
64. $\{8, 9, 6\} \cup \{9, 8, 6\} = \{8, 9\}$
65. $\{3, 5, 9, 10\} \cap \varnothing = \{3, 5, 9, 10\}$
66. $\{3, 5, 9, 10\} \cup \varnothing = \{3, 5, 9, 10\}$
67. $\{1, 2, 4\} \cup \{1, 2, 4\} = \{1, 2, 4\}$
68. $\{1, 2, 4\} \cap \{1, 2, 4\} = \varnothing$
69. $\varnothing \cup \varnothing = \varnothing$
70. $\varnothing \cap \varnothing = \varnothing$

Let $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$, $M = \{0, 2, 4, 6, 8\}$, $N = \{1, 3, 5, 7, 9, 11, 13\}$, $Q = \{0, 2, 4, 6, 8, 10, 12\}$, *and* $R = \{0, 1, 2, 3, 4\}$.

Use these sets to find each of the following. Identify any disjoint sets. ***See Examples 4–6.***

71. $M \cap R$
72. $M \cap U$
73. $M \cup N$
74. $M \cup R$
75. $M \cap N$
76. $U \cap N$
77. $N \cup R$
78. $M \cup Q$
79. N'
80. Q'
81. $M' \cap Q$
82. $Q \cap R'$
83. $\varnothing \cap R$
84. $\varnothing \cap Q$
85. $N \cup \varnothing$
86. $R \cup \varnothing$
87. $(M \cap N) \cup R$
88. $(N \cup R) \cap M$
89. $(Q \cap M) \cup R$
90. $(R \cup N) \cap M'$
91. $(M' \cup Q) \cap R$
92. $Q \cap (M \cup N)$
93. $Q' \cap (N' \cap U)$
94. $(U \cap \varnothing') \cup R$

NOTE As seen in **Example 9(b)**, absolute value bars can also act as symbols of inclusion. Remember this when applying the rules for order of operations.

EXAMPLE 9 Evaluating Absolute Value Expressions

Let $x = -6$ and $y = 10$. Evaluate each expression.

(a) $|2x - 3y|$ **(b)** $\dfrac{2|x| - |3y|}{|xy|}$

SOLUTION

(a)

$$\begin{aligned} |2x - 3y| &= |2(-6) - 3(10)| && \text{Substitute.} \\ &= |-12 - 30| && \text{Work inside absolute value bars. Multiply.} \\ &= |-42| && \text{Subtract.} \\ &= 42 && \text{Definition of absolute value} \end{aligned}$$

(b)

$$\begin{aligned} \frac{2|x| - |3y|}{|xy|} &= \frac{2|-6| - |3(10)|}{|-6(10)|} && \text{Substitute.} \\ &= \frac{2 \cdot 6 - |30|}{|-60|} && |-6| = 6;\ \text{multiply.} \\ &= \frac{12 - 30}{60} && \text{Multiply. } |30| = 30,\ |-60| = 60 \\ &= \frac{-18}{60} && \text{Subtract.} \\ &= -\frac{3}{10} && \text{Write in lowest terms. } \tfrac{-a}{b} = -\tfrac{a}{b}. \end{aligned}$$

✔ *Now Try Exercises 93 and 95.*

Distance between Points on a Number Line

If P and Q are points on a number line with coordinates a and b, respectively, then the distance $d(P, Q)$ between them is given by the following.

$$d(P, Q) = |b - a| \quad \text{or} \quad d(P, Q) = |a - b|$$

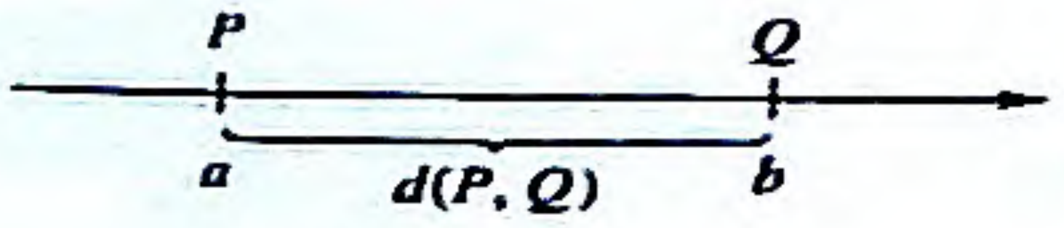


Figure 12

***That is, the distance between two points on a number line is the absolute value of the difference between their coordinates in either order. See* Figure 12.**

EXAMPLE 10 Finding the Distance between Two Points

Find the distance between -5 and 8.

SOLUTION Use the first formula above, with $a = -5$ and $b = 8$.

$$|b - a| = |8 - (-5)| = |8 + 5| = |13| = 13$$

Alternatively, for $a = 8$ and $b = -5$, we obtain the same result.

$$|b - a| = |(-5) - 8| = |-13| = 13$$

✔ *Now Try Exercise 105*

R.2 Exercises

1. ***Concept Check*** Match each number from Column I with the letter or letters of the sets of numbers from Column II to which the number belongs. There may be more than one choice, so give all choices.

I		II	
(a) 0	(b) 34	A. Natural numbers	B. Whole numbers
(c) $-\frac{9}{4}$	(d) $\sqrt{36}$	C. Integers	D. Rational numbers
(e) $\sqrt{13}$	(f) 2.16	E. Irrational numbers	F. Real numbers

2. Explain why no answer in Exercise 1 can contain both D and E as choices.

Concept Check *Decide whether each statement is* true *or* false. *If it is false, tell why.*

3. Every integer is a whole number.
4. Every natural number is an integer.
5. Every irrational number is an integer.
6. Every integer is a rational number.
7. Every natural number is a whole number.
8. Some rational numbers are irrational.
9. Some rational numbers are whole numbers.
10. Some real numbers are integers.

Let set $A = \left\{-6, -\frac{12}{4}, -\frac{5}{8}, -\sqrt{3}, 0, \frac{1}{4}, 1, 2\pi, 3, \sqrt{12}\right\}$. *List all the elements of A that belong to each set. See Example 1.*

11. Natural numbers
12. Whole numbers
13. Integers
14. Rational numbers
15. Irrational numbers
16. Real numbers

Evaluate each expression. See Example 2.

17. -2^4
18. -3^5
19. $(-2)^4$
20. $(-2)^6$
21. $(-3)^5$
22. $(-2)^5$
23. $-2 \cdot 3^4$
24. $-4 \cdot 5^3$

Evaluate each expression. See Example 3.

25. $-2 \cdot 5 + 12 \div 3$
26. $9 \cdot 3 - 16 \div 4$
27. $-4(9-8) + (-7)(2)^3$
28. $6(-5) - (-3)(2)^4$
29. $(4 - 2^3)(-2 + \sqrt{25})$
30. $(5 - 3^2)(\sqrt{16} - 2^3)$
31. $\left(-\frac{2}{9} - \frac{1}{4}\right) - \left[-\frac{5}{18} - \left(-\frac{1}{2}\right)\right]$
32. $\left[-\frac{5}{8} - \left(-\frac{2}{5}\right)\right] - \left(\frac{3}{2} - \frac{11}{10}\right)$
33. $\dfrac{-8 + (-4)(-6) \div 12}{4 - (-3)}$
34. $\dfrac{15 \div 5 \cdot 4 \div 6 - 8}{-6 - (-5) - 8 \div 2}$

Evaluate each expression for $p = -4$, $q = 8$, *and* $r = -10$. *See Example 4.*

35. $-p^2 - 7q + r^2$
36. $-p^2 - 2q + r$
37. $\dfrac{q + r}{q + p}$
38. $\dfrac{p + r}{p + q}$
39. $\dfrac{3q}{r} - \dfrac{5}{p}$
40. $\dfrac{3r}{q} - \dfrac{2}{r}$
41. $\dfrac{5r}{2p - 3r}$
42. $\dfrac{3q}{3p - 2r}$
43. $\dfrac{\frac{q}{2} - \frac{r}{3}}{\frac{3p}{4} + \frac{q}{8}}$
44. $\dfrac{\frac{q}{4} - \frac{r}{5}}{\frac{p}{2} + \frac{q}{2}}$
45. $\dfrac{-(p + 2)^2 - 3r}{2 - q}$
46. $\dfrac{-(q - 6)^2 - 2p}{4 - p}$
47. $\dfrac{3p + 3(4 + p)^3}{r + 8}$
48. $\dfrac{5q + 2(1 + p)^3}{r + 3}$

Identify the property illustrated in each statement. Assume all variables represent real numbers. See Examples 5 and 6.

49. $6 \cdot 12 + 6 \cdot 15 = 6(12 + 15)$

50. $8(m + 4) = 8m + 32$

51. $(t - 6) \cdot \left(\frac{1}{t - 6}\right) = 1$, if $t - 6 \neq 0$

52. $\frac{2 + m}{2 - m} \cdot \frac{2 - m}{2 + m} = 1$, if $m \neq 2$ or -2

53. $(7.5 - y) + 0 = 7.5 - y$

54. $1 \cdot (3x - 7) = 3x - 7$

55. $5(t + 3) = (t + 3) \cdot 5$

56. $-7 + (x + 3) = (x + 3) + (-7)$

57. $(5x)\left(\frac{1}{x}\right) = 5\left(x \cdot \frac{1}{x}\right)$

58. $(38 + 99) + 1 = 38 + (99 + 1)$

59. $5 + \sqrt{3}$ is a real number.

60. 5π is a real number.

61. Is there a commutative property for subtraction? That is, in general, is $a - b$ equal to $b - a$? Support your answer with examples.

62. Is there an associative property for subtraction? That is, does $(a - b) - c$ equal $a - (b - c)$ in general? Support your answer with examples.

Simplify each expression. See Examples 5 and 6.

63. $\frac{10}{11}(22z)$

64. $\left(\frac{3}{4}r\right)(-12)$

65. $(m + 5) + 6$

66. $8 + (a + 7)$

67. $\frac{3}{8}\left(\frac{16}{9}y + \frac{32}{27}z - \frac{40}{9}\right)$

68. $-\frac{1}{4}(20m + 8y - 32z)$

Use the distributive property to rewrite sums as products and products as sums. See Example 6.

69. $8p - 14p$

70. $15x - 10x$

71. $-4(z - y)$

72. $-3(m + n)$

Concept Check *Use the distributive property to calculate each value mentally.*

73. $72 \cdot 17 + 28 \cdot 17$

74. $32 \cdot 80 + 32 \cdot 20$

75. $123\frac{5}{8} \cdot 1\frac{1}{2} - 23\frac{5}{8} \cdot 1\frac{1}{2}$

76. $17\frac{2}{5} \cdot 14\frac{3}{4} - 17\frac{2}{5} \cdot 4\frac{3}{4}$

Concept Check *Decide whether each statement is* true *or* false. *If false, correct the statement so it is true.*

77. $|6 - 8| = |6| - |8|$

78. $|(-3)^3| = -|3^3|$

79. $|-5| \cdot |6| = |-5 \cdot 6|$

80. $\frac{|-14|}{|2|} = \left|\frac{-14}{2}\right|$

81. $|a - b| = |a| - |b|$, if $b > a > 0$

82. If a is negative, then $|a| = -a$.

Evaluate each expression. See Example 7.

83. $|-10|$

84. $|-15|$

85. $-\left|\frac{4}{7}\right|$

86. $-\left|\frac{7}{2}\right|$

87. $-|-8|$

88. $-|-12|$

Solve each problem. See Example 8.

89. ***Blood Pressure Difference*** Calculate the P_d value for a woman whose actual systolic pressure is 116 and whose normal value should be 125.

90. ***Systolic Blood Pressure*** If a patient's P_d value is 17 and the normal pressure for his gender and age should be 130, what are the two possible values for his systolic blood pressure?

Let $x = -4$ and $y = 2$. Evaluate each expression. See Example 9.

91. $|3x - 2y|$ **92.** $|2x - 5y|$ **93.** $|-3x + 4y|$ **94.** $|-5y + x|$

95. $\dfrac{2|y| - 3|x|}{|xy|}$ **96.** $\dfrac{4|x| + 4|y|}{|x|}$ **97.** $\dfrac{|-8y + x|}{-|x|}$ **98.** $\dfrac{|x| + 2|y|}{-|x|}$

Justify each statement by giving the correct property of absolute value from this section. Assume all variables represent real numbers.

99. $|m| = |-m|$ **100.** $|-k| \geq 0$

101. $|9| \cdot |-6| = |-54|$ **102.** $|k - m| \leq |k| + |-m|$

103. $|12 + 11r| \geq 0$ **104.** $\left|\dfrac{-12}{5}\right| = \dfrac{|-12|}{|5|}$

Find the given distances between points P, Q, R, and S on a number line, with coordinates −4, −1, 8, and 12, respectively. See Example 10.

105. $d(P, Q)$ **106.** $d(P, R)$ **107.** $d(Q, R)$ **108.** $d(Q, S)$

Concept Check *Determine what signs on values of x and y would make each statement true. Assume that x and y are not 0. (You should be able to work these mentally.)*

109. $xy > 0$ **110.** $x^2y > 0$ **111.** $\dfrac{x}{y} < 0$

112. $\dfrac{y^2}{x} < 0$ **113.** $\dfrac{x^3}{y} > 0$ **114.** $-\dfrac{x}{y} > 0$

Solve each problem.

115. ***Golf Scores*** Phil Mickelson won the 2010 Masters Golf Tournament with a total score that was 16 under par, and Zach Johnson won the 2007 tournament with a total score that was 1 above par. Using −16 to represent 16 below par and +1 to represent 1 over par, find the difference between these scores (in either order) and take the absolute value of this difference. What does this final number represent? (*Source:* www.masters.org)

116. ***Total Football Yardage*** During his 16 years in the NFL, Marcus Allen gained 12,243 yd rushing, 5411 yd receiving, and −6 yd returning fumbles. Find his total yardage (called *all-purpose yards*). Is this the same as the sum of the absolute values of the three categories? Explain. (*Source: The Sports Illustrated Sports Almanac.*)

Blood Alcohol Concentration *The blood alcohol concentration (BAC) of a person who has been drinking is approximated by the following expression.*

number of oz × % alcohol × 0.075 ÷ body weight in lb − hr of drinking × 0.015

(*Source:* Lawlor, J., *Auto Math Handbook: Mathematical Calculations, Theory, and Formulas for Automotive Enthusiasts,* HP Books.)

117. Suppose a policeman stops a 190-lb man who, in 2 hr, has ingested four 12-oz beers (48 oz), each having a 3.2% alcohol content. Calculate the man's BAC to the nearest thousandth. Follow the order of operations.

118. Find the BAC to the nearest thousandth for a 135-lb woman who, in 3 hr, has drunk three 12-oz beers (36 oz), each having a 4.0% alcohol content.

119. Calculate the BACs in **Exercises 117 and 118** if each person weighs 25 lb more and the rest of the variables stay the same. How does increased weight affect a person's BAC?

When a polynomial has a missing term, we allow for that term by inserting a term with a 0 coefficient for it.

EXAMPLE 11 Dividing Polynomials with Missing Terms

Divide $3x^3 - 2x^2 - 150$ by $x^2 - 4$.

SOLUTION Both polynomials have missing first-degree terms. Insert each missing term with a 0 coefficient.

$$\begin{array}{r} 3x - 2 \\ x^2 + 0x - 4\overline{)3x^3 - 2x^2 + 0x - 150} \\ \underline{3x^3 + 0x^2 - 12x} \\ -2x^2 + 12x - 150 \\ \underline{-2x^2 + 0x + 8} \\ 12x - 158 \leftarrow \text{Remainder} \end{array}$$

Missing term

Insert placeholders for missing terms.

Missing term

The division process ends when the remainder is 0 or the degree of the remainder is less than that of the divisor. Since $12x - 158$ has lesser degree than the divisor $x^2 - 4$, it is the remainder. Thus, the entire quotient is written as follows.

$$\frac{3x^3 - 2x^2 - 150}{x^2 - 4} = 3x - 2 + \frac{12x - 158}{x^2 - 4}$$

✔ *Now Try Exercise 93.*

R.3 Exercises

Simplify each expression. ***See Example 1.***

1. $(-4x^5)(4x^2)$
2. $(3y^4)(-6y^3)$
3. $n^6 \cdot n^4 \cdot n$
4. $a^8 \cdot a^5 \cdot a$
5. $9^3 \cdot 9^5$
6. $4^2 \cdot 4^8$
7. $(-3m^4)(6m^2)(-4m^5)$
8. $(-8t^3)(2t^6)(-5t^4)$
9. $(5x^2y)(-3x^3y^4)$

10. ***Concept Check*** Decide whether each expression has been simplified correctly. If not, correct it. Assume all variables represent nonzero real numbers.

(a) $(mn)^2 = mn^2$
(b) $y^2 \cdot y^5 = y^7$
(c) $\left(\frac{k}{5}\right)^3 = \frac{k^3}{5}$
(d) $3^0y = 0$
(e) $4^5 \cdot 4^2 = 16^7$
(f) $(a^2)^3 = a^5$
(g) $cd^0 = 1$
(h) $(2b)^4 = 8b^4$

Simplify each expression. Assume variables represent nonzero real numbers. ***See Examples 1–3.***

11. $(2^2)^5$
12. $(6^4)^3$
13. $(-6x^2)^3$
14. $(-2x^5)^5$
15. $-(4m^3n^0)^2$
16. $-(2x^0y^4)^3$
17. $\left(\frac{r^8}{s^2}\right)^3$
18. $\left(\frac{p^4}{q}\right)^2$
19. $\left(\frac{-4m^2}{tp^2}\right)^4$
20. $\left(\frac{-5n^4}{r^2}\right)^3$
21. $-\left(\frac{x^3y^5}{z}\right)^0$
22. $-\left(\frac{p^2q^3}{r^3}\right)^0$

Match each expression in Column I with its equivalent in Column II. See Example 3.

	I	II
23.	**(a)** 6^0	**A.** 0
	(b) -6^0	**B.** 1
	(c) $(-6)^0$	**C.** -1
	(d) $-(-6)^0$	**D.** 6
		E. -6

	I	II
24.	**(a)** $3p^0$	**A.** 0
	(b) $-3p^0$	**B.** 1
	(c) $(3p)^0$	**C.** -1
	(d) $(-3p)^0$	**D.** 3
		E. -3

25. Explain why $x^2 + x^2$ is not equivalent to x^4.

26. Explain why $(x + y)^2$ is not equivalent to $x^2 + y^2$.

Identify each expression as a polynomial *or* not a polynomial. *For each polynomial, give the degree and identify it as a* monomial, binomial, trinomial, *or* none of these. *See Example 4.*

27. $-5x^{11}$

28. $-4y^5$

29. $6x + 3x^4$

30. $-9y + 5y^3$

31. $-7z^5 - 2z^3 + 1$

32. $-9t^4 + 8t^3 - 7$

33. $15a^2b^3 + 12a^3b^8 - 13b^5 + 12b^6$

34. $-16x^5y^7 + 12x^3y^8 - 4xy^9 + 18x^{10}$

35. $\frac{3}{8}x^5 - \frac{1}{x^2} + 9$

36. $\frac{2}{3}t^6 + \frac{3}{t^5} + 1$

37. 5

38. 9

Find each sum or difference. See Example 5.

39. $(5x^2 - 4x + 7) + (-4x^2 + 3x - 5)$

40. $(3m^3 - 3m^2 + 4) + (-2m^3 - m^2 + 6)$

41. $2(12y^2 - 8y + 6) - 4(3y^2 - 4y + 2)$

42. $3(8p^2 - 5p) - 5(3p^2 - 2p + 4)$

43. $(6m^4 - 3m^2 + m) - (2m^3 + 5m^2 + 4m) + (m^2 - m)$

44. $-(8x^3 + x - 3) + (2x^3 + x^2) - (4x^2 + 3x - 1)$

Find each product. See Examples 6–8.

45. $(4r - 1)(7r + 2)$

46. $(5m - 6)(3m + 4)$

47. $x^2\left(3x - \frac{2}{3}\right)\left(5x + \frac{1}{3}\right)$

48. $m^3\left(2m - \frac{1}{4}\right)\left(3m + \frac{1}{2}\right)$

49. $4x^2(3x^3 + 2x^2 - 5x + 1)$

50. $2b^3(b^2 - 4b + 3)$

51. $(2z - 1)(-z^2 + 3z - 4)$

52. $(3w + 2)(-w^2 + 4w - 3)$

53. $(m - n + k)(m + 2n - 3k)$

54. $(r - 3s + t)(2r - s + t)$

55. $(2x + 3)(2x - 3)(4x^2 - 9)$

56. $(3y - 5)(3y + 5)(9y^2 - 25)$

57. $(x + 1)(x + 1)(x - 1)(x - 1)$

58. $(t + 4)(t + 4)(t - 4)(t - 4)$

Find each product. See Examples 8 and 9.

59. $(2m + 3)(2m - 3)$

60. $(8s - 3t)(8s + 3t)$

61. $(4x^2 - 5y)(4x^2 + 5y)$

62. $(2m^3 + n)(2m^3 - n)$

63. $(4m + 2n)^2$

64. $(a - 6b)^2$

65. $(5r - 3t^2)^2$

66. $(2z^4 - 3y)^2$

67. $[(2p - 3) + q]^2$

68. $[(4y - 1) + z]^2$

69. $[(3q + 5) - p][(3q + 5) + p]$

70. $[(9r - s) + 2][(9r - s) - 2]$

71. $[(3a + b) - 1]^2$ **72.** $[(2m + 7) - n]^2$ **73.** $(y + 2)^3$

74. $(z - 3)^3$ **75.** $(q - 2)^4$ **76.** $(r + 3)^4$

Perform the indicated operations. ***See Examples 5–9.***

77. $(p^3 - 4p^2 + p) - (3p^2 + 2p + 7)$ **78.** $(x^4 - 3x^2 + 2) - (-2x^4 + x^2 - 3)$

79. $(7m + 2n)(7m - 2n)$ **80.** $(3p + 5)^2$

81. $-3(4q^2 - 3q + 2) + 2(-q^2 + q - 4)$ **82.** $2(3r^2 + 4r + 2) - 3(-r^2 + 4r - 5)$

83. $p(4p - 6) + 2(3p - 8)$ **84.** $m(5m - 2) + 9(5 - m)$

85. $-y(y^2 - 4) + 6y^2(2y - 3)$ **86.** $-z^3(9 - z) + 4z(2 + 3z)$

Perform each division. ***See Examples 10 and 11.***

87. $\dfrac{-4x^7 - 14x^6 + 10x^4 - 14x^2}{-2x^2}$ **88.** $\dfrac{-8r^3s - 12r^2s^2 + 20rs^3}{-4rs}$

89. $\dfrac{4x^3 - 3x^2 + 1}{x - 2}$ **90.** $\dfrac{3x^3 - 2x + 5}{x - 3}$

91. $\dfrac{6m^3 + 7m^2 - 4m + 2}{3m + 2}$ **92.** $\dfrac{10x^3 + 11x^2 - 2x + 3}{5x + 3}$

93. $\dfrac{x^4 + 5x^2 + 5x + 27}{x^2 + 3}$ **94.** $\dfrac{k^4 - 4k^2 + 2k + 5}{k^2 + 1}$

Relating Concepts

For individual or collaborative investigation *(Exercises 95–98)*

The special products can be used to perform selected multiplications. On the left, we use $(x + y)(x - y) = x^2 - y^2$. *On the right,* $(x - y)^2 = x^2 - 2xy + y^2$.

$$\begin{aligned} 51 \times 49 &= (50 + 1)(50 - 1) \\ &= 50^2 - 1^2 \\ &= 2500 - 1 \\ &= 2499 \end{aligned}$$

$$\begin{aligned} 47^2 &= (50 - 3)^2 \\ &= 50^2 - 2(50)(3) + 3^2 \\ &= 2500 - 300 + 9 \\ &= 2209 \end{aligned}$$

Use special products to evaluate each expression.

95. 99×101 **96.** 63×57 **97.** 102^2 **98.** 71^2

Solve each problem.

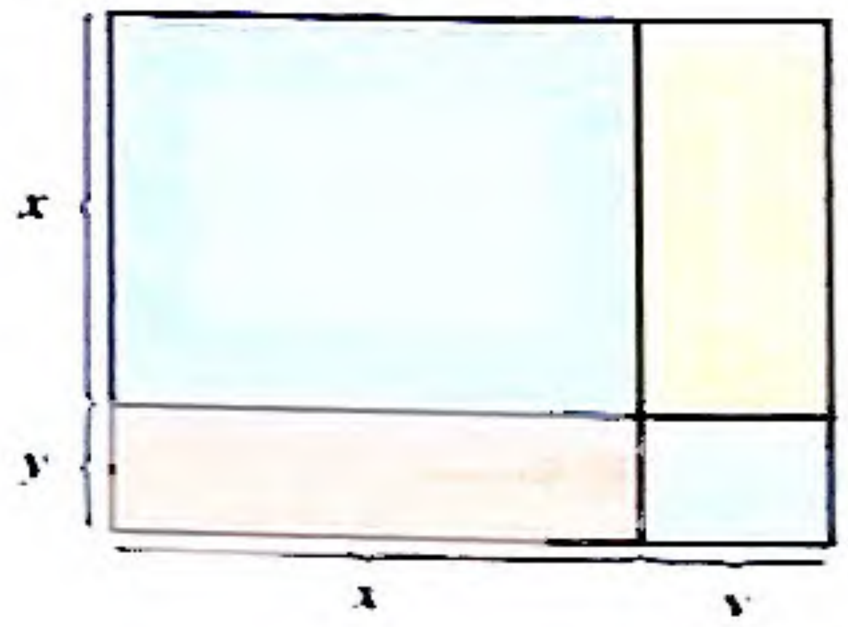
99. ***Geometric Modeling*** **Consider the figure, which is a square divided into two squares and two rectangles.**

(a) The length of each side of the largest square is $x + y$. Use the formula for the area of a square to write the area of the largest square as a power.

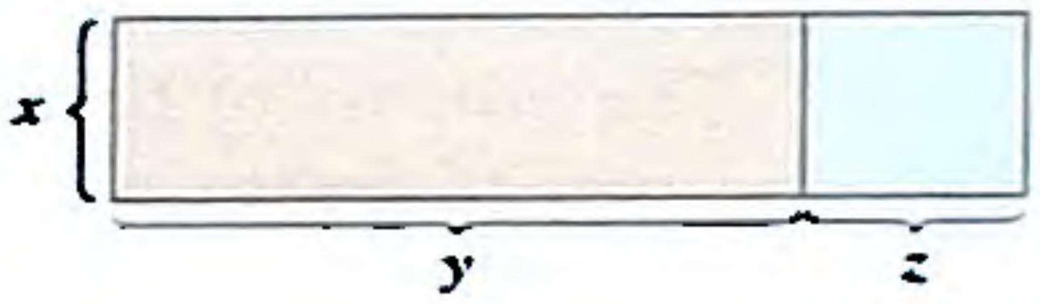
(b) Use the formulas for the area of a square and the area of a rectangle to write the area of the largest square as a trinomial that represents the sum of the areas of the four figures that comprise it.

(c) Explain why the expressions in parts (a) and (b) must be equivalent.

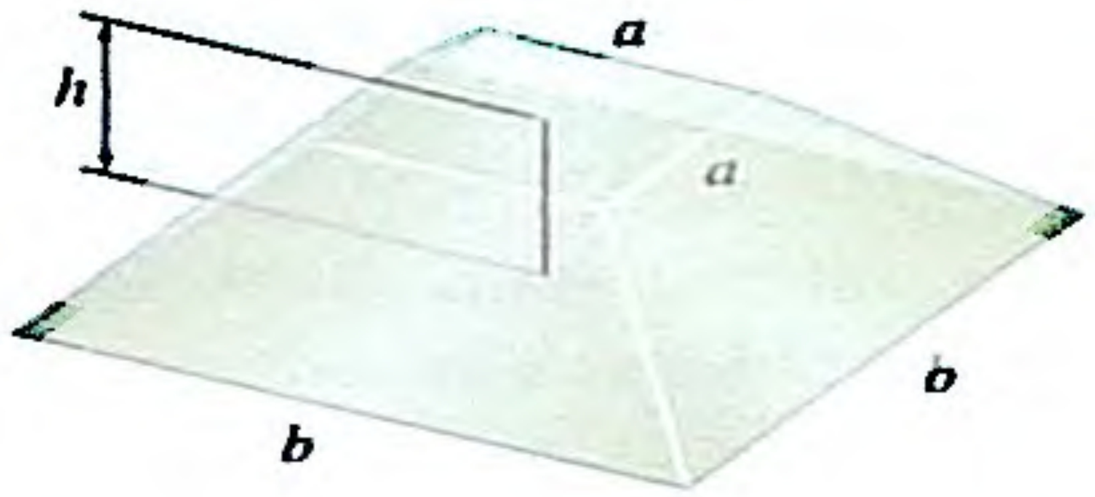
(d) What special product formula from this section does this exercise reinforce geometrically?



100. ***Geometric Modeling*** Use the reasoning process of **Exercise 99** and the accompanying figure to geometrically support the distributive property. Write a short paragraph explaining this process.



101. ***Volume of the Great Pyramid*** An amazing formula from ancient mathematics was used by the Egyptians to find the volume of the frustum of a square pyramid, as shown in the figure. Its volume is given by



$$V = \frac{1}{3}h(a^2 + ab + b^2),$$

where b is the length of the base, a is the length of the top, and h is the height. (*Source:* Freebury, H. A., *A History of Mathematics,* Macmillan Company, New York.)

(a) When the Great Pyramid in Egypt was partially completed to a height h of 200 ft, b was 756 ft, and a was 314 ft. Calculate its volume at this stage of construction.
(b) Try to visualize the figure if $a = b$. What is the resulting shape? Find its volume.
(c) Let $a = b$ in the Egyptian formula and simplify. Are the results the same?

102. ***Volume of the Great Pyramid*** Refer to the formula and the discussion in Exercise 101.

(a) Use $V = \frac{1}{3}h(a^2 + ab + b^2)$ to determine a formula for the volume of a pyramid with square base of length b and height h by letting $a = 0$.
(b) The Great Pyramid in Egypt had a square base of length 756 ft and a height of 481 ft. Find the volume of the Great Pyramid. Compare it with the 273-ft-tall Superdome in New Orleans, which has an approximate volume of 100 million ft^3. (*Source: Guinness Book of World Records.*)
(c) The Superdome covers an area of 13 acres. How many acres does the Great Pyramid cover? (*Hint:* 1 acre = 43,560 ft^2)

(Modeling) Number of Farms in the United States *The graph shows the number of farms in the United States since 1940 for selected years. The polynomial*

$$0.001147x^2 - 4.5905x + 4595$$

provides a good approximation of the number of farms for these years by substituting the year for x and evaluating the polynomial. For example, if $x = 1960$, *the value of the polynomial is approximately* 3.9, *which differs from the data in the bar graph by only* 0.1.

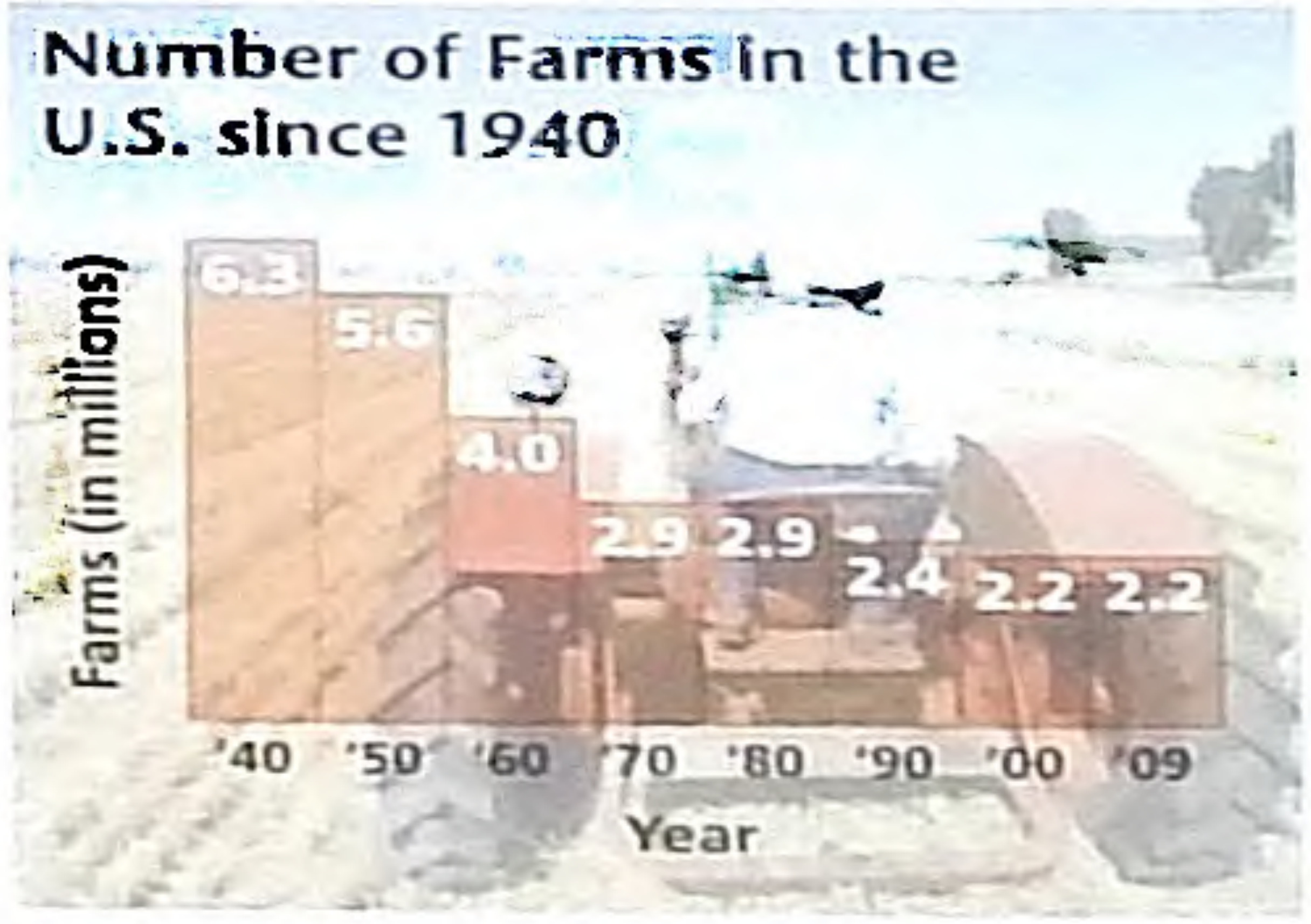
Evaluate the polynomial for each year and then give the difference from the value in the graph.

103. 1940

104. 1970

105. 1990

106. 2009



Source U.S. Department of Agriculture

Concept Check *Perform each operation mentally.*

107. $(0.25^3)(400^3)$ **108.** $(24^2)(0.5^2)$ **109.** $\frac{4.2^5}{2.1^5}$ **110.** $\frac{15^4}{5^4}$

R.4 Exercises

Factor out the greatest common factor from each polynomial. See Examples 1 and 2.

1. $12m + 60$
2. $15r - 27$
3. $8k^3 + 24k$
4. $9z^4 + 81z$
5. $xy - 5xy^2$
6. $5h^2j + hj$
7. $-4p^3q^4 - 2p^2q^5$
8. $-3z^5w^2 - 18z^3w^4$
9. $4k^2m^3 + 8k^4m^3 - 12k^2m^4$
10. $28r^4s^2 + 7r^3s - 35r^4s^3$
11. $2(a + b) + 4m(a + b)$
12. $6x(a + b) - 4y(a + b)$
13. $(5r - 6)(r + 3) - (2r - 1)(r + 3)$
14. $(4z - 5)(3z - 2) - (3z - 9)(3z - 2)$
15. $2(m - 1) - 3(m - 1)^2 + 2(m - 1)^3$
16. $5(a + 3)^3 - 2(a + 3) + (a + 3)^2$
17. **Concept Check** When directed to completely factor the polynomial $4x^2y^5 - 8xy^3$, a student wrote $2xy^3(2xy^2 - 4)$. When the teacher did not give him full credit, he complained because when his answer is multiplied out, the result is the original polynomial. Give the correct answer.
18. **Concept Check** Kurt factored $16a^2 - 40a - 6a + 15$ by grouping and obtained $(8a - 3)(2a - 5)$. Callie factored the same polynomial and gave an answer of $(3 - 8a)(5 - 2a)$. Which answer is correct?

Factor each polynomial by grouping. See Example 2.

19. $6st + 9t - 10s - 15$
20. $10ab - 6b + 35a - 21$
21. $2m^4 + 6 - am^4 - 3a$
22. $15 - 5m^2 - 3r^2 + m^2r^2$
23. $p^2q^2 - 10 - 2q^2 + 5p^2$
24. $20z^2 - 8x + 5pz^2 - 2px$

Factor each trinomial, if possible. See Examples 3 and 4.

25. $6a^2 - 11a + 4$
26. $8h^2 - 2h - 21$
27. $3m^2 + 14m + 8$
28. $9y^2 - 18y + 8$
29. $15p^2 + 24p + 8$
30. $9x^2 + 4x - 2$
31. $12a^3 + 10a^2 - 42a$
32. $36x^3 + 18x^2 - 4x$
33. $6k^2 + 5kp - 6p^2$
34. $14m^2 + 11mr - 15r^2$
35. $5a^2 - 7ab - 6b^2$
36. $12s^2 + 11st - 5t^2$
37. $12x^2 - xy - y^2$
38. $30a^2 + am - m^2$
39. $24a^4 + 10a^3b - 4a^2b^2$
40. $18x^5 + 15x^4z - 75x^3z^2$
41. $9m^2 - 12m + 4$
42. $16p^2 - 40p + 25$
43. $32a^2 + 48ab + 18b^2$
44. $20p^2 - 100pq + 125q^2$
45. $4x^2y^2 + 28xy + 49$
46. $9m^2n^2 + 12mn + 4$
47. $(a - 3b)^2 - 6(a - 3b) + 9$
48. $(2p + q)^2 - 10(2p + q) + 25$
49. **Concept Check** Match each polynomial in Column I with its factored form in Column II.

I	II
(a) $x^2 + 10xy + 25y^2$	A. $(x + 5y)(x - 5y)$
(b) $x^2 - 10xy + 25y^2$	B. $(x + 5y)^2$
(c) $x^2 - 25y^2$	C. $(x - 5y)^2$
(d) $25y^2 - x^2$	D. $(5y + x)(5y - x)$

50. **Concept Check** Match each polynomial in Column I with its factored form in Column II.

I	II
(a) $8x^3 - 27$	A. $(3 - 2x)(9 + 6x + 4x^2)$
(b) $8x^3 + 27$	B. $(2x - 3)(4x^2 + 6x + 9)$
(c) $27 - 8x^3$	C. $(2x + 3)(4x^2 - 6x + 9)$

Factor each polynomial. See Examples 5 and 6.

51. $9a^2 - 16$

52. $16q^2 - 25$

53. $x^4 - 16$

54. $y^4 - 81$

55. $25s^4 - 9t^2$

56. $36z^2 - 81y^4$

57. $(a + b)^2 - 16$

58. $(p - 2q)^2 - 100$

59. $p^4 - 625$

60. $m^4 - 1296$

61. $8 - a^3$

62. $27 - r^3$

63. $125x^3 - 27$

64. $8m^3 - 27n^3$

65. $27y^9 + 125z^6$

66. $27z^9 + 64y^{12}$

67. $(r + 6)^3 - 216$

68. $(b + 3)^3 - 27$

69. $27 - (m + 2n)^3$

70. $125 - (4a - b)^3$

71. ***Concept Check*** Which of the following is the correct complete factorization of $x^4 - 1$?

A. $(x^2 - 1)(x^2 + 1)$

B. $(x^2 + 1)(x + 1)(x - 1)$

C. $(x^2 - 1)^2$

D. $(x - 1)^2(x + 1)^2$

72. ***Concept Check*** Which of the following is the correct factorization of $x^3 + 8$?

A. $(x + 2)^3$

B. $(x + 2)(x^2 + 2x + 4)$

C. $(x + 2)(x^2 - 2x + 4)$

D. $(x + 2)(x^2 - 4x + 4)$

Relating Concepts

For individual or collaborative investigation *(Exercises 73–78)*

The polynomial $x^6 - 1$ can be considered either a difference of squares or a difference of cubes. ***Work Exercises 73–78 in order,*** *to connect the results obtained when two different methods of factoring are used.*

73. Factor $x^6 - 1$ by first factoring as a difference of squares, and then factor further by using the patterns for a sum of cubes and a difference of cubes.

74. Factor $x^6 - 1$ by first factoring as a difference of cubes, and then factor further by using the pattern for a difference of squares.

75. Compare your answers in **Exercises 73 and 74.** Based on these results, what is the factorization of $x^4 + x^2 + 1$?

76. The polynomial $x^4 + x^2 + 1$ cannot be factored using the methods described in this section. However, there is a technique that enables us to factor it, as shown here. Supply the reason why each step is valid.

$x^4 + x^2 + 1 = x^4 + 2x^2 + 1 - x^2$ ________________

$= (x^4 + 2x^2 + 1) - x^2$ ________________

$= (x^2 + 1)^2 - x^2$ ________________

$= (x^2 + 1 - x)(x^2 + 1 + x)$ ________________

$= (x^2 - x + 1)(x^2 + x + 1)$ ________________

77. Compare your answer in **Exercise 75** with the final line in **Exercise 76.** What do you notice?

78. Factor $x^8 + x^4 + 1$ using the technique outlined in **Exercise 76.**

Factor each polynomial by substitution. See Example 7.

79. $7(3k-1)^2+26(3k-1)-8$

80. $6(4z-3)^2+7(4z-3)-3$

81. $9(a-4)^2+30(a-4)+25$

82. $4(5x+7)^2+12(5x+7)+9$

83. m^4-3m^2-10

84. a^4-2a^2-48

Factor by any method. See Examples 1–7.

85. $4b^2+4bc+c^2-16$

86. $(2y-1)^2-4(2y-1)+4$

87. $x^2+xy-5x-5y$

88. $8r^2-3rs+10s^2$

89. $p^4(m-2n)+q(m-2n)$

90. $36a^2+60a+25$

91. $4z^2+28z+49$

92. $6p^4+7p^2-3$

93. $1000x^3+343y^3$

94. $b^2+8b+16-a^2$

95. $125m^6-216$

96. $q^2+6q+9-p^2$

97. $64+(3x+2)^3$

98. $216p^3+125q^3$

99. $(x+y)^3-(x-y)^3$

100. $100r^2-169s^2$

101. $144z^2+121$

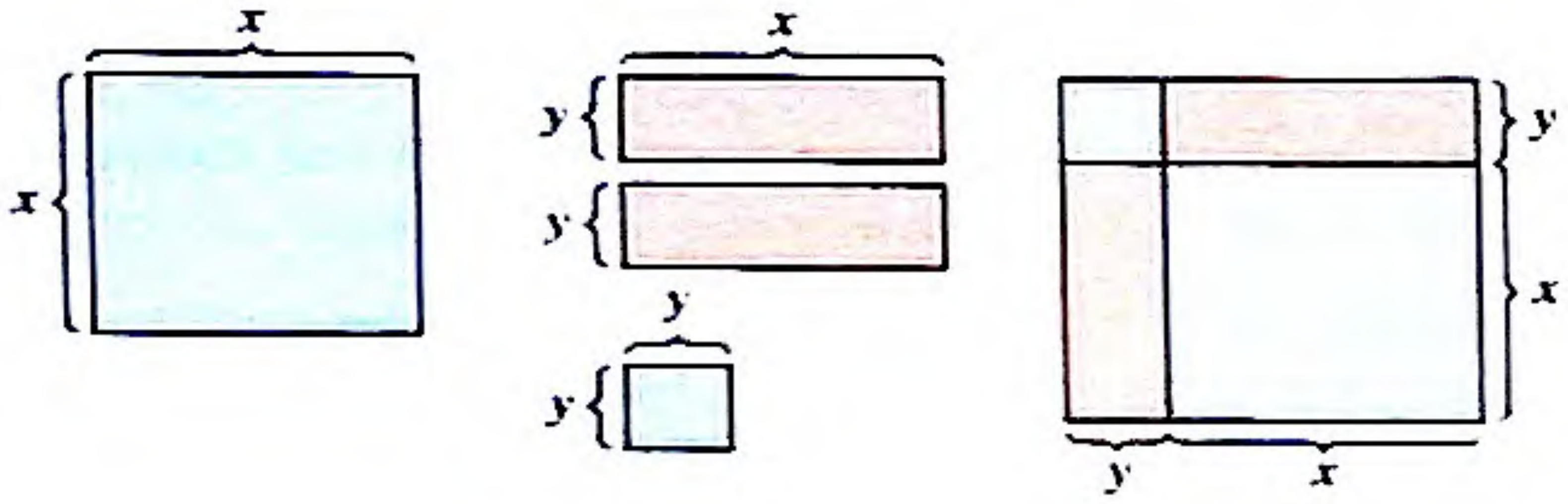
102. $(3a+5)^2-18(3a+5)+81$

103. $(x+y)^2-(x-y)^2$

104. $4z^4-7z^2-15$

105. Are there any conditions under which a sum of squares can be factored? If so, give an example.

106. ***Geometric Modeling*** Explain how the figures give geometric interpretation to the formula $x^2+2xy+y^2=(x+y)^2$.



Factor each polynomial over the set of rational number coefficients.

107. $49x^2-\dfrac{1}{25}$

108. $81y^2-\dfrac{1}{49}$

109. $\dfrac{25}{9}x^4-9y^2$

110. $\dfrac{121}{25}y^4-49x^2$

Concept Check *Find all values of b or c that will make the polynomial a perfect square trinomial.*

111. $4z^2+bz+81$

112. $9p^2+bp+25$

113. $100r^2-60r+c$

114. $49x^2+70x+c$

R.5 Rational Expressions

- Rational Expressions
- Lowest Terms of a Rational Expression
- Multiplication and Division
- Addition and Subtraction
- Complex Fractions

Rational Expressions The quotient of two polynomials P and Q, with $Q \neq 0$, is a **rational expression.**

$$\frac{x+6}{x+2}, \quad \frac{(x+6)(x+4)}{(x+2)(x+4)}, \quad \frac{2p^2+7p-4}{5p^2+20p} \qquad \text{Rational expressions}$$

The **domain** of a rational expression is the set of real numbers for which the expression is defined. Because the denominator of a fraction cannot be 0, the domain consists of all real numbers except those that make the denominator 0. We find these numbers by setting the denominator equal to 0 and solving the resulting equation. For example, in the rational expression

$$\frac{x+6}{x+2},$$

the solution to the equation $x + 2 = 0$ is excluded from the domain. Since this solution is -2, the domain is the set of all real numbers x not equal to -2, or

$$\{x \mid x \neq -2\}. \qquad \text{Set-builder notation (Section R.1)}$$

If the denominator of a rational expression contains a product, we determine the domain with the **zero-factor property** (covered in more detail in **Section 1.4**), which states that $ab = 0$ if and only if $a = 0$ or $b = 0$.

EXAMPLE 1 Finding the Domain

Find the domain of the rational expression.

$$\frac{(x+6)(x+4)}{(x+2)(x+4)}$$

SOLUTION

$$\begin{aligned} (x+2)(x+4) &= 0 && \text{Set the denominator equal to zero.} \\ x+2=0 \quad \text{or} \quad x+4 &= 0 && \text{Zero-factor property} \\ x=-2 \quad \text{or} \quad x &= -4 && \text{Solve each equation.} \end{aligned}$$

The domain is the set of real numbers *not equal to* -2 or -4, written

$$\{x \mid x \neq -2, -4\}.$$

✔ ***Now Try Exercises 1, 3, and 7.***

Lowest Terms of a Rational Expression A rational expression is written in **lowest terms** when the greatest common factor of its numerator and its denominator is 1. We use the following **fundamental principle of fractions.**

Fundamental Principle of Fractions

$$\frac{ac}{bc} = \frac{a}{b} \quad (b \neq 0, c \neq 0)$$

Complex Fractions The quotient of two rational expressions is a **complex fraction.** There are two methods for simplifying a complex fraction.

EXAMPLE 5 Simplifying Complex Fractions

Simplify each complex fraction. In part (b), use two methods.

(a) $\dfrac{6-\dfrac{5}{k}}{1+\dfrac{5}{k}}$ (b) $\dfrac{\dfrac{a}{a+1}+\dfrac{1}{a}}{\dfrac{1}{a}+\dfrac{1}{a+1}}$

SOLUTION

(a) Method 1 for simplifying uses the identity property for multiplication. We multiply both numerator and denominator by the LCD of all the fractions, k.

$$\frac{6-\frac{5}{k}}{1+\frac{5}{k}}=\frac{k\left(6-\frac{5}{k}\right)}{k\left(1+\frac{5}{k}\right)}=\frac{6k-k\left(\frac{5}{k}\right)}{k+k\left(\frac{5}{k}\right)}=\frac{6k-5}{k+5}$$

Distribute k to *all* terms within the parentheses.

(b) $$\frac{\frac{a}{a+1}+\frac{1}{a}}{\frac{1}{a}+\frac{1}{a+1}}=\frac{\left(\frac{a}{a+1}+\frac{1}{a}\right)a(a+1)}{\left(\frac{1}{a}+\frac{1}{a+1}\right)a(a+1)}$$

For Method 1, multiply both numerator and denominator by the LCD of all the fractions, $a(a+1)$.

(Method 1)

$$=\frac{\frac{a}{a+1}(a)(a+1)+\frac{1}{a}(a)(a+1)}{\frac{1}{a}(a)(a+1)+\frac{1}{a+1}(a)(a+1)}$$

Distributive property

$$=\frac{a^2+(a+1)}{(a+1)+a}$$

Multiply

$$=\frac{a^2+a+1}{2a+1}$$

Combine like terms

$$\frac{\frac{a}{a+1}+\frac{1}{a}}{\frac{1}{a}+\frac{1}{a+1}}=\frac{\frac{a^2+1(a+1)}{a(a+1)}}{\frac{1(a+1)+1(a)}{a(a+1)}}$$

For Method 2, find the LCD, and add terms in the numerator and denominator of the complex fraction.

(Method 2)

$$=\frac{\frac{a^2+a+1}{a(a+1)}}{\frac{2a+1}{a(a+1)}}$$

Combine terms in the numerator and denominator

$$=\frac{a^2+a+1}{a(a+1)}\cdot\frac{a(a+1)}{2a+1}$$

Definition of division

$$=\frac{a^2+a+1}{2a+1}$$

The result is the same as in Method 1.

Multiply fractions and write in lowest terms.

✔ *Now Try Exercises 61 and 73*

R.5 Exercises

Find the domain of each rational expression. See Example 1.

1. $\dfrac{x+3}{x-6}$
2. $\dfrac{2x-4}{x+7}$
3. $\dfrac{3x+7}{(4x+2)(x-1)}$
4. $\dfrac{9x+12}{(2x+3)(x-5)}$
5. $\dfrac{12}{x^2+5x+6}$
6. $\dfrac{3}{x^2-5x-6}$
7. $\dfrac{x^2-1}{x+1}$
8. $\dfrac{x^2-25}{x-5}$
9. $\dfrac{x^3-1}{x-1}$

10. ***Concept Check*** Use specific values for x and y to show that in general, $\frac{1}{x}+\frac{1}{y}$ is not equivalent to $\frac{1}{x+y}$.

Write each rational expression in lowest terms. See Example 2.

11. $\dfrac{8x^2+16x}{4x^2}$
12. $\dfrac{36y^2+72y}{9y^2}$
13. $\dfrac{3(3-t)}{(t+5)(t-3)}$
14. $\dfrac{-8(4-y)}{(y+2)(y-4)}$
15. $\dfrac{8k+16}{9k+18}$
16. $\dfrac{20r+10}{30r+15}$
17. $\dfrac{m^2-4m+4}{m^2+m-6}$
18. $\dfrac{r^2-r-6}{r^2+r-12}$
19. $\dfrac{8m^2+6m-9}{16m^2-9}$
20. $\dfrac{6y^2+11y+4}{3y^2+7y+4}$
21. $\dfrac{x^3+64}{x+4}$
22. $\dfrac{y^3-27}{y-3}$

Find each product or quotient. See Example 3.

23. $\dfrac{15p^3}{9p^2}\div\dfrac{6p}{10p^2}$
24. $\dfrac{8r^3}{6r}\div\dfrac{5r^2}{9r^3}$
25. $\dfrac{2k+8}{6}\div\dfrac{3k+12}{2}$
26. $\dfrac{5m+25}{10}\div\dfrac{6m+30}{12}$
27. $\dfrac{x^2+x}{5}\cdot\dfrac{25}{xy+y}$
28. $\dfrac{y^3+y^2}{7}\cdot\dfrac{49}{y^4+y^3}$
29. $\dfrac{4a+12}{2a-10}\div\dfrac{a^2-9}{a^2-a-20}$
30. $\dfrac{6r-18}{9r^2+6r-24}\div\dfrac{4r-12}{12r-16}$
31. $\dfrac{p^2-p-12}{p^2-2p-15}\cdot\dfrac{p^2-9p+20}{p^2-8p+16}$
32. $\dfrac{x^2+2x-15}{x^2+11x+30}\cdot\dfrac{x^2+2x-24}{x^2-8x+15}$
33. $\dfrac{m^2+3m+2}{m^2+5m+4}\div\dfrac{m^2+5m+6}{m^2+10m+24}$
34. $\dfrac{y^2+y-2}{y^2+3y-4}\div\dfrac{y^2+3y+2}{y^2+4y+3}$
35. $\dfrac{x^3+y^3}{x^3-y^3}\cdot\dfrac{x^2-y^2}{x^2+2xy+y^2}$
36. $\dfrac{x^2-y^2}{(x-y)^2}\cdot\dfrac{x^2-xy+y^2}{x^2-2xy+y^2}\div\dfrac{x^3+y^3}{(x-y)^4}$
37. $\dfrac{xz-xw+2yz-2yw}{z^2-w^2}\cdot\dfrac{4z+4w+xz+wx}{16-x^2}$
38. $\dfrac{ac+ad+bc+bd}{a^2-b^2}\cdot\dfrac{a^3-b^3}{2a^2+2ab+2b^2}$

39. ***Concept Check*** Which of the following rational expressions is equivalent to -1? In choices A, B, and D, $x \neq -4$, and in choice C, $x \neq 4$. (*Hint:* There may be more than one answer.)

A. $\dfrac{x-4}{x+4}$ **B.** $\dfrac{-x-4}{x+4}$ **C.** $\dfrac{x-4}{4-x}$ **D.** $\dfrac{x-4}{-x-4}$

40. Explain how to find the least common denominator of several fractions.

Perform each addition or subtraction. See Example 4.

41. $\frac{3}{2k}+\frac{5}{3k}$

42. $\frac{8}{5p}+\frac{3}{4p}$

43. $\frac{1}{6m}+\frac{2}{5m}+\frac{4}{m}$

44. $\frac{8}{3p}+\frac{5}{4p}+\frac{9}{2p}$

45. $\frac{1}{a}-\frac{b}{a^2}$

46. $\frac{3}{z}+\frac{x}{z^2}$

47. $\frac{5}{12x^2y}-\frac{11}{6xy}$

48. $\frac{7}{18a^3b^2}-\frac{2}{9ab}$

49. $\frac{17y+3}{9y+7}-\frac{-10y-18}{9y+7}$

50. $\frac{7x+8}{3x+2}-\frac{x+4}{3x+2}$

51. $\frac{1}{x+z}+\frac{1}{x-z}$

52. $\frac{m+1}{m-1}+\frac{m-1}{m+1}$

53. $\frac{3}{a-2}-\frac{1}{2-a}$

54. $\frac{4}{p-q}-\frac{2}{q-p}$

55. $\frac{x+y}{2x-y}-\frac{2x}{y-2x}$

56. $\frac{m-4}{3m-4}-\frac{5m}{4-3m}$

57. $\frac{4}{x+1}+\frac{1}{x^2-x+1}-\frac{12}{x^3+1}$

58. $\frac{5}{x+2}+\frac{2}{x^2-2x+4}-\frac{60}{x^3+8}$

59. $\frac{3x}{x^2+x-12}-\frac{x}{x^2-16}$

60. $\frac{p}{2p^2-9p-5}-\frac{2p}{6p^2-p-2}$

Simplify each expression. See Example 5.

61. $\dfrac{1+\frac{1}{x}}{1-\frac{1}{x}}$

62. $\dfrac{2-\frac{2}{y}}{2+\frac{2}{y}}$

63. $\dfrac{\frac{1}{x+1}-\frac{1}{x}}{\frac{1}{x}}$

64. $\dfrac{\frac{1}{y+3}-\frac{1}{y}}{\frac{1}{y}}$

65. $\dfrac{1+\frac{1}{1-b}}{1-\frac{1}{1+b}}$

66. $\dfrac{2+\frac{2}{1+x}}{2-\frac{2}{1-x}}$

67. $\dfrac{\frac{1}{a^3+b^3}}{\frac{1}{a^2+2ab+b^2}}$

68. $\dfrac{\frac{1}{x^3-y^3}}{\frac{1}{x^2-y^2}}$

69. $\dfrac{m-\frac{1}{m^2-4}}{\frac{1}{m+2}}$

70. $\dfrac{y+\frac{1}{y^2-9}}{\frac{1}{y+3}}$

71. $\dfrac{\frac{3}{p^2-16}+p}{\frac{1}{p-4}}$

72. $\dfrac{\frac{6}{x^2-25}+x}{\frac{1}{x-5}}$

73. $\dfrac{\frac{y+3}{y}-\frac{4}{y-1}}{\frac{y}{y-1}+\frac{1}{y}}$

74. $\dfrac{\frac{x+4}{x}-\frac{3}{x-2}}{\frac{x}{x-2}+\frac{1}{x}}$

75. $\dfrac{\frac{1}{x+h}-\frac{1}{x}}{h}$

76. $\dfrac{\frac{-2}{x+h}-\frac{-2}{x}}{h}$

77. $\dfrac{\frac{1}{(x+h)^2+9}-\frac{1}{x^2+9}}{h}$

78. $\dfrac{\frac{2}{(x+h)^2+16}-\frac{2}{x^2+16}}{h}$

(Modeling) Distance from the Origin of the Nile River The Nile River in Africa is about 4000 mi *long. The Nile begins as an outlet of Lake Victoria at an altitude of* 7000 ft *above sea level and empties into the Mediterranean Sea at sea level* (0 ft). *The distance from its origin in thousands of miles is related to its height above sea level in thousands of feet* (x) *by the following rational expression.*

$$\frac{7-x}{0.639x+1.75}$$

For example, when the river is at an altitude of 600 ft, $x = 0.6$ *(thousand), and the distance from the origin is*

$$\frac{7-0.6}{0.639(0.6)+1.75} \approx 3, \quad \textit{which represents } 3000 \text{ mi.}$$

(Source: World Almanac and Book of Facts.)

79. What is the distance from the origin of the Nile when the river has an altitude of 7000 ft?

80. What is the distance from the origin of the Nile when the river has an altitude of 1200 ft?

(Modeling) Cost-Benefit Model for a Pollutant In situations involving environmental pollution, a cost-benefit model expresses cost in terms of the percentage of pollutant removed from the environment. Suppose a cost-benefit model is expressed as

$$y = \frac{6.7x}{100-x},$$

where y is the cost in thousands of dollars of removing x percent of a certain pollutant. Find the value of y for each given value of x.

81. $x = 75$ (75%)

82. $x = 95$ (95%)

R.6 Rational Exponents

- **Negative Exponents and the Quotient Rule**
- **Rational Exponents**
- **Complex Fractions Revisited**

Negative Exponents and the Quotient Rule In Section R.2, we justified the definition $a^0 = 1$ for $a \neq 0$ using the product rule for exponents. Suppose that n is a positive integer, and we wish to define a^{-n} to be consistent with the application of the product rule. Consider the product $a^n \cdot a^{-n}$, and apply the rule.

$$\begin{aligned} a^n \cdot a^{-n} &= a^{n+(-n)} && \text{Product rule} \\ &= a^0 && n \text{ and } -n \text{ are additive inverses.} \\ &= 1 && \text{Definition of } a^0 \end{aligned}$$

The expression a^{-n} acts as the *reciprocal* of a^n, which is written $\frac{1}{a^n}$. Thus, these two expressions must be equivalent.

Negative Exponent

Suppose that a is a nonzero real number and n is any integer.

$$a^{-n} = \frac{1}{a^n}$$

EXAMPLE 6 Using the Rules for Exponents

Simplify each expression. Assume all variables represent positive real numbers.

(a) $\dfrac{27^{1/3} \cdot 27^{5/3}}{27^3}$ **(b)** $81^{5/4} \cdot 4^{-3/2}$ **(c)** $6y^{2/3} \cdot 2y^{1/2}$

(d) $\left(\dfrac{3m^{5/6}}{y^{3/4}}\right)^2\left(\dfrac{8y^3}{m^6}\right)^{2/3}$ **(e)** $m^{2/3}(m^{7/3} + 2m^{1/3})$

SOLUTION

(a)

$$\begin{aligned}\frac{27^{1/3} \cdot 27^{5/3}}{27^3} &= \frac{27^{1/3+5/3}}{27^3} && \text{Product rule}\\ &= \frac{27^2}{27^3} && \text{Simplify.}\\ &= 27^{2-3} && \text{Quotient rule}\\ &= 27^{-1}, \quad \text{or} \quad \frac{1}{27} && \text{Negative exponent}\end{aligned}$$

(b)

$$\begin{aligned}81^{5/4} \cdot 4^{-3/2} &= (81^{1/4})^5(4^{1/2})^{-3}\\ &= 3^5 \cdot 2^{-3}\\ &= \frac{3^5}{2^3}, \quad \text{or} \quad \frac{243}{8}\end{aligned}$$

(c)

$$\begin{aligned}6y^{2/3} \cdot 2y^{1/2} &= 12y^{2/3+1/2}\\ &= 12y^{7/6}\end{aligned}$$

(d)

$$\begin{aligned}\left(\frac{3m^{5/6}}{y^{3/4}}\right)^2\left(\frac{8y^3}{m^6}\right)^{2/3} &= \frac{9m^{5/3}}{y^{3/2}} \cdot \frac{4y^2}{m^4} && \text{Power rules}\\ &= 36m^{5/3-4}y^{2-3/2} && \text{Quotient rule}\\ &= 36m^{-7/3}y^{1/2} && \text{Simplify the exponents.}\\ &= \frac{36y^{1/2}}{m^{7/3}} && \text{Simplify.}\end{aligned}$$

(e)

$$\begin{aligned}m^{2/3}(m^{7/3} + 2m^{1/3}) &= m^{2/3} \cdot m^{7/3} + m^{2/3} \cdot 2m^{1/3} && \text{Distributive property (Section R.2)}\\ &= m^{2/3+7/3} + 2m^{2/3+1/3} && \text{Product rule}\\ &= m^3 + 2m && \text{Simplify.}\end{aligned}$$

Do *not* multiply the exponents.

✔ ***Now Try Exercises 61, 63, 67, 69, and 75.***

EXAMPLE 7 Factoring Expressions with Negative or Rational Exponents

Factor out the least power of the variable or variable expression. Assume all variables represent positive real numbers.

(a) $12x^{-2} - 8x^{-3}$ **(b)** $4m^{1/2} + 3m^{3/2}$ **(c)** $(y - 2)^{-1/3} + (y - 2)^{2/3}$

SOLUTION

(a) The least exponent on $12x^{-2} - 8x^{-3}$ is -3. Since 4 is a common numerical factor, factor out $4x^{-3}$.

$$\begin{aligned}12x^{-2} - 8x^{-3} &= 4x^{-3}(3x^{-2-(-3)} - 2x^{-3-(-3)}) && \text{Factor.}\\ &= 4x^{-3}(3x - 2) && \text{Simplify the exponents.}\end{aligned}$$

Check by multiplying on the right.

LOOKING AHEAD TO CALCULUS
The technique of Example 7(c) is used often in calculus.

(b) $4m^{1/2} + 3m^{3/2} = m^{1/2}(4 + 3m)$ Factor out $m^{1/2}$.

To *check*, multiply $m^{1/2}$ by $4 + 3m$.

(c)
$$\begin{aligned}(y-2)^{-1/3} + (y-2)^{2/3} &= (y-2)^{-1/3}[1 + (y-2)]\\ &= (y-2)^{-1/3}(y-1)\end{aligned}$$

✔ *Now Try Exercises 83, 89, and 93.*

Complex Fractions Revisited Negative exponents are sometimes used to write complex fractions. Recall that complex fractions are simplified either by first multiplying the numerator and denominator by the LCD of all the denominators, or by performing any indicated operations in the numerator and the denominator and then using the definition of division for fractions.

EXAMPLE 8 Simplifying a Fraction with Negative Exponents

Simplify $\dfrac{(x+y)^{-1}}{x^{-1}+y^{-1}}$. Write the result with only positive exponents.

SOLUTION

$$\frac{(x+y)^{-1}}{x^{-1}+y^{-1}} = \frac{\frac{1}{x+y}}{\frac{1}{x}+\frac{1}{y}} \qquad \text{Definition of negative exponent}$$

$$= \frac{\frac{1}{x+y}}{\frac{y+x}{xy}} \qquad \text{Add fractions in the denominator. (Section R.5)}$$

$$= \frac{1}{x+y} \cdot \frac{xy}{x+y} \qquad \text{Multiply by the reciprocal of the denominator of the complex fraction.}$$

$$= \frac{xy}{(x+y)^2} \qquad \text{Multiply fractions.}$$

✔ *Now Try Exercise 99.*

CAUTION Remember that if $r \neq 1$, then $(x+y)^r \neq x^r + y^r$. In particular, this means that $(x+y)^{-1} \neq x^{-1} + y^{-1}$.

R.6 Exercises

Concept Check In Exercises 1 and 2, match each expression in Column I with its equivalent expression in Column II. Choices may be used once, more than once, or not at all.

I	II	I	II
1. (a) 4^{-2}	**A.** 16	**2. (a)** 5^{-3}	**A.** 125
(b) -4^{-2}	**B.** $\frac{1}{16}$	**(b)** -5^{-3}	**B.** -125
(c) $(-4)^{-2}$	**C.** -16	**(c)** $(-5)^{-3}$	**C.** $\frac{1}{125}$
(d) $-(-4)^{-2}$	**D.** $-\frac{1}{16}$	**(d)** $-(-5)^{-3}$	**D.** $-\frac{1}{125}$

Write each expression with only positive exponents and evaluate if possible. Assume all variables represent nonzero real numbers. **See Example 1.**

3. $(-4)^{-3}$ **4.** $(-5)^{-2}$ **5.** -5^{-4}

6. -7^{-2} **7.** $\left(\frac{1}{3}\right)^{-2}$ **8.** $\left(\frac{4}{3}\right)^{-3}$

9. $(4x)^{-2}$ **10.** $(5t)^{-3}$ **11.** $4x^{-2}$

12. $5t^{-3}$ **13.** $-a^{-3}$ **14.** $-b^{-4}$

Perform the indicated operations. Write each answer using only positive exponents. Assume all variables represent nonzero real numbers. **See Examples 2 and 3.**

15. $\frac{4^8}{4^6}$ **16.** $\frac{5^9}{5^7}$ **17.** $\frac{x^{12}}{x^8}$ **18.** $\frac{y^{14}}{y^{10}}$

19. $\frac{r^7}{r^{10}}$ **20.** $\frac{y^8}{y^{12}}$ **21.** $\frac{6^4}{6^{-2}}$ **22.** $\frac{7^5}{7^{-3}}$

23. $\frac{4r^{-3}}{6r^{-6}}$ **24.** $\frac{15s^{-4}}{5s^{-8}}$ **25.** $\frac{16m^{-5}n^4}{12m^2n^{-3}}$ **26.** $\frac{15a^{-5}b^{-1}}{25a^{-2}b^4}$

27. $-4r^{-2}(r^4)^2$ **28.** $-2m^{-1}(m^3)^2$ **29.** $(5a^{-1})^4(a^2)^{-3}$ **30.** $(3p^{-4})^2(p^3)^{-1}$

31. $\frac{(p^{-2})^0}{5p^{-4}}$ **32.** $\frac{(m^4)^0}{9m^{-3}}$ **33.** $\frac{(3pq)q^2}{6p^2q^4}$ **34.** $\frac{(-8xy)y^3}{4x^5y^4}$

35. $\frac{4a^5(a^{-1})^3}{(a^{-2})^{-2}}$ **36.** $\frac{12k^{-2}(k^{-3})^{-4}}{6k^5}$ **37.** $\frac{(5x)^{-2}(5x^3)^{-3}}{(5^{-2}x^{-3})^3}$ **38.** $\frac{(8y^2)^{-4}(8y^5)^{-2}}{(8^{-3}y^{-4})^2}$

Evaluate each expression. **See Example 4.**

39. $169^{1/2}$ **40.** $121^{1/2}$ **41.** $16^{1/4}$ **42.** $625^{1/4}$

43. $\left(-\frac{64}{27}\right)^{1/3}$ **44.** $\left(-\frac{8}{27}\right)^{1/3}$ **45.** $(-4)^{1/2}$ **46.** $(-64)^{1/4}$

Concept Check *In Exercises 47 and 48, match each expression from Column I with its equivalent expression from Column II. Choices may be used once, more than once, or not at all.*

I

47. (a) $\left(\frac{4}{9}\right)^{3/2}$
(b) $\left(\frac{4}{9}\right)^{-3/2}$
(c) $-\left(\frac{9}{4}\right)^{3/2}$
(d) $-\left(\frac{4}{9}\right)^{-3/2}$

48. (a) $\left(\frac{8}{27}\right)^{2/3}$
(b) $\left(\frac{8}{27}\right)^{-2/3}$
(c) $-\left(\frac{27}{8}\right)^{2/3}$
(d) $-\left(\frac{27}{8}\right)^{-2/3}$

II

A. $\frac{9}{4}$ **B.** $-\frac{9}{4}$
C. $-\frac{4}{9}$ **D.** $\frac{4}{9}$
E. $\frac{8}{27}$ **F.** $-\frac{27}{8}$
G. $\frac{27}{8}$ **H.** $-\frac{8}{27}$

Perform the indicated operations. Write each answer using only positive exponents. Assume all variables represent positive real numbers. **See Examples 5 and 6.**

49. $8^{2/3}$ **50.** $27^{4/3}$ **51.** $100^{3/2}$

52. $64^{3/2}$ **53.** $-81^{3/4}$ **54.** $(-32)^{-4/5}$

55. $\left(\frac{27}{64}\right)^{-4/3}$ **56.** $\left(\frac{121}{100}\right)^{-3/2}$ **57.** $3^{1/2} \cdot 3^{3/2}$

58. $6^{4/3} \cdot 6^{2/3}$ **59.** $\frac{64^{5/3}}{64^{4/3}}$ **60.** $\frac{125^{7/3}}{125^{5/3}}$

61. $y^{7/3} \cdot y^{-4/3}$ **62.** $r^{-8/9} \cdot r^{17/9}$ **63.** $\frac{k^{1/3}}{k^{2/3} \cdot k^{-1}}$

64. $\frac{z^{3/4}}{z^{5/4} \cdot z^{-2}}$ **65.** $\frac{(x^{1/4}y^{2/5})^{20}}{x^2}$ **66.** $\frac{(r^{1/5}s^{2/3})^{15}}{r^2}$

67. $\frac{(x^{2/3})^2}{(x^2)^{7/3}}$ **68.** $\frac{(p^3)^{1/4}}{(p^{5/4})^2}$ **69.** $\left(\frac{16m^3}{n}\right)^{1/4}\left(\frac{9n^{-1}}{m^2}\right)^{1/2}$

70. $\left(\frac{25^4a^3}{b^2}\right)^{1/8}\left(\frac{4^2b^{-5}}{a^2}\right)^{1/4}$ **71.** $\frac{p^{1/5}p^{7/10}p^{1/2}}{(p^3)^{-1/5}}$ **72.** $\frac{z^{1/3}z^{-2/3}z^{1/6}}{(z^{-1/6})^3}$

Solve each applied problem.

73. ***(Modeling) Holding Time of Athletes*** A group of ten athletes were tested for isometric endurance by measuring the length of time they could resist a load pulling on their legs while seated. The approximate amount of time (called the **holding time**) that they could resist the load was given by the formula

$$t = 31{,}293w^{-1.5},$$

where w is the weight of the load in pounds and the holding time t is measured in seconds. (*Source:* Townend, M. Stewart, *Mathematics in Sport,* Chichester, Ellis Horwood Limited.)

(a) Determine the holding time for a load of 25 lb.
(b) When the weight of the load is doubled, by what factor is the holding time changed?

74. ***Duration of a Storm*** Suppose that meteorologists approximate the duration of a particular storm by using the formula

$$T = 0.07D^{3/2},$$

where T is the time in hours that a storm of diameter D (in miles) lasts.

(a) The National Weather Service reports that a storm 4 mi in diameter is headed toward New Haven. How long can the residents expect the storm to last?
(b) After weeks of dry weather, a thunderstorm is predicted for the farming community of Apple Valley. The crops need at least 1.5 hr of rain. Local radar shows that the storm is 7 mi in diameter. Will it rain long enough to meet the farmers' need?

Find each product. Assume all variables represent positive real numbers. ***See Example 6(e) in this section and Example 7 in Section R.3.***

75. $y^{5/8}(y^{3/8} - 10y^{11/8})$ **76.** $p^{11/5}(3p^{4/5} + 9p^{19/5})$

77. $-4k(k^{7/3} - 6k^{1/3})$ **78.** $-5y(3y^{9/10} + 4y^{3/10})$

79. $(x + x^{1/2})(x - x^{1/2})$ **80.** $(2z^{1/2} + z)(z^{1/2} - z)$

81. $(r^{1/2} - r^{-1/2})^2$ **82.** $(p^{1/2} - p^{-1/2})(p^{1/2} + p^{-1/2})$

Factor, using the given common factor. Assume all variables represent positive real numbers. ***See Example 7.***

83. $4k^{-1} + k^{-2}$; k^{-2} **84.** $y^{-5} - 3y^{-3}$; y^{-5} **85.** $4t^{-2} + 8t^{-4}$; $4t^{-4}$

86. $5r^{-6} - 10r^{-8}$; $5r^{-8}$ **87.** $9z^{-1/2} + 2z^{1/2}$; $z^{-1/2}$ **88.** $3m^{2/3} - 4m^{-1/3}$; $m^{-1/3}$

89. $p^{-3/4} - 2p^{-7/4}$; $p^{-7/4}$ **90.** $6r^{-2/3} - 5r^{-5/3}$; $r^{-5/3}$

91. $-4a^{-2/5} + 16a^{-7/5}$; $4a^{-7/5}$ **92.** $-3p^{-3/4} - 30p^{-7/4}$; $-3p^{-7/4}$

93. $(p + 4)^{-3/2} + (p + 4)^{-1/2} + (p + 4)^{1/2}$; $(p + 4)^{-3/2}$

94. $(3r + 1)^{-2/3} + (3r + 1)^{1/3} + (3r + 1)^{4/3}$; $(3r + 1)^{-2/3}$

95. $2(3x + 1)^{-3/2} + 4(3x + 1)^{-1/2} + 6(3x + 1)^{1/2}$; $2(3x + 1)^{-3/2}$

96. $7(5t + 3)^{-5/3} + 14(5t + 3)^{-2/3} - 21(5t + 3)^{1/3}$; $7(5t + 3)^{-5/3}$

97. $4x(2x + 3)^{-5/9} + 6x^2(2x + 3)^{4/9} - 8x^3(2x + 3)^{13/9}$; $2x(2x + 3)^{-5/9}$

98. $6y^3(4y - 1)^{-3/7} - 8y^2(4y - 1)^{4/7} + 16y(4y - 1)^{11/7}$; $2y(4y - 1)^{-3/7}$

Perform all indicated operations and write each answer with positive integer exponents. ***See Example 8.***

99. $\dfrac{a^{-1} + b^{-1}}{(ab)^{-1}}$ **100.** $\dfrac{p^{-1} - q^{-1}}{(pq)^{-1}}$

101. $\dfrac{r^{-1} + q^{-1}}{r^{-1} - q^{-1}} \cdot \dfrac{r - q}{r + q}$ **102.** $\dfrac{x^{-2} + y^{-2}}{x^{-2} - y^{-2}} \cdot \dfrac{x + y}{x - y}$

103. $\dfrac{x - 9y^{-1}}{(x - 3y^{-1})(x + 3y^{-1})}$ **104.** $\dfrac{a - 16b^{-1}}{(a + 4b^{-1})(a - 4b^{-1})}$

Simplify each rational expression. Use factoring, and refer to Section R.5 as needed. Assume all variable expressions represent positive real numbers.

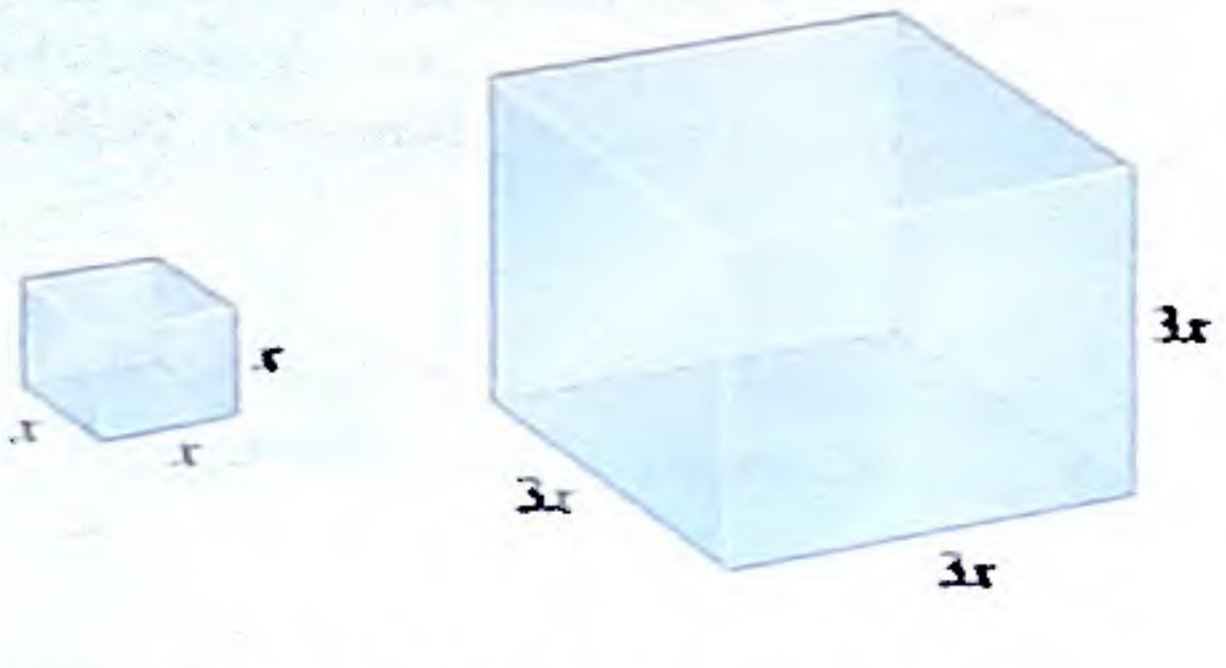
105. $\dfrac{(x^2 + 1)^4(2x) - x^2(4)(x^2 + 1)^3(2x)}{(x^2 + 1)^8}$ **106.** $\dfrac{(y^2 + 2)^5(3y) - y^3(6)(y^2 + 2)^4(3y)}{(y^2 + 2)^7}$

107. $\dfrac{4(x^2 - 1)^3 + 8x(x^2 - 1)^4}{16(x^2 - 1)^3}$ **108.** $\dfrac{10(4x^2 - 9)^2 - 25x(4x^2 - 9)^3}{15(4x^2 - 9)^6}$

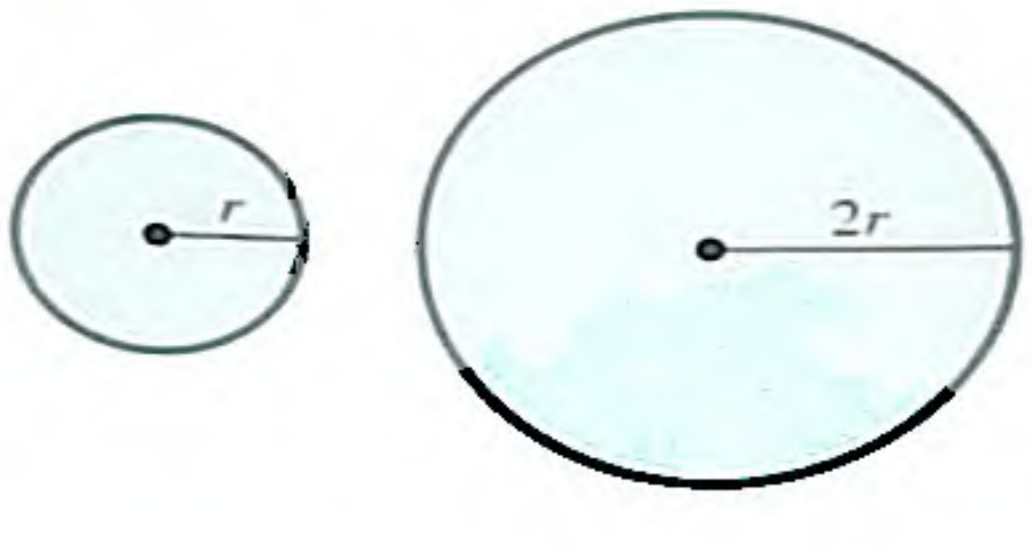
109. $\dfrac{2(2x - 3)^{1/3} - (x - 1)(2x - 3)^{-2/3}}{(2x - 3)^{2/3}}$ **110.** $\dfrac{7(3t + 1)^{1/4} - (t - 1)(3t + 1)^{-3/4}}{(3t + 1)^{3/4}}$

Concept Check *Answer each question.*

111. If the lengths of the sides of a cube are tripled, by what factor will the volume change?



112. If the radius of a circle is doubled, by what factor will the area change?



Concept Check *Calculate each value mentally.*

113. $0.2^{2/3} \cdot 40^{2/3}$ **114.** $0.1^{3/2} \cdot 90^{3/2}$ **115.** $\dfrac{2^{2/3}}{2000^{2/3}}$ **116.** $\dfrac{20^{3/2}}{5^{3/2}}$

R.7 Radical Expressions

- Radical Notation
- Simplified Radicals
- Operations with Radicals
- Rationalizing Denominators

Radical Notation In **Section R.6** we used rational exponents to express roots. An alternative notation for roots is **radical notation.**

Radical Notation for $a^{1/n}$

Suppose that a is a real number, n is a positive integer, and $a^{1/n}$ is a real number.

$$\sqrt[n]{a} = a^{1/n}$$

Radical Notation for $a^{m/n}$

Suppose that a is a real number, m is an integer, n is a positive integer, and $\sqrt[n]{a}$ is a real number.

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

In the radical $\sqrt[n]{a}$, the symbol $\sqrt[n]{}$ is a **radical symbol,** the number a is the **radicand,** and n is the **index.** We use the familiar notation $\sqrt{a}$ instead of $\sqrt[2]{a}$ for the square root.

For even values of n (square roots, fourth roots, and so on), when a is positive, there are two nth roots, one positive and one negative. In such cases, the notation $\sqrt[n]{a}$ represents the positive root, the **principal nth root.** We write the negative root as $-\sqrt[n]{a}$.

EXAMPLE 1 **Evaluating Roots**

Write each root using exponents and evaluate.

(a) $\sqrt[4]{16}$ **(b)** $-\sqrt[4]{16}$ **(c)** $\sqrt[5]{-32}$

(d) $\sqrt[3]{1000}$ **(e)** $\sqrt[6]{\dfrac{64}{729}}$ **(f)** $\sqrt[4]{-16}$

SOLUTION

(a) $\sqrt[4]{16} = 16^{1/4}$
$= 2$

(b) $-\sqrt[4]{16} = -16^{1/4}$
$= -2$

(c) $\sqrt[5]{-32} = (-32)^{1/5}$
$= -2$

(d) $\sqrt[3]{1000} = 1000^{1/3}$
$= 10$

(e) $\sqrt[6]{\dfrac{64}{729}} = \left(\dfrac{64}{729}\right)^{1/6}$
$= \dfrac{2}{3}$

(f) $\sqrt[4]{-16}$ is not a real number.

✔ *Now Try Exercises 1, 3, 7, and 11.*

LOOKING AHEAD TO CALCULUS

Another standard problem in calculus is investigating the value that an expression such as $\frac{\sqrt{x^2+9}-3}{x^2}$ approaches as x approaches 0. This cannot be done by simply substituting 0 for x, since the result is $\frac{0}{0}$. However, by **rationalizing the numerator,** we can show that for $x \neq 0$ the expression is equivalent to $\frac{1}{\sqrt{x^2+9}+3}$. Then, by substituting 0 for x, we find that the original expression approaches $\frac{1}{6}$ as x approaches 0.

(b) $\sqrt[3]{\frac{5}{x^6}} - \sqrt[3]{\frac{4}{x^9}} = \frac{\sqrt[3]{5}}{\sqrt[3]{x^6}} - \frac{\sqrt[3]{4}}{\sqrt[3]{x^9}}$ Quotient rule

$= \frac{\sqrt[3]{5}}{x^2} - \frac{\sqrt[3]{4}}{x^3}$ Simplify the denominators.

$= \frac{x\sqrt[3]{5}}{x^3} - \frac{\sqrt[3]{4}}{x^3}$ Write with a common denominator. (Section R.5)

$= \frac{x\sqrt[3]{5} - \sqrt[3]{4}}{x^3}$ Subtract the numerators.

✓ ***Now Try Exercises 85 and 87.***

In **Example 9(a),** we saw that the product

$(\sqrt{7}-\sqrt{10})(\sqrt{7}+\sqrt{10})$ equals -3, a rational number.

This suggests a way to rationalize a denominator that is a binomial in which one or both terms is a square root radical. The expressions $a-b$ and $a+b$ are **conjugates.**

EXAMPLE 12 Rationalizing a Binomial Denominator

Rationalize the denominator of $\frac{1}{1-\sqrt{2}}$.

SOLUTION

$\frac{1+\sqrt{2}}{1+\sqrt{2}} = 1$

$$\frac{1}{1-\sqrt{2}} = \frac{1(1+\sqrt{2})}{(1-\sqrt{2})(1+\sqrt{2})}$$

Multiply numerator and denominator by the conjugate of the denominator: $1+\sqrt{2}$.

$$= \frac{1+\sqrt{2}}{1-2}$$

$(x-y)(x+y) = x^2-y^2$

$$= -1-\sqrt{2}$$

$\frac{1+\sqrt{2}}{-1} = -1-\sqrt{2}$

✓ ***Now Try Exercise 93.***

R.7 Exercises

Write each root using exponents and evaluate. See Example 1.

1. $\sqrt[3]{125}$

2. $\sqrt[3]{216}$

3. $\sqrt[4]{81}$

4. $\sqrt[4]{256}$

5. $\sqrt[3]{-125}$

6. $\sqrt[3]{-343}$

7. $\sqrt[4]{-81}$

8. $\sqrt[4]{-256}$

9. $\sqrt[5]{32}$

10. $\sqrt[7]{128}$

11. $-\sqrt[5]{-32}$

12. $-\sqrt[3]{-343}$

Concept Check In Exercises 13 and 14, match the rational exponent expression in Column I with the equivalent radical expression in Column II. Assume that x is not 0. See Example 2.

I		II	
13. (a) $(-3x)^{1/3}$	**14.** (a) $-3x^{1/3}$	**A.** $\frac{3}{\sqrt[3]{x}}$	**B.** $-3\sqrt[3]{x}$
(b) $(-3x)^{-1/3}$	(b) $-3x^{-1/3}$	**C.** $\frac{1}{\sqrt[3]{3x}}$	**D.** $\frac{-3}{\sqrt[3]{x}}$
(c) $(3x)^{1/3}$	(c) $3x^{-1/3}$	**E.** $3\sqrt[3]{x}$	**F.** $\sqrt[3]{-3x}$
(d) $(3x)^{-1/3}$	(d) $3x^{1/3}$	**G.** $\sqrt[3]{3x}$	**H.** $\frac{1}{\sqrt[3]{-3x}}$

If the expression is in exponential form, write it in radical form. If it is in radical form, write it in exponential form. Assume all variables represent positive real numbers. See Examples 2 and 3.

15. $m^{2/3}$
16. $p^{5/4}$
17. $(2m + p)^{2/3}$
18. $(5r + 3t)^{4/7}$
19. $\sqrt[5]{k^2}$
20. $\sqrt[4]{z^5}$
21. $-3\sqrt{5p^3}$
22. $-m\sqrt{2y^5}$

Concept Check Answer each question.

23. For which of the following cases is $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$ a true statement?

A. a and b both positive
B. a and b both negative

24. For what positive integers n greater than or equal to 2 is $\sqrt[n]{a^n} = a$ always a true statement?

25. For what values of x is $\sqrt{9ax^2} = 3x\sqrt{a}$ a true statement? Assume $a \geq 0$.

26. Which of the following expressions is *not* simplified? Give the simplified form.

A. $\sqrt[3]{2y}$
B. $\frac{\sqrt{5}}{2}$
C. $\sqrt[4]{m^3}$
D. $\sqrt{\frac{3}{4}}$

Simplify each expression. See Example 4.

27. $\sqrt[4]{x^4}$
28. $\sqrt[6]{x^6}$
29. $\sqrt{25k^4m^2}$
30. $\sqrt[4]{81p^{12}q^4}$
31. $\sqrt{(4x - y)^2}$
32. $\sqrt[4]{(5 + 2m)^4}$

Simplify each expression. Assume all variables represent positive real numbers. See Examples 1, 4–6, and 8–11.

33. $\sqrt[3]{81}$
34. $\sqrt[3]{250}$
35. $-\sqrt[4]{32}$
36. $-\sqrt[4]{243}$
37. $\sqrt{14} \cdot \sqrt{3pqr}$
38. $\sqrt{7} \cdot \sqrt{5xt}$
39. $\sqrt[3]{7x} \cdot \sqrt[3]{2y}$
40. $\sqrt[3]{9x} \cdot \sqrt[3]{4y}$
41. $-\sqrt{\frac{9}{25}}$
42. $-\sqrt{\frac{16}{49}}$
43. $-\sqrt[3]{\frac{5}{8}}$
44. $-\sqrt[4]{\frac{3}{16}}$
45. $\sqrt[4]{\frac{m}{n^4}}$
46. $\sqrt[6]{\frac{r}{s^6}}$
47. $3\sqrt[5]{-3125}$
48. $5\sqrt[3]{-343}$
49. $\sqrt[3]{16(-2)^4(2)^8}$
50. $\sqrt[3]{25(-3)^4(5)^3}$
51. $\sqrt{8x^5z^8}$
52. $\sqrt{24m^6n^5}$
53. $\sqrt[4]{x^4 + y^4}$

54. $\sqrt[3]{27 + a^3}$

55. $\sqrt{\frac{2}{3x}}$

56. $\sqrt{\frac{5}{3p}}$

57. $\sqrt{\frac{x^5y^3}{z^2}}$

58. $\sqrt{\frac{g^3h^5}{r^3}}$

59. $\sqrt[3]{\frac{8}{x^4}}$

60. $\sqrt[3]{\frac{9}{16p^4}}$

61. $\sqrt[4]{\frac{g^3h^5}{9r^6}}$

62. $\sqrt[4]{\frac{32x^5}{y^5}}$

63. $\sqrt[8]{3^4}$

64. $\sqrt[9]{5^3}$

65. $\sqrt[3]{\sqrt{4}}$

66. $\sqrt[4]{\sqrt{25}}$

67. $\sqrt[4]{\sqrt[3]{2}}$

68. $\sqrt[5]{\sqrt[3]{9}}$

Simplify each expression. Assume all variables represent positive real numbers. See Examples 7, 9, and 11.

69. $8\sqrt{2x} - \sqrt{8x} + \sqrt{72x}$

70. $4\sqrt{18k} - \sqrt{72k} + \sqrt{50k}$

71. $2\sqrt[3]{3} + 4\sqrt[3]{24} - \sqrt[3]{81}$

72. $\sqrt[3]{32} - 5\sqrt[3]{4} + 2\sqrt[3]{108}$

73. $\sqrt[4]{81x^6y^3} - \sqrt[4]{16x^{10}y^3}$

74. $\sqrt[4]{256x^5y^6} + \sqrt[4]{625x^9y^2}$

75. $5\sqrt{6} + 2\sqrt{10}$

76. $3\sqrt{11} - 5\sqrt{13}$

77. $(\sqrt{2} + 3)(\sqrt{2} - 3)$

78. $(\sqrt{5} + \sqrt{2})(\sqrt{5} - \sqrt{2})$

79. $(\sqrt[3]{11} - 1)(\sqrt[3]{11^2} + \sqrt[3]{11} + 1)$

80. $(\sqrt[3]{7} + 3)(\sqrt[3]{7^2} - 3\sqrt[3]{7} + 9)$

81. $(\sqrt{3} + \sqrt{8})^2$

82. $(\sqrt{5} + \sqrt{10})^2$

83. $(3\sqrt{2} + \sqrt{3})(2\sqrt{3} - \sqrt{2})$

84. $(4\sqrt{5} + \sqrt{2})(3\sqrt{2} - \sqrt{5})$

85. $\frac{\sqrt[3]{mn} \cdot \sqrt[3]{m^2}}{\sqrt[3]{n^2}}$

86. $\frac{\sqrt[3]{8m^2n^3} \cdot \sqrt[3]{2m^2}}{\sqrt[3]{32m^4n^3}}$

87. $\sqrt[3]{\frac{2}{x^6}} - \sqrt[3]{\frac{5}{x^9}}$

88. $\sqrt[4]{\frac{7}{t^{12}}} + \sqrt[4]{\frac{9}{t^4}}$

89. $\frac{1}{\sqrt{2}} + \frac{3}{\sqrt{8}} + \frac{1}{\sqrt{32}}$

90. $\frac{2}{\sqrt{12}} - \frac{1}{\sqrt{27}} - \frac{5}{\sqrt{48}}$

91. $\frac{-4}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{24}} - \frac{2}{\sqrt[3]{81}}$

92. $\frac{5}{\sqrt[3]{2}} - \frac{2}{\sqrt[3]{16}} + \frac{1}{\sqrt[3]{54}}$

Rationalize the denominator of each radical expression. Assume all variables represent nonnegative numbers and that no denominators are 0. See Example 12.

93. $\frac{\sqrt{3}}{\sqrt{5} + \sqrt{3}}$

94. $\frac{\sqrt{7}}{\sqrt{3} - \sqrt{7}}$

95. $\frac{\sqrt{7} - 1}{2\sqrt{7} + 4\sqrt{2}}$

96. $\frac{1 + \sqrt{3}}{3\sqrt{5} + 2\sqrt{3}}$

97. $\frac{p - 4}{\sqrt{p} + 2}$

98. $\frac{9 - r}{3 - \sqrt{r}}$

99. $\frac{3m}{2 + \sqrt{m + n}}$

100. $\frac{a}{\sqrt{a + b} - 1}$

101. $\frac{5\sqrt{x}}{2\sqrt{x} + \sqrt{y}}$

102. ***Concept Check*** What should the numerator and denominator of

$$\frac{1}{\sqrt[3]{3} - \sqrt[3]{5}}$$

be multiplied by in order to rationalize the denominator? Write this fraction with a rationalized denominator

(Modeling) Solve each problem.

103. *Rowing Speed* Olympic rowing events have one-, two-, four-, or eight-person crews, with each person pulling a single oar. Increasing the size of the crew increases the speed of the boat. An analysis of Olympic rowing events concluded that the approximate speed, s, of the boat (in feet per second) was given by the formula

$$s = 15.18\sqrt[9]{n},$$

where n is the number of oarsmen. Estimate the speed of a boat with a four-person crew. (*Source:* Townend, M. Stewart, *Mathematics in Sport*, Chichester, Ellis Horwood Limited.)

104. *Rowing Speed* See Exercise **103**. Estimate the speed of a boat with an eight-person crew.

(Modeling) Windchill The National Weather Service has used the formula

$$\text{Windchill temperature} = 35.74 + 0.6215T - 35.75V^{0.16} + 0.4275TV^{0.16},$$

where T is the temperature in °F and V is the wind speed in miles per hour, to calculate windchill. (Source: National Oceanic and Atmospheric Administration, National Weather Service.) Use the formula to calculate the windchill to the nearest degree given the following conditions.

105. 10°F, 30 mph wind

106. 30°F, 15 mph wind

Concept Check Simplify each expression mentally.

107. $\sqrt[4]{8} \cdot \sqrt[4]{2}$

108. $\dfrac{\sqrt[3]{54}}{\sqrt[3]{2}}$

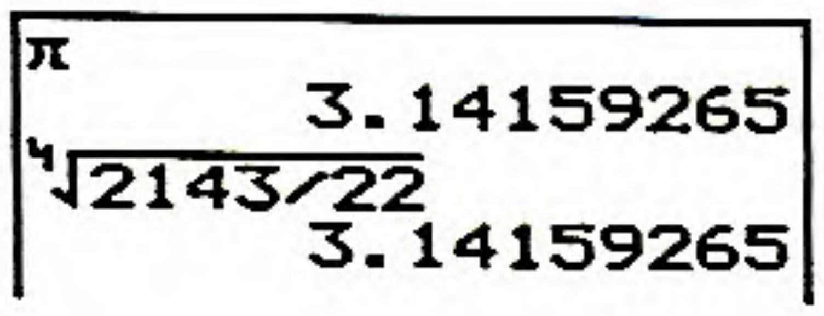
109. $\dfrac{\sqrt[5]{320}}{\sqrt[5]{10}}$

110. $\sqrt{0.1} \cdot \sqrt{40}$

111. $\dfrac{\sqrt[3]{15}}{\sqrt[3]{5}} \cdot \sqrt[3]{9}$

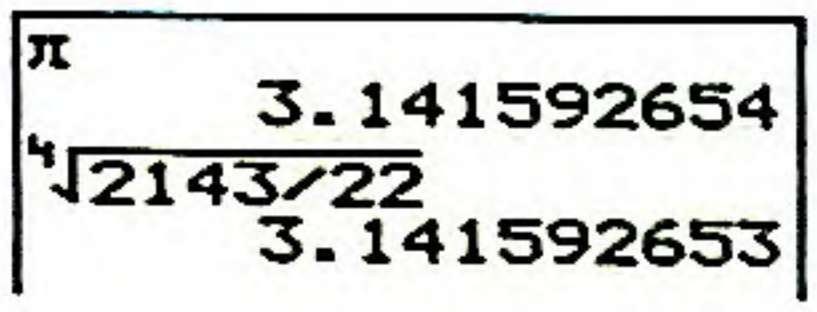
112. $\sqrt[6]{2} \cdot \sqrt[6]{4} \cdot \sqrt[6]{8}$

The screen in **Figure A** *seems to indicate that π and $\sqrt[4]{\frac{2143}{22}}$ are exactly equal, since the eight decimal values given by the calculator agree. However, as shown in* **Figure B** *using one more decimal place in the display, they differ in the ninth decimal place.* ***The radical expression is a very good approximation for π, but it is still only an approximation.***



```
π
              3.14159265
⁴√2143/22
              3.14159265
```

Figure A



```
π
             3.141592654
⁴√2143/22
             3.141592653
```

Figure B

Use your calculator to answer each question. Refer to the display for π in **Figure B.**

113. The Chinese of the fifth century used $\frac{355}{113}$ as an approximation for π. How many decimal places of accuracy does this fraction give?

114. A value for π that the Greeks used circa A.D. 150 is equivalent to $\frac{377}{120}$. In which decimal place does this value first differ from π?

115. The Hindu mathematician Bhaskara used $\frac{3927}{1250}$ as an approximation for π circa A.D. 1150. In which decimal place does this value first differ from π?

Concepts	Examples
Order	
$a > b$ if a is to the right of b on a number line.	$7 > -5$
$a < b$ if a is to the left of b on a number line.	$0 < 15$
Absolute Value $\|a\| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$	$\|3\| = 3$ and $\|-3\| = 3$

R.3 Polynomials

Special Products	
Product of the Sum and Difference of Two Terms $(x + y)(x - y) = x^2 - y^2$	$(7 - x)(7 + x) = 7^2 - x^2 = 49 - x^2$
Square of a Binomial $(x + y)^2 = x^2 + 2xy + y^2$	$(3a + b)^2 = (3a)^2 + 2(3a)(b) + b^2$ $= 9a^2 + 6ab + b^2$
$(x - y)^2 = x^2 - 2xy + y^2$	$(2m - 5)^2 = (2m)^2 - 2(2m)(5) + 5^2$ $= 4m^2 - 20m + 25$

R.4 Factoring Polynomials

Factoring Patterns	
Difference of Squares $x^2 - y^2 = (x + y)(x - y)$	$4t^2 - 9 = (2t + 3)(2t - 3)$
Perfect Square Trinomial $x^2 + 2xy + y^2 = (x + y)^2$	$p^2 + 4pq + 4q^2 = (p + 2q)^2$
$x^2 - 2xy + y^2 = (x - y)^2$	$9m^2 - 12mn + 4n^2 = (3m - 2n)^2$
Difference of Cubes $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$	$r^3 - 8 = (r - 2)(r^2 + 2r + 4)$
Sum of Cubes $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$	$27x^3 + 64 = (3x + 4)(9x^2 - 12x + 16)$

R.5 Rational Expressions

Operations Let $\frac{a}{b}$ and $\frac{c}{d}$ $(b \neq 0, d \neq 0)$ represent fractions.	
$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$	$\frac{2}{x} + \frac{5}{y} = \frac{2y + 5x}{xy}$ $\quad \frac{x}{6} - \frac{2y}{5} = \frac{5x - 12y}{30}$
$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ and $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$ $(c \neq 0)$	$\frac{3}{q} \cdot \frac{3}{2p} = \frac{9}{2pq}$ $\quad \frac{z}{4} \div \frac{z}{2t} = \frac{z}{4} \cdot \frac{2t}{z} = \frac{2zt}{4z} = \frac{t}{2}$

Concepts	Examples
R.6 Rational Exponents	
Rules for Exponents Let r and s be rational numbers. The following results are valid for all positive numbers a and b. $a^r \cdot a^s = a^{r+s}$ $\quad (ab)^r = a^r b^r$ $\quad (a^r)^s = a^{rs}$ $\frac{a^r}{a^s} = a^{r-s}$ $\quad \left(\frac{a}{b}\right)^r = \frac{a^r}{b^r}$ $\quad a^{-r} = \frac{1}{a^r}$	$6^2 \cdot 6^3 = 6^5$ $\quad (3x)^4 = 3^4x^4$ $\quad (m^2)^3 = m^6$ $\frac{p^5}{p^2} = p^3$ $\quad \left(\frac{x}{3}\right)^2 = \frac{x^2}{3^2}$ $\quad 4^{-3} = \frac{1}{4^3}$
R.7 Radical Expressions	
Radical Notation Suppose that a is a real number, n is a positive integer, and $a^{1/n}$ is defined. $\sqrt[n]{a} = a^{1/n}$	$\sqrt[4]{16} = 16^{1/4} = 2$
Suppose that m is an integer, n is a positive integer, and a is a real number for which $\sqrt[n]{a}$ is defined. $a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$	$8^{2/3} = (\sqrt[3]{8})^2 = \sqrt[3]{8^2} = 4$
Operations Operations with radical expressions are performed like operations with polynomials.	$\sqrt{8x} + \sqrt{32x} = 2\sqrt{2x} + 4\sqrt{2x}$ $= 6\sqrt{2x}$ $(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3}) = 5 - 3$ $= 2$ $(\sqrt{2} + \sqrt{7})(\sqrt{3} - \sqrt{6})$ $= \sqrt{6} - 2\sqrt{3} + \sqrt{21} - \sqrt{42}$ FOIL; $\sqrt{12} = 2\sqrt{3}$
Rationalize the denominator by multiplying numerator and denominator by a form of 1.	$\frac{\sqrt{7y}}{\sqrt{5}} = \frac{\sqrt{7y}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}$ $= \frac{\sqrt{35y}}{5}$

Chapter R Review Exercises

1. Use set notation to list all the elements of the set $\{6, 8, 10, \ldots, 20\}$.

2. Is the set $\{x \mid x$ is a decimal between 0 and 1$\}$ finite or infinite?

3. *Concept Check* *True* or *false:* The set of negative integers and the set of whole numbers are disjoint sets.

4. *Concept Check* *True* or *false:* 9 is an element of $\{999\}$.

Determine whether each statement is true or false.

5. $1 \in \{6, 2, 5, 1\}$

6. $7 \notin \{1, 3, 5, 7\}$

7. $\{8, 11, 4\} = \{8, 11, 4, 0\}$

8. $\{0\} = \emptyset$

Let $A = \{1, 3, 4, 5, 7, 8\}$, $B = \{2, 4, 6, 8\}$, $C = \{1, 3, 5, 7\}$, $D = \{1, 2, 3\}$, $E = \{3, 7\}$, *and* $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Determine whether each statement is true *or* false.

9. $\emptyset \subseteq A$
10. $E \subseteq C$
11. $D \not\subseteq B$
12. $E \not\subseteq A$

Refer to the sets given for Exercises 9–12. Specify each set.

13. A'
14. $B \cap A$
15. $B \cap E$
16. $C \cup E$
17. $D \cap \emptyset$
18. $B \cup \emptyset$
19. $(C \cap D) \cup B$
20. $(D' \cap U) \cup E$
21. $\emptyset'$

22. ***Concept Check*** *True* or *false:* For all sets A and B, $(A \cap B) \subseteq (A \cup B)$.

For Exercises 23 and 24, let set $K = \left\{-12, -6, -0.9, -\sqrt{7}, -\sqrt{4}, 0, \frac{1}{8}, \frac{\pi}{4}, 6, \sqrt{11}\right\}$. *List all elements of K that belong to each set.*

23. Integers
24. Rational numbers

For Exercises 25–28, choose all words from the following list that apply.

natural number	whole number	integer
rational number	irrational number	real number

25. $\frac{4\pi}{5}$
26. $\frac{\pi}{0}$
27. 0
28. $-\sqrt{36}$

Write each algebraic identity (true statement) as a complete English sentence without using the names of the variables. For instance, $z(x + y) = zx + zy$ *can be stated as "The multiple of a sum is the sum of the multiples."*

29. $\frac{1}{xy} = \frac{1}{x} \cdot \frac{1}{y}$
30. $a(b - c) = ab - ac$
31. $(ab)^n = a^n b^n$
32. $a^2 - b^2 = (a + b)(a - b)$
33. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
34. $|st| = |s| \cdot |t|$

Identify by name each property illustrated.

35. $8(5 + 9) = (5 + 9)8$
36. $4 \cdot 6 + 4 \cdot 12 = 4(6 + 12)$
37. $3 \cdot (4 \cdot 2) = (3 \cdot 4) \cdot 2$
38. $-8 + 8 = 0$
39. $(9 + p) + 0 = 9 + p$
40. $\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

41. ***Ages of College Undergraduates*** The following table shows the age distribution of college students in 2008. In a random sample of 5000 such students, how many would you expect to be over 19?

Age	Percent
14–19	29
20–24	43
25–34	16
35 and older	12

Source: U.S. Census Bureau

42. *Counting Marshmallows* In early 2011, there were media reports about students providing a correction to the following question posed on boxes of Swiss Miss Chocolate: *On average, how many mini-marshmallows are in one serving?*

$$3 + 2 \times 4 \div 2 - 3 \times 7 - 4 + 47 = \underline{\qquad\qquad}$$

The company provided 92 as the answer. What is the *correct* calculation provided by the students? (*Source:* Swiss Miss Chocolate box.)

Simplify each expression.

43. $(-4 - 1)(-3 - 5) - 2^3$

44. $(6 - 9)(-2 - 7) \div (-4)$

45. $\left(-\frac{5}{9} - \frac{2}{3}\right) - \frac{5}{6}$

46. $\left(-\frac{2^3}{5} - \frac{3}{4}\right) - \left(-\frac{1}{2}\right)$

47. $\dfrac{6(-4) - 3^2(-2)^3}{-5[-2 - (-6)]}$

48. $\dfrac{(-7)(-3) - (-2^3)(-5)}{(-2^2 - 2)(-1 - 6)}$

Evaluate each expression for $a = -1$, $b = -2$, and $c = 4$.

49. $-c(2a - 5b)$

50. $(a - 2) \div 5 \cdot b + c$

51. $\dfrac{9a + 2b}{a + b + c}$

52. $\dfrac{3|b| - 4|c|}{|ac|}$

Perform the indicated operations.

53. $(3q^3 - 9q^2 + 6) + (4q^3 - 8q + 3)$

54. $2(3y^6 - 9y^2 + 2y) - (5y^6 - 4y)$

55. $(8y - 7)(2y + 7)$

56. $(2r + 11s)(4r - 9s)$

57. $(3k - 5m)^2$

58. $(4a - 3b)^2$

Perform each division.

59. $\dfrac{30m^3 - 9m^2 + 22m + 5}{5m + 1}$

60. $\dfrac{72r^2 + 59r + 12}{8r + 3}$

61. $\dfrac{3b^3 - 8b^2 + 12b - 30}{b^2 + 4}$

62. $\dfrac{5m^3 - 7m^2 + 14}{m^2 - 2}$

Factor as completely as possible.

63. $3(z - 4)^2 + 9(z - 4)^3$

64. $7z^2 - 9z^3 + z$

65. $z^2 - 6zk - 16k^2$

66. $r^2 + rp - 42p^2$

67. $48a^8 - 12a^7b - 90a^6b^2$

68. $6m^2 - 13m - 5$

69. $49m^8 - 9n^2$

70. $169y^4 - 1$

71. $6(3r - 1)^2 + (3r - 1) - 35$

72. $8y^3 - 1000z^6$

73. $xy + 2x - y - 2$

74. $15mp + 9mq - 10np - 6nq$

Factor each expression. (These expressions arise in calculus from a technique called the product rule that is used to determine the shape of a curve.)

75. $(3x - 4)^2 + (x - 5)(2)(3x - 4)(3)$

76. $(5 - 2x)(3)(7x - 8)^2(7) + (7x - 8)^3(-2)$

Perform the indicated operations.

77. $\dfrac{k^2+k}{8k^3} \cdot \dfrac{4}{k^2-1}$

78. $\dfrac{3r^3-9r^2}{r^2-9} \div \dfrac{8r^3}{r+3}$

79. $\dfrac{x^2+x-2}{x^2+5x+6} \div \dfrac{x^2+3x-4}{x^2+4x+3}$

80. $\dfrac{27m^3-n^3}{3m-n} \div \dfrac{9m^2+3mn+n^2}{9m^2-n^2}$

81. $\dfrac{p^2-36q^2}{p^2-12pq+36q^2} \cdot \dfrac{p^2-5pq-6q^2}{p^2+2pq+q^2}$

82. $\dfrac{1}{4y} + \dfrac{8}{5y}$

83. $\dfrac{m}{4-m} + \dfrac{3m}{m-4}$

84. $\dfrac{3}{x^2-4x+3} - \dfrac{2}{x^2-1}$

85. $\dfrac{p^{-1}+q^{-1}}{1-(pq)^{-1}}$

86. $\dfrac{3+\dfrac{2m}{m^2-4}}{\dfrac{5}{m-2}}$

Simplify each expression. Write the answer with only positive exponents. Assume all variables represent positive real numbers.

87. $\left(-\dfrac{5}{4}\right)^{-2}$

88. $3^{-1}-4^{-1}$

89. $(5z^3)(-2z^5)$

90. $(8p^2q^3)(-2p^5q^{-4})$

91. $(-6p^5w^4m^{12})^0$

92. $(-6x^2y^{-3}z^2)^{-2}$

93. $\dfrac{-8y^7p^{-2}}{y^{-4}p^{-3}}$

94. $\dfrac{a^{-6}(a^{-8})}{a^{-2}(a^{11})}$

95. $\dfrac{(p+q)^4(p+q)^{-3}}{(p+q)^6}$

96. $\dfrac{[p^2(m+n)^3]^{-2}}{p^{-2}(m+n)^{-5}}$

97. $(7r^{1/2})(2r^{3/4})(-r^{1/6})$

98. $(a^{3/4}b^{2/3})(a^{5/8}b^{-5/6})$

99. $\dfrac{y^{5/3}\cdot y^{-2}}{y^{-5/6}}$

100. $\left(\dfrac{25m^3n^5}{m^{-2}n^6}\right)^{-1/2}$

101. $\dfrac{(p^{15}q^{12})^{-4/3}}{(p^{24}q^{16})^{-3/4}}$

102. Simplify the product $-m^{3/4}(8m^{1/2}+4m^{-3/2})$. Assume the variable represents a positive real number.

Simplify. Assume all variables represent positive real numbers.

103. $\sqrt{200}$

104. $\sqrt[3]{16}$

105. $\sqrt[4]{1250}$

106. $-\sqrt{\dfrac{16}{3}}$

107. $-\sqrt[3]{\dfrac{2}{5p^2}}$

108. $\sqrt{\dfrac{2^7y^8}{m^3}}$

109. $\sqrt[4]{\sqrt[3]{m}}$

110. $\dfrac{\sqrt[4]{8p^2q^5}\cdot\sqrt[4]{2p^3q}}{\sqrt[4]{p^5q^2}}$

111. $(\sqrt[3]{2}+4)(\sqrt[3]{2^2}-4\sqrt[3]{2}+16)$

112. $\dfrac{3}{\sqrt{5}} - \dfrac{2}{\sqrt{45}} + \dfrac{6}{\sqrt{80}}$

113. $\sqrt{18m^3} - 3m\sqrt{32m} + 5\sqrt{m^3}$

114. $\dfrac{2}{7-\sqrt{3}}$

115. $\dfrac{6}{3-\sqrt{2}}$

116. $\dfrac{k}{\sqrt{k}-3}$

Concept Check *Correct each INCORRECT statement by changing the right-hand side of the equation.*

117. $x(x^2+5) = x^3+5$

118. $-3^2 = 9$

119. $(m^2)^3 = m^5$

120. $(3x)(3y) = 3xy$

121. $\dfrac{\left(\frac{a}{b}\right)}{2} = \dfrac{2a}{b}$

122. $\dfrac{m}{r} \cdot \dfrac{n}{r} = \dfrac{mn}{r}$

123. $\dfrac{1}{(-2)^3} = 2^{-3}$

124. $(-5)^2 = -5^2$

125. $\left(\dfrac{8}{7} + \dfrac{a}{b}\right)^{-1} = \dfrac{7}{8} + \dfrac{b}{a}$

Chapter R Test

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{1, 3, 5\}$, $C = \{1, 6\}$, *and* $D = \{4\}$. *Tell whether each statement is* true *or* false.

1. $B' = \{2, 4, 6, 8\}$

2. $C \subseteq A$

3. $(B \cap C) \cup D = \{1, 3, 4, 5, 6\}$

4. $(A' \cup C) \cap B' = \{6, 7, 8\}$

5. Let $A = \left\{-13, -\frac{12}{4}, 0, \frac{3}{5}, \frac{\pi}{4}, 5.9, \sqrt{49}\right\}$. List the elements of A that belong to the given set.

(a) Integers **(b)** Rational numbers **(c)** Real numbers

6. Evaluate the expression $\left|\dfrac{x^2 + 2yz}{3(x + z)}\right|$ for $x = -2$, $y = -4$, and $z = 5$.

7. Identify each property illustrated. Let a, b, and c represent any real numbers.

(a) $a + (b + c) = (a + b) + c$

(b) $a + (c + b) = a + (b + c)$

(c) $a(b + c) = ab + ac$

(d) $a + [b + (-b)] = a + 0$

8. *Passer Rating for NFL Quarterbacks* Approximate the quarterback rating (to the nearest tenth) of Drew Brees of the New Orleans Saints during the 2008 regular season. He attempted 514 passes, completed 363, had 4388 total yards, threw for 34 touchdowns, and had 11 interceptions. (*Source: World Almanac and Book of Facts.*)

$$\text{Rating} = \frac{\left(250 \cdot \frac{C}{A}\right) + \left(1000 \cdot \frac{T}{A}\right) + \left(12.5 \cdot \frac{Y}{A}\right) + 6.25 - \left(1250 \cdot \frac{I}{A}\right)}{3},$$

where $A =$ attempted passes, $C =$ completed passes, $T =$ touchdown passes, $Y =$ yards gained passing, and $I =$ interceptions.

In addition to the weighting factors appearing in the formula, the four category ratios are limited to nonnegative values with the following maximums.

$$0.775 \text{ for } \frac{C}{A}, \quad 0.11875 \text{ for } \frac{T}{A}, \quad 12.5 \text{ for } \frac{Y}{A}, \quad 0.095 \text{ for } \frac{I}{A}$$

Perform the indicated operations.

9. $(x^2 - 3x + 2) - (x - 4x^2) + 3x(2x + 1)$

10. $(6r - 5)^2$

11. $(t + 2)(3t^2 - t + 4)$

12. $\dfrac{2x^3 - 11x^2 + 28}{x - 5}$

(Modeling) Adjusted Poverty Threshold The adjusted poverty threshold for a single person between the years 1995 and 2009 can be approximated by the polynomial

$$4.872x^2 + 170.2x + 7787,$$

where $x = 0$ corresponds to 1995, $x = 1$ corresponds to 1996, and so on, and the amount is in dollars. According to this model, what was the adjusted poverty threshold, to the nearest dollar, in each given year? (Source: U.S. Census Bureau.)

13. 2000

14. 2008

Factor completely.

15. $6x^2 - 17x + 7$

16. $x^4 - 16$

17. $24m^3 - 14m^2 - 24m$

18. $x^3y^2 - 9x^3 - 8y^2 + 72$

19. $(a - b)^2 + 2(a - b)$

20. $1 - 27x^6$

Perform the indicated operations.

21. $\dfrac{5x^2 - 9x - 2}{30x^3 + 6x^2} - \dfrac{x^4 - 3x^2 - 4}{2x^8 + 6x^7 + 4x^6}$

22. $\dfrac{x}{x^2 + 3x + 2} + \dfrac{2x}{2x^2 - x - 3}$

23. $\dfrac{a + b}{2a - 3} - \dfrac{a - b}{3 - 2a}$

24. $\dfrac{y - 2}{y - \dfrac{4}{y}}$

Simplify or evaluate as appropriate. Assume all variables represent positive real numbers.

25. $\sqrt{18x^5y^8}$

26. $\sqrt{32x} + \sqrt{2x} - \sqrt{18x}$

27. $(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})$

28. $\dfrac{14}{\sqrt{11} - \sqrt{7}}$

29. $\left(\dfrac{x^{-2}y^{-1/3}}{x^{-5/3}y^{-2/3}}\right)^3$

30. $\left(-\dfrac{64}{27}\right)^{-2/3}$

31. *Concept Check True* or *false*: For all real numbers x, $\sqrt{x^2} = x$.

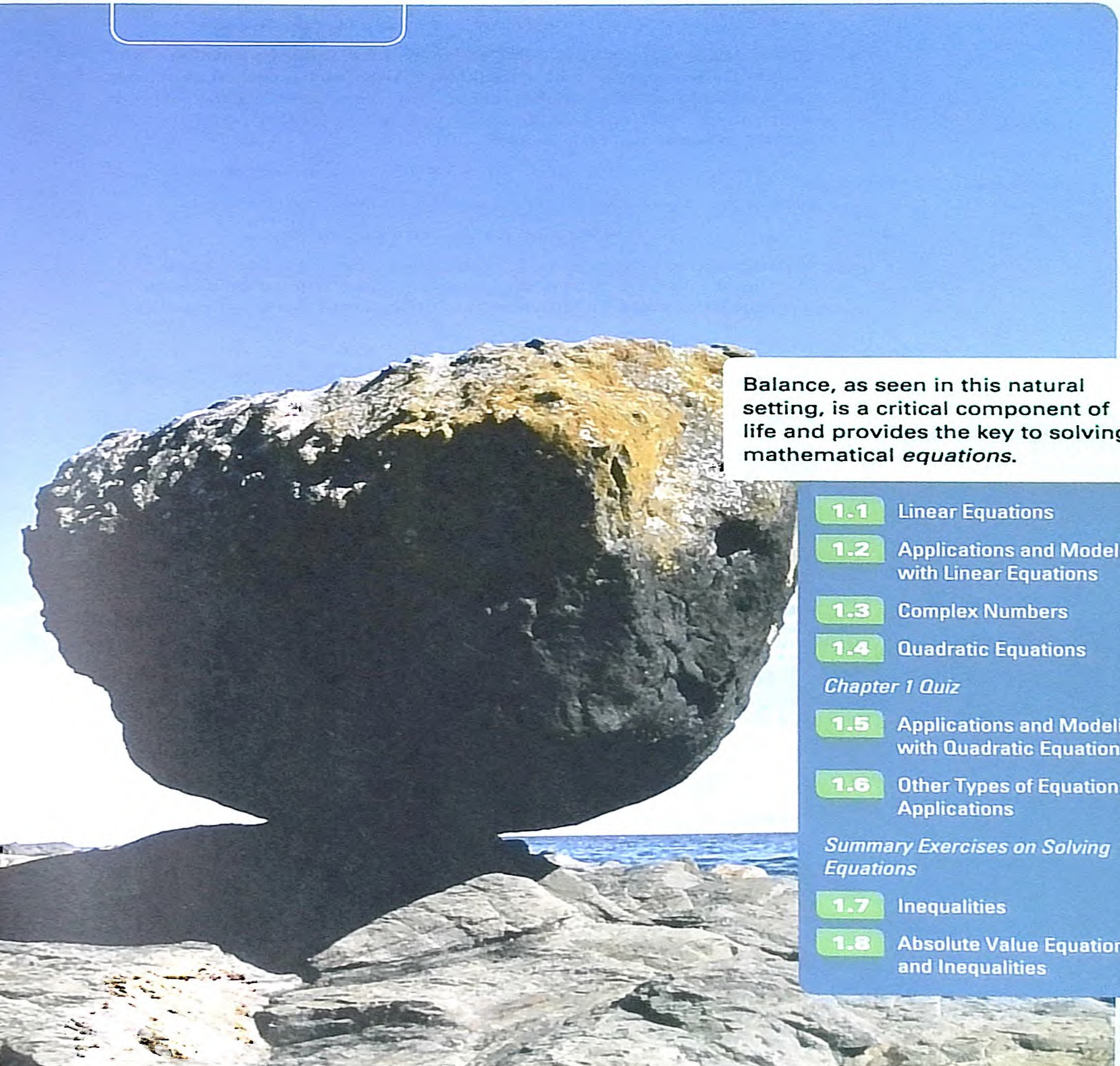
32. *(Modeling) Period of a Pendulum* The period t, in seconds, of the swing of a pendulum is given by the equation

$$t = 2\pi\sqrt{\frac{L}{32}},$$

where L is the length of the pendulum in feet. Find the period of a pendulum 3.5 ft long. Use a calculator.

1

Equations and Inequalities



Balance, as seen in this natural setting, is a critical component of life and provides the key to solving mathematical *equations*.

1.1 Linear Equations

1.2 Applications and Modeling with Linear Equations

1.3 Complex Numbers

1.4 Quadratic Equations

Chapter 1 Quiz

1.5 Applications and Modeling with Quadratic Equations

1.6 Other Types of Equations and Applications

Summary Exercises on Solving Equations

1.7 Inequalities

1.8 Absolute Value Equations and Inequalities

EXAMPLE 5 **Applying the Simple Interest Formula**

Becky Brugman borrowed \$5240 for new furniture. She will pay it off in 11 months at an annual simple interest rate of 4.5%. How much interest will she pay?

SOLUTION Use the simple interest formula $I = Prt$.

$$I = Prt = 5240(0.045)\left(\frac{11}{12}\right) = \$216.15 \qquad P = 5240,\ r = 0.045, \text{ and } t = \tfrac{11}{12} \text{ (year)}$$

She will pay \$216.15 interest on her purchase.

Now Try Exercise 59.

1.1 Exercises

Concept Check In Exercises 1–4, decide whether each statement is true or false.

1. The solution set of $2x + 5 = x - 3$ is $\{-8\}$.
2. The equation $5(x - 8) = 5x - 40$ is an example of an identity.
3. The equations $x^2 = 4$ and $x + 2 = 4$ are equivalent equations.
4. It is possible for a linear equation to have exactly two solutions.
5. Explain the difference between an identity and a conditional equation.
6. Make a complete list of the steps needed to solve a linear equation. (Some equations will not require every step.)
7. *Concept Check* Which one is not a linear equation?

   A. $5x + 7(x - 1) = -3x$ B. $9x^2 - 4x + 3 = 0$

   C. $7x + 8x = 13x$ D. $0.04x - 0.08x = 0.40$
8. In solving the equation $3(2x - 8) = 6x - 24$, a student obtains the result $0 = 0$ and gives the solution set $\{0\}$. Is this correct? Explain.

Solve each equation. See Examples 1 and 2.

9. $5x + 4 = 3x - 4$
10. $9x + 11 = 7x + 1$
11. $6(3x - 1) = 8 - (10x - 14)$
12. $4(-2x + 1) = 6 - (2x - 4)$
13. $\frac{5}{6}x - 2x + \frac{4}{3} = \frac{5}{3}$
14. $\frac{7}{4} + \frac{1}{5}x - \frac{3}{2} = \frac{4}{5}x$
15. $3x + 5 - 5(x + 1) = 6x + 7$
16. $5(x + 3) + 4x - 3 = -(2x - 4) + 2$
17. $2[x - (4 + 2x) + 3] = 2x + 2$
18. $4[2x - (3 - x) + 5] = -6x - 28$
19. $\frac{1}{14}(3x - 2) = \frac{x + 10}{10}$
20. $\frac{1}{15}(2x + 5) = \frac{x + 2}{9}$
21. $0.2x - 0.5 = 0.1x + 7$
22. $0.01x + 3.1 = 2.03x - 2.96$
23. $-4(2x - 6) + 8x = 5x + 24 + x$
24. $-8(3x + 4) + 6x = 4(x - 8) + 4x$
25. $0.5x + \frac{4}{3}x = x + 10$
26. $\frac{2}{3}x + 0.25x = x + 2$
27. $0.08x + 0.06(x + 12) = 7.72$
28. $0.04(x - 12) + 0.06x = 1.52$

Determine whether each equation is an identity, a conditional equation, or a contradiction. Give the solution set. See Example 3.

29. $4(2x + 7) = 2x + 22 + 3(2x + 2)$
30. $\frac{1}{2}(6x + 20) = x + 4 + 2(x + 3)$

31. $2(x-8)=3x-16$

32. $-8(x+5)=-8x-5(x+8)$

33. $0.3(x+2)-0.5(x+2)=-0.2x-0.4$

34. $-0.6(x-5)+0.8(x-6)=0.2x-1.8$

35. $4(x+7)=2(x+12)+2(x+1)$

36. $-6(2x+1)-3(x-4)=-15x+1$

37. A student claims that the equation $5x=4x$ is a contradiction, since dividing both sides by x leads to $5=4$, a false statement. Explain why the student is incorrect.

38. If $k \neq 0$, is the equation $x+k=x$ a contradiction, a conditional equation, or an identity? Explain.

Solve each formula for the indicated variable. Assume that the denominator is not 0 if variables appear in the denominator. See Examples 4(a) and (b).

39. $V=lwh$, for l (volume of a rectangular box)

40. $I=Prt$, for P (simple interest)

41. $P=a+b+c$, for c (perimeter of a triangle)

42. $P=2l+2w$, for w (perimeter of a rectangle)

43. $\mathcal{A}=\frac{1}{2}h(B+b)$, for B (area of a trapezoid)

44. $\mathcal{A}=\frac{1}{2}h(B+b)$, for h (area of a trapezoid)

45. $S=2\pi rh+2\pi r^2$, for h (surface area of a right circular cylinder)

46. $s=\frac{1}{2}gt^2$, for g (distance traveled by a falling object)

47. $S=2lw+2wh+2hl$, for h (surface area of a rectangular box)

48. Refer to Exercise 45. Why is it not possible to solve this formula for r using the methods of this section?

Solve each equation for x. See Example 4(c).

49. $2(x-a)+b=3x+a$

50. $5x-(2a+c)=4(x+c)$

51. $ax+b=3(x-a)$

52. $4a-ax=3b+bx$

53. $\frac{x}{a-1}=ax+3$

54. $\frac{x-1}{2a}=2x-a$

55. $a^2x+3x=2a^2$

56. $ax+b^2=bx-a^2$

57. $3x=(2x-1)(m+4)$

58. $-x=(5x+3)(3k+1)$

Work each problem. See Example 5.

59. *Simple Interest* Elmer Velasquez borrowed \$3150 from his brother Julio to pay for books and tuition. He agreed to repay Julio in 6 months with simple annual interest at 4%.

(a) How much will the interest amount to?

(b) What amount must Elmer pay Julio at the end of the 6 months?

60. *Simple Interest* Levada Qualls borrows \$30,900 from her bank to open a florist shop. She agrees to repay the money in 18 months with simple annual interest of 5.5%.

(a) How much must she pay the bank in 18 months?

(b) How much of the amount in part (a) is interest?

Celsius and Fahrenheit Temperatures ***In the metric system of weights and measures, temperature is measured in degrees Celsius (°C) instead of degrees Fahrenheit (°F). To convert between the two systems, we use the equations***

$$C = \frac{5}{9}(F - 32) \quad \text{and} \quad F = \frac{9}{5}C + 32.$$

In each exercise, convert to the other system. Round answers to the nearest tenth of a degree if necessary.

61. 40°C

62. 200°C

63. 50°F

64. 77°F

65. 100°F

66. 350°F

Work each problem.

67. *Temperature of Venus* Venus is the hottest planet with a surface temperature of 867°F. What is this temperature in Celsius? (*Source: World Almanac and Book of Facts.*)

68. *Temperature at Soviet Antarctica Station* A record low temperature of −89.4°C was recorded at the Soviet Antarctica Station of Vostok on July 21, 1983. Find the corresponding Fahrenheit temperature. (*Source: World Almanac and Book of Facts.*)

69. *Temperature in Montreal* The average low temperature in Montreal, Canada, for the date January 30 is 7°F. What is the corresponding Celsius temperature to the nearest tenth of a degree? (*Source:* www.wunderground.com)

70. *Temperature in Haiti* The average annual temperature in Port-au-Prince, Haiti, is approximately 28.1°C. What is the corresponding Fahrenheit temperature to the nearest tenth of a degree? (*Source:* www.climatetemp.info/haiti)

1.2 Applications and Modeling with Linear Equations

- **Solving Applied Problems**
- **Geometry Problems**
- **Motion Problems**
- **Mixture Problems**
- **Modeling with Linear Equations**

Solving Applied Problems One of the main reasons for learning mathematics is to be able use it to solve application problems. While there is no one method that enables us to solve all types of applied problems, the following six steps provide a useful guide.

Solving an Applied Problem

Step 1 **Read** the problem carefully until you understand what is given and what is to be found.

Step 2 **Assign a variable** to represent the unknown value, using diagrams or tables as needed. Write down what the variable represents. If necessary, express any other unknown values in terms of the variable.

Step 3 **Write an equation** using the variable expression(s).

Step 4 **Solve** the equation.

Step 5 **State the answer** to the problem. Does it seem reasonable?

Step 6 **Check** the answer in the words of the original problem.

Geometry Problems

EXAMPLE 1 **Finding the Dimensions of a Square**

If the length of each side of a square is increased by 3 cm, the perimeter of the new square is 40 cm more than twice the length of each side of the original square. Find the dimensions of the original square.

SOLUTION

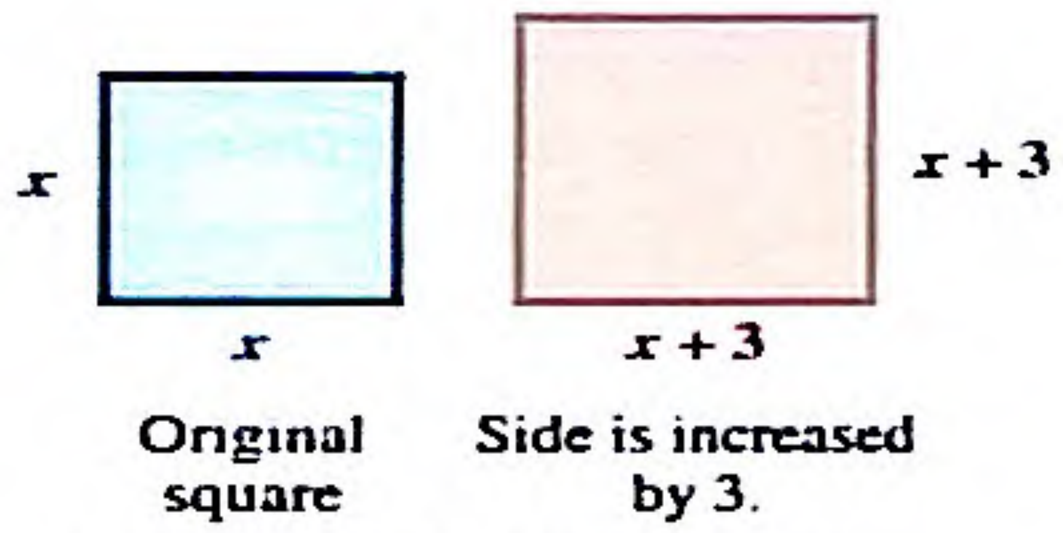
Step 1 **Read** the problem. We must find the length of each side of the original square.

Step 2 **Assign a variable.** Since the length of a side of the original square is to be found, let the variable represent this length.

Let $x =$ the length of a side of the original square in centimeters.

The length of a side of the new square is 3 cm more than the length of a side of the old square.

Then $x + 3 =$ the length of a side of the new square.



Original square

Side is increased by 3.

x and $x + 3$ are in centimeters.

Figure 1

See **Figure 1**. Now write a variable expression for the perimeter of the new square. The perimeter of a square is 4 times the length of a side.

Thus, $4(x + 3) =$ the perimeter of the new square.

Step 3 **Write an equation.** Translate the English sentence that follows into its equivalent algebraic equation.

The new perimeter	is	40	more than	twice the length of a side of the original square.
$4(x+3)$	$=$	40	$+$	$2x$

Step 4 **Solve** the equation.

$4x + 12 = 40 + 2x$ Distributive property (Section R.2)

$2x = 28$ Subtract $2x$ and 12. (Section 1.1)

$x = 14$ Divide by 2. (Section 1.1)

Step 5 **State the answer.** Each side of the original square measures 14 cm.

Step 6 **Check.** Go back to the words of the original problem to see that all necessary conditions are satisfied. The length of a side of the new square would be $14 + 3 = 17$ cm. The perimeter of the new square would be $4(17) = 68$ cm. Twice the length of a side of the original square would be $2(14) = 28$ cm. Since $40 + 28 = 68$, the answer checks.

✔ ***Now Try Exercise 13.***

Motion Problems

LOOKING AHEAD TO CALCULUS

In calculus the concept of the **definite integral** is used to find the distance traveled by an object traveling at a *non-constant* velocity.

PROBLEM-SOLVING HINT In a motion problem, the components *distance, rate,* and *time* are denoted by the letters d, r, and t, respectively. (The *rate* is also called the *speed* or *velocity*. Here, rate is understood to be constant.) These variables are related by the equation

$$d = rt, \quad \text{and its related forms} \quad r = \frac{d}{t} \quad \text{and} \quad t = \frac{d}{r}.$$

1.2 Exercises

Concept Check Exercises 1–8 should be done mentally. They will prepare you for some of the applications found in this exercise set.

1. If a train travels at 80 mph for 15 min, what is the distance traveled?

2. If 120 L of an acid solution is 75% acid, how much pure acid is there in the mixture?

3. If a person invests $500 at 2% simple interest for 4 yr, how much interest is earned?

4. If a jar of coins contains 40 half-dollars and 200 quarters, what is the monetary value of the coins?

5. *Acid Mixture* Suppose two acid solutions are mixed. One is 26% acid and the other is 34% acid. Which one of the following concentrations cannot possibly be the concentration of the mixture?

   A. 24% B. 30% C. 31% D. 33%

6. *Sale Price* Suppose that a computer that originally sold for x dollars has been discounted 60%. Which one of the following expressions does not represent its sale price?

   A. $x - 0.60x$ B. $0.40x$ C. $\frac{4}{10}x$ D. $x - 0.60$

7. ***Unknown Numbers*** Consider the following problem.

   One number is 3 less than 6 times a second number. Their sum is 46. Find the numbers.

   If x represents the second number, which equation is correct for solving this problem?

   A. $46 - (x + 3) = 6x$ B. $(3 - 6x) + x = 46$

   C. $46 - (3 - 6x) = x$ D. $(6x - 3) + x = 46$

8. ***Unknown Numbers*** Consider the following problem.

   The difference between seven times a number and 9 is equal to five times the sum of the number and 2. Find the number.

   If x represents the number, which equation is correct for solving this problem?

   A. $7x - 9 = 5(x + 2)$ B. $9 - 7x = 5(x + 2)$

   C. $7x - 9 = 5x + 2$ D. $9 - 7x = 5x + 2$

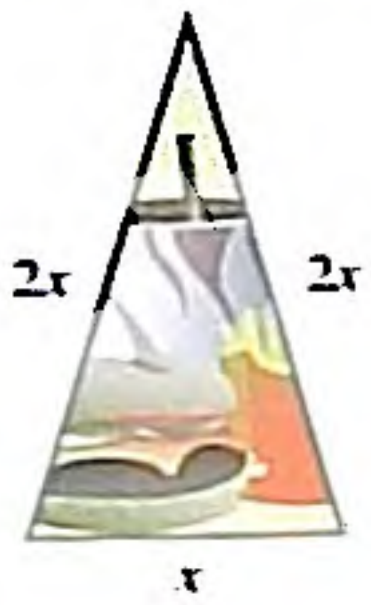
***Note:* Geometry formulas can be found on the back inside cover of this book.**

Solve each problem. See Example 1.

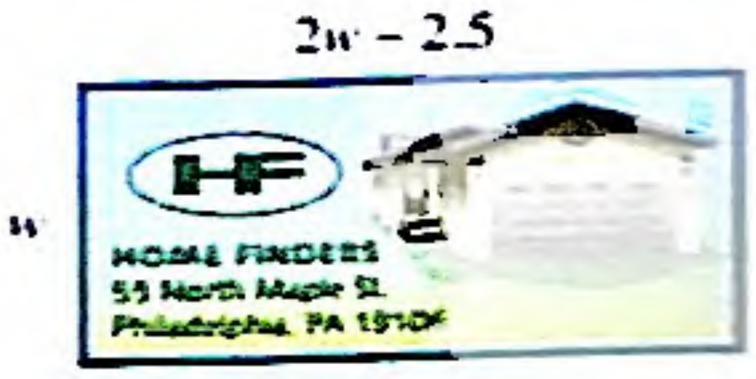
9. ***Perimeter of a Rectangle*** The perimeter of a rectangle is 294 cm. The width is 57 cm. Find the length.

10. ***Perimeter of a Storage Shed*** Michael Gomski must build a rectangular storage shed. He wants the length to be 6 ft greater than the width, and the perimeter will be 44 ft. Find the length and the width of the shed.

11. ***Dimensions of a Puzzle Piece*** A puzzle piece in the shape of a triangle has perimeter 30 cm. Two sides of the triangle are each twice as long as the shortest side. Find the length of the shortest side. (Side lengths in the figure are in centimeters.)



12. ***Dimensions of a Label*** The length of a rectangular label is 2.5 cm less than twice the width. The perimeter is 40.6 cm. Find the width. (Side lengths in the figure are in centimeters.)

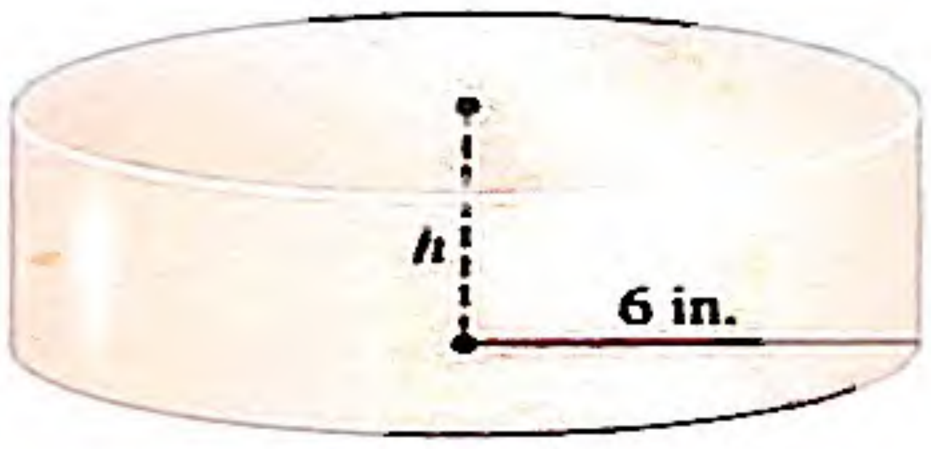


13. ***Perimeter of a Plot of Land*** The perimeter of a triangular plot of land is 2400 ft. The longest side is 200 ft less than twice the shortest. The middle side is 200 ft less than the longest side. Find the lengths of the three sides of the triangular plot.

14. ***World Largest Ice Cream Cake*** The world's largest ice cream cake, made at the Baxy ice cream factory in Beijing, China, on January 16, 2006, had length 5.9 ft greater than its width. Its perimeter was 51 ft. What were the length and width of this 8-ton cake? (*Sources:* www.chinadaily.com.cn, www.foodmall.org)

15. ***World's Smallest Quran*** The world's smallest Quran, published in Cairo, Egypt, in 1982 has width 0.42 cm shorter than its length. The book's perimeter is 5.96 cm. What are the width and length of this Quran, in centimeters? (*Source:* www.guinnessworldrecords.com)

16. ***Cylinder Dimensions*** A right circular cylinder has radius 6 in. and volume 144π in.^{3}. What is its height? (In the figure, h = height.)



h is in inches.

17. ***Recycling Bin Dimensions*** A recycling bin is in the shape of a rectangular box. Find the height of the box if its length is 18 ft, its width is 8 ft, and its surface area is 496 ft^2. (In the figure, h = height. Assume that the given surface area includes that of the top lid of the box.)

18. ***Concept Check*** Which one or more of the following cannot be a correct equation to solve a geometry problem, if x represents the length of a rectangle? (*Hint:* Solve each equation and consider the solution.)

A. $2x + 2(x - 1) = 14$ **B.** $-2x + 7(5 - x) = 52$
C. $5(x + 2) + 5x = 10$ **D.** $2x + 2(x - 3) = 22$

Solve each problem. See Example 2.

19. ***Distance to an Appointment*** Margaret drove to a business appointment at 50 mph. Her average speed on the return trip was 40 mph. The return trip took $\frac{1}{4}$ hr longer because of heavy traffic. How far did she travel to the appointment?

	r	t	d
Morning	50	x	
Afternoon	40	$x + \frac{1}{4}$	

20. ***Distance between Cities*** On a vacation, Elwyn averaged 50 mph traveling from Denver to Minneapolis. Returning by a different route that covered the same number of miles, he averaged 55 mph. What is the distance between the two cities if his total traveling time was 32 hr?

	r	t	d
Going	50	x	
Returning	55	$32 - x$	

21. *Distance to Work* David gets to work in 20 min when he drives his car. Riding his bike (by the same route) takes him 45 min. His average driving speed is 4.5 mph greater than his average speed on his bike. How far does he travel to work?

22. *Speed of a Plane* Two planes leave Los Angeles at the same time. One heads south to San Diego, while the other heads north to San Francisco. The San Diego plane flies 50 mph slower than the San Francisco plane. In $\frac{1}{2}$ hr, the planes are 275 mi apart. What are their speeds?

23. *Running Times* Mary and Janet are running in the Apple Hill Fun Run. Mary runs at 7 mph, Janet at 5 mph. If they start at the same time, how long will it be before they are 1.5 mi apart?

24. *Running Times* If the run in Exercise 23 has a staggered start, and Janet starts first, with Mary starting 10 min later, how long will it be before Mary catches up with Janet?

25. *Track Event Speeds* At the 2008 Summer Olympics in Beijing, China, Usain Bolt (Jamaica) set a new Olympic and world record in the 100-m dash with a time of 9.69 sec. If this pace could be maintained for an entire 26-mi marathon, what would his time be? How would this time compare to the fastest time for a marathon, which is 2 hr, 3 min, 59 sec, also set in 2008? (*Hint:* 1 m ≈ 3.281 ft.) (*Source: World Almanac and Book of Facts.*)

26. *Track Event Speeds* On August 19, 2009, at the World Track and Field Championship in Berlin, Usain Bolt broke his own record in the 100-m dash with a time of 9.58 sec. Refer to Exercise 25 and answer the questions using Bolt's 2009 time. (*Source:* www.sports.espn.go.com)

27. *Boat Speed* Callie took 20 min to drive her boat upstream to water-ski at her favorite spot. Coming back later in the day, at the same boat speed, took her 15 min. If the current in that part of the river is 5 km per hr, what was her boat speed?

28. *Wind Speed* Joe traveled against the wind in a small plane for 3 hr. The return trip with the wind took 2.8 hr. Find the speed of the wind if the speed of the plane in still air is 180 mph.

Solve each problem. See Example 3.

29. *Acid Mixture* How many gallons of a 5% acid solution must be mixed with 5 gal of a 10% solution to obtain a 7% solution?

Strength	Gallons of Solution	Gallons of Pure Acid
5%	x	
10%	5	
7%	$x + 5$	

30. *Acid Mixture* Marin Caswell needs 10% hydrochloric acid for a chemistry experiment. How much 5% acid should she mix with 60 mL of 20% acid to get a 10% solution?

Strength	mL of Solution	mL of Pure Acid
5%	x	
20%	60	
10%	$x + 60$	

31. *Alcohol Mixture* Beau Glaser wishes to strengthen a mixture from 10% alcohol to 30% alcohol. How much pure alcohol should be added to 7 L of the 10% mixture?

32. *Alcohol Mixture* How many gallons of pure alcohol should be mixed with 20 gal of a 15% alcohol solution to obtain a mixture that is 25% alcohol?

33. *Saline Solution* How much water should be added to 8 mL of 6% saline solution to reduce the concentration to 4%?

34. *Acid Mixture* How much pure acid should be added to 18 L of 30% acid to increase the concentration to 50% acid?

Solve each problem. See Example 4.

35. *Real Estate Financing* Cody Westmoreland wishes to sell a piece of property for $240,000. He wants the money to be paid off in two ways—a short-term note at 2% interest and a long-term note at 2.5%. Find the amount of each note if the total annual interest paid is $5500.

Note Amount	Interest Rate (%)	Time (in years)	Interest Paid
x	2	1	$x(0.02)(1)$
$240{,}000 - x$	2.5	1	$(240{,}000 - x)(0.025)(1)$

36. *Buying and Selling Land* Roger bought two plots of land for a total of $120,000. When he sold the first plot, he made a profit of 15%. When he sold the second, he lost 10%. His total profit was $5500. How much did he pay for each piece of land?

37. *Retirement Planning* In planning her retirement, Janet Karrenbrock deposits some money at 2.5% interest, with twice as much deposited at 3%. Find the amount deposited at each rate if the total annual interest income is $850.

38. *Investing a Building Fund* A church building fund has invested some money in two ways: part of the money at 3% interest and four times as much at 2.75%. Find the amount invested at each rate if the total annual income from interest is $2800.

39. *Lottery Winnings* Linda won $200,000 in a state lottery. She first paid income tax of 30% on the winnings. She invested some of the rest at 1.5% and some at 4%, earning $4350 interest per year. How much did she invest at each rate?

40. *Cookbook Royalties* Becky Schantz earned $48,000 from royalties on her cookbook. She paid a 28% income tax on these royalties. The balance was invested in two ways, some of it at 3.25% interest and some at 1.75%. The investments produced $904.80 interest per year. Find the amount invested at each rate.

(Modeling) Solve each problem. See Examples 5 and 6.

41. *Warehouse Club Membership* Membership warehouse clubs offer shoppers low prices, along with rewards of cash back on club purchases. If the yearly fee for a warehouse club membership is $100 and the reward rate is 2% on club purchases for the year, then the linear equation

$$y = 100 - 0.02x$$

models the actual yearly cost of the membership y, in dollars. Here x represents the yearly amount of club purchases, also in dollars.

(a) Determine the actual yearly cost of the membership if club purchases for the year are $2400.

(b) What amount of club purchases would reduce the actual yearly cost of the membership to $50?

(c) How much would a member have to spend in yearly club purchases to reduce the yearly membership cost to $0?

42. *Warehouse Club Membership* Suppose that the yearly fee for a warehouse club membership is \$50 and that the reward rate on club purchases for the year is 1.6%. Then the actual yearly cost of a membership y, in dollars, for an amount of yearly club purchases x, in dollars, can be modeled by the following linear equation.

$$y = 50 - 0.016x$$

(a) Determine the actual yearly cost of the membership if club purchases for the year are \$1500.
(b) What amount of club purchases would reduce the actual yearly cost of the membership to \$0?
(c) If club purchases for the year exceed \$3125, how is the actual yearly membership cost affected?

43. ***Indoor Air Pollution*** Formaldehyde is an indoor air pollutant formerly found in plywood, foam insulation, and carpeting. When concentrations in the air reach 33 micrograms per cubic foot ($\mu g/ft^3$), eye irritation can occur. One square foot of new plywood could emit 140 μg per hr. (*Source:* A. Hines, *Indoor Air Quality & Control.*)

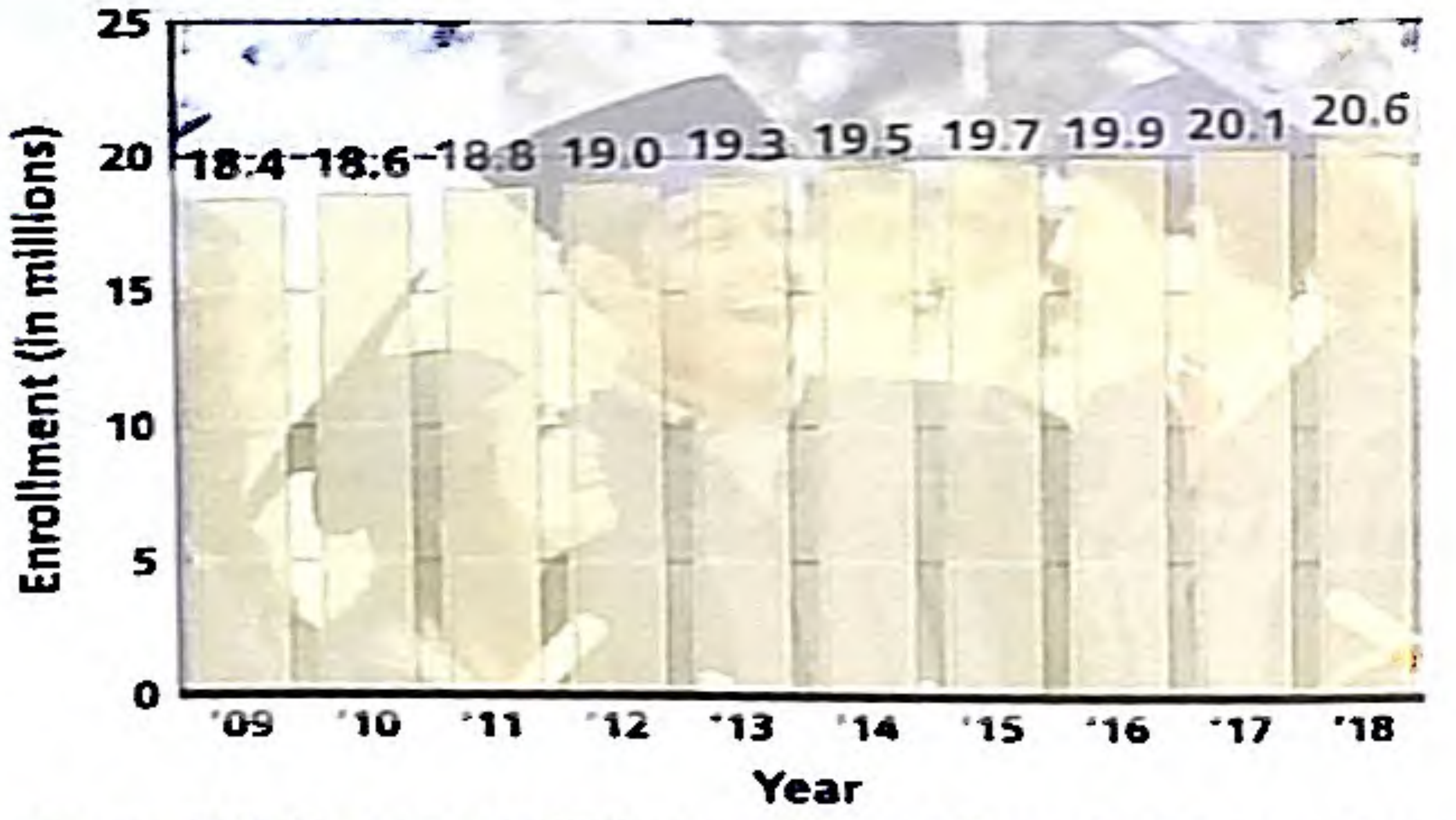
(a) A room has 100 ft^2 of new plywood flooring. Find a linear equation F that computes the amount of formaldehyde, in micrograms, emitted in x hours.
(b) The room contains 800 ft^3 of air and has no ventilation. Determine how long it would take for concentrations to reach 33 $\mu g/ft^3$.

44. ***Classroom Ventilation*** According to the American Society of Heating, Refrigerating and Air-Conditioning Engineers, Inc. (ASHRAE), a nonsmoking classroom should have a ventilation rate of 15 ft^3 per min for each person in the room.

(a) Write an equation that models the total ventilation V (in cubic feet per hour) necessary for a classroom with x students.
(b) A common unit of ventilation is air change per hour (ach). 1 ach is equivalent to exchanging all of the air in a room every hour. If x students are in a classroom having volume 15,000 ft^3, determine how many air exchanges per hour (A) are necessary to keep the room properly ventilated.
(c) Find the necessary number of ach (A) if the classroom has 40 students in it.
(d) In areas like bars and lounges that allow smoking, the ventilation rate should be increased to 50 ft^3 per min per person. Compared to classrooms, ventilation should be increased by what factor in heavy smoking areas?

45. ***College Enrollments*** The graph shows the projections in total enrollment at degree-granting institutions from fall 2009 to fall 2018.

Enrollments at Degree-Granting Institutions



Source: U.S. Department of Education, National Center for Education Statistics

The following linear model provides the approximate enrollment, in millions, between the years 2009 and 2018, where $x = 0$ corresponds to 2009, $x = 1$ to 2010, and so on, and y is in millions of students.

$$y = 0.2309x + 18.35$$

(a) Use the model to determine projected enrollment for fall 2014.
(b) Use the model to determine the year in which enrollment is projected to reach 20 million.
(c) How do your answers to parts (a) and (b) compare to the corresponding values shown in the graph?
(d) The actual enrollment in fall 1997 was 14.5 million. The model here is based on data from 2009 to 2018. If you were to use the model for 1997, what would the projected enrollment be?
(e) Why do you think there is such a discrepancy between the actual value and the value based on the model in part (d)? Discuss the pitfalls of using the model to predict enrollment for years preceding 2009.

1.3 Complex Numbers

- **Basic Concepts of Complex Numbers**
- **Operations on Complex Numbers**

Basic Concepts of Complex Numbers The set of real numbers does not include all the numbers needed in algebra. For example, there is no real number solution of the equation

$$x^2 = -1,$$

since no real number, when squared, gives -1. To extend the real number system to include solutions of equations of this type, the number i is defined to have the following property.

The Imaginary Unit i

$$i = \sqrt{-1}, \quad \text{and therefore,} \quad i^2 = -1.$$

(Note that $-i$ is also a square root of -1.)

Square roots of negative numbers were not incorporated into an integrated number system until the 16th century. They were then used as solutions of equations and later (in the 18th century) in surveying. Today, such numbers are used extensively in science and engineering.

Complex numbers are formed by adding real numbers and multiples of i.

Complex Number

If a and b are real numbers, then any number of the form $a + bi$ is a **complex number.** In the complex number $a + bi$, a is the **real part** and b is the **imaginary part.***

Two complex numbers $a + bi$ and $c + di$ are equal provided that their real parts are equal and their imaginary parts are equal; that is, they are equal if and only if $a = c$ and $b = d$.

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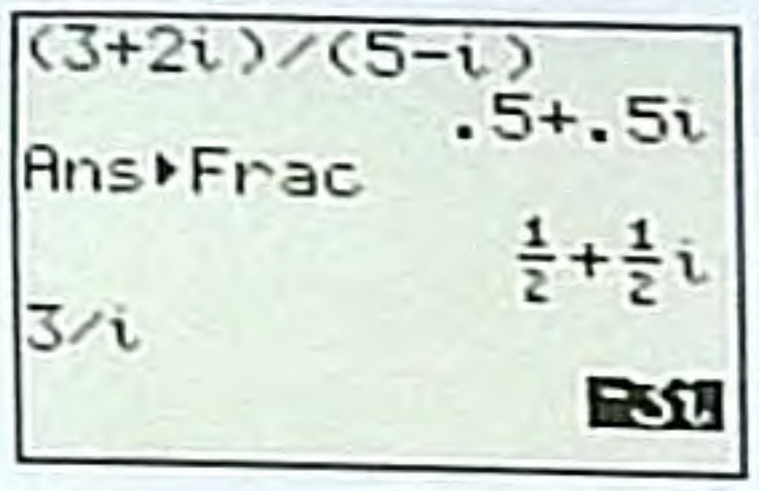
* In some texts, the term bi is defined to be the imaginary part



```
(3+2i)/(5-i)
               .5+.5i
Ans▸Frac
               1/2+1/2i
3/i
```

This screen supports the results in **Example 6.**

(b) $\frac{3}{i} = \frac{3(-i)}{i(-i)}$ $\quad -i$ is the conjugate of i.

$= \frac{-3i}{-i^2}$ $\quad$ Multiply.

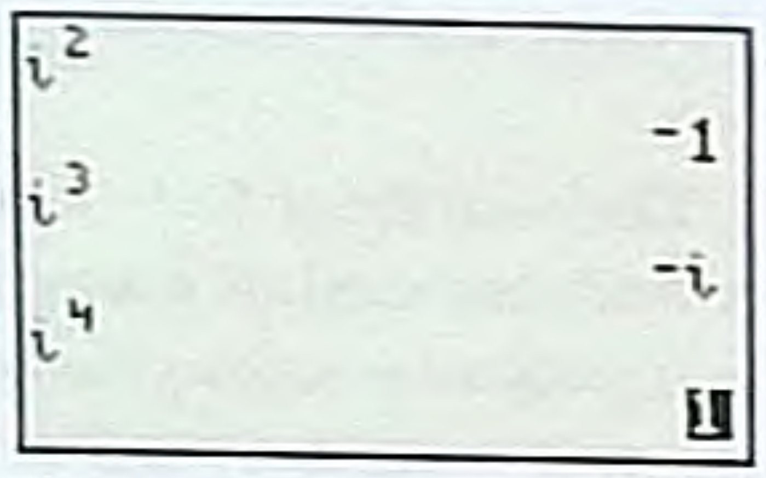
$= \frac{-3i}{1}$ $\quad -i^2 = -(-1) = 1$

$= -3i$, or $0 - 3i$ $\quad$ Standard form

Now Try Exercises 69 and 75.

Powers of i can be simplified using the facts

$$i^2 = -1 \quad \text{and} \quad i^4 = (i^2)^2 = (-1)^2 = 1.$$



```
i^2
                 -1
i^3
                 -i
i^4
```

Powers of i can be found on the TI–83/84 Plus calculator

Consider the following powers of i.

$i^1 = i$ $\qquad\qquad$ $i^5 = i^4 \cdot i = 1 \cdot i = i$

$i^2 = -1$ $\qquad\qquad$ $i^6 = i^4 \cdot i^2 = 1(-1) = -1$

$i^3 = i^2 \cdot i = (-1) \cdot i = -i$ $\qquad\qquad$ $i^7 = i^4 \cdot i^3 = 1 \cdot (-i) = -i$

$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$ $\qquad\qquad$ $i^8 = i^4 \cdot i^4 = 1 \cdot 1 = 1$ and so on.

Powers of i cycle through the same four outcomes (i, -1, $-i$, and 1) since i^4 has the same multiplicative property as 1. Also, any power of i with an exponent that is a multiple of 4 has value 1. As with real numbers, $i^0 = 1$.

EXAMPLE 7 Simplifying Powers of *i*

Simplify each power of i.

(a) i^{15} $\qquad\qquad$ **(b)** i^{-3}

SOLUTION

(a) Since $i^4 = 1$, write the given power as a product involving i^4.

$$i^{15} = i^{12} \cdot i^3 = (i^4)^3 \cdot i^3 = 1^3(-i) = -i$$

(b) Multiply i^{-3} by 1 in the form of i^4 to create the least positive exponent for i.

$$i^{-3} = i^{-3} \cdot 1 = i^{-3} \cdot i^4 = i \qquad i^4 = 1$$

Now Try Exercises 85 and 93.

1.3 Exercises

Concept Check *Determine whether each statement is true or false. If it is false, tell why.*

1. Every real number is a complex number.
2. No real number is a pure imaginary number.
3. Every pure imaginary number is a complex number.
4. A number can be both real and complex.
5. There is no real number that is a complex number.
6. A complex number might not be a pure imaginary number.

Identify each number as real, complex, pure imaginary, *or* nonreal complex. *(More than one of these descriptions will apply.)*

7. -4 **8.** 0 **9.** $13i$ **10.** $-7i$ **11.** $5 + i$

12. $-6 - 2i$ **13.** π **14.** $\sqrt{24}$ **15.** $\sqrt{-25}$ **16.** $\sqrt{-36}$

Write each number as the product of a real number and i. See Example 1.

17. $\sqrt{-25}$ **18.** $\sqrt{-36}$ **19.** $\sqrt{-10}$ **20.** $\sqrt{-15}$

21. $\sqrt{-288}$ **22.** $\sqrt{-500}$ **23.** $-\sqrt{-18}$ **24.** $-\sqrt{-80}$

Multiply or divide, as indicated. Simplify each answer. See Example 2.

25. $\sqrt{-13} \cdot \sqrt{-13}$ **26.** $\sqrt{-17} \cdot \sqrt{-17}$ **27.** $\sqrt{-3} \cdot \sqrt{-8}$

28. $\sqrt{-5} \cdot \sqrt{-15}$ **29.** $\dfrac{\sqrt{-30}}{\sqrt{-10}}$ **30.** $\dfrac{\sqrt{-70}}{\sqrt{-7}}$

31. $\dfrac{\sqrt{-24}}{\sqrt{8}}$ **32.** $\dfrac{\sqrt{-54}}{\sqrt{27}}$ **33.** $\dfrac{\sqrt{-10}}{\sqrt{-40}}$

34. $\dfrac{\sqrt{-8}}{\sqrt{-72}}$ **35.** $\dfrac{\sqrt{-6} \cdot \sqrt{-2}}{\sqrt{3}}$ **36.** $\dfrac{\sqrt{-12} \cdot \sqrt{-6}}{\sqrt{8}}$

Write each number in standard form $a + bi$. See Example 3.

37. $\dfrac{-6 - \sqrt{-24}}{2}$ **38.** $\dfrac{-9 - \sqrt{-18}}{3}$ **39.** $\dfrac{10 + \sqrt{-200}}{5}$

40. $\dfrac{20 + \sqrt{-8}}{2}$ **41.** $\dfrac{-3 + \sqrt{-18}}{24}$ **42.** $\dfrac{-5 + \sqrt{-50}}{10}$

Find each sum or difference. Write the answer in standard form. See Example 4.

43. $(3 + 2i) + (9 - 3i)$ **44.** $(4 - i) + (8 + 5i)$

45. $(-2 + 4i) - (-4 + 4i)$ **46.** $(-3 + 2i) - (-4 + 2i)$

47. $(2 - 5i) - (3 + 4i) - (-2 + i)$ **48.** $(-4 - i) - (2 + 3i) + (-4 + 5i)$

49. $-i\sqrt{2} - 2 - \left(6 - 4i\sqrt{2}\right) - \left(5 - i\sqrt{2}\right)$

50. $3\sqrt{7} - \left(4\sqrt{7} - i\right) - 4i + \left(-2\sqrt{7} + 5i\right)$

Find each product. Write the answer in standard form. See Example 5.

51. $(2 + i)(3 - 2i)$ **52.** $(-2 + 3i)(4 - 2i)$ **53.** $(2 + 4i)(-1 + 3i)$

54. $(1 + 3i)(2 - 5i)$ **55.** $(3 - 2i)^2$ **56.** $(2 + i)^2$

57. $(3 + i)(3 - i)$ **58.** $(5 + i)(5 - i)$ **59.** $(-2 - 3i)(-2 + 3i)$

60. $(6 - 4i)(6 + 4i)$ **61.** $\left(\sqrt{6} + i\right)\left(\sqrt{6} - i\right)$ **62.** $\left(\sqrt{2} - 4i\right)\left(\sqrt{2} + 4i\right)$

63. $i(3 - 4i)(3 + 4i)$ **64.** $i(2 + 7i)(2 - 7i)$ **65.** $3i(2 - i)^2$

66. $-5i(4 - 3i)^2$ **67.** $(2 + i)(2 - i)(4 + 3i)$ **68.** $(3 - i)(3 + i)(2 - 6i)$

Find each quotient. Write the answer in standard form $a + bi$. See Example 6.

69. $\dfrac{6 + 2i}{1 + 2i}$ **70.** $\dfrac{14 + 5i}{3 + 2i}$ **71.** $\dfrac{2 - i}{2 + i}$ **72.** $\dfrac{4 - 3i}{4 + 3i}$

73. $\dfrac{1 - 3i}{1 + i}$ **74.** $\dfrac{-3 + 4i}{2 - i}$ **75.** $\dfrac{-5}{i}$ **76.** $\dfrac{-6}{i}$

77. $\dfrac{8}{-i}$ **78.** $\dfrac{12}{-i}$ **79.** $\dfrac{2}{3i}$ **80.** $\dfrac{5}{9i}$

(Modeling) Alternating Current Complex numbers are used to describe current I, voltage E, and impedance Z (the opposition to current). These three quantities are related by the equation

$$E = IZ, \quad \text{which is known as } \textit{Ohm's Law}.$$

Thus, if any two of these quantities are known, the third can be found. In each exercise, solve the equation $E = IZ$ for the remaining value.

81. $I = 5 + 7i$, $Z = 6 + 4i$

82. $I = 20 + 12i$, $Z = 10 - 5i$

83. $I = 10 + 4i$, $E = 88 + 128i$

84. $E = 57 + 67i$, $Z = 9 + 5i$

Simplify each power of i. See Example 7.

85. i^{25}

86. i^{29}

87. i^{22}

88. i^{26}

89. i^{23}

90. i^{27}

91. i^{32}

92. i^{40}

93. i^{-13}

94. i^{-14}

95. $\frac{1}{i^{-11}}$

96. $\frac{1}{i^{-12}}$

97. Suppose that your friend, Kathy Strautz, tells you that she has discovered a method of simplifying a positive power of i. "Just divide the exponent by 2. Your answer is then the simplified form of i^2 raised to the quotient times i raised to the remainder." Explain why her method works.

98. Explain why the following method of simplifying i^{-42} works.

$$i^{-42} = \frac{1}{i^{42}} = \frac{1}{(i^2)^{21}} = \frac{1}{(-1)^{21}} = \frac{1}{-1} = -1$$

99. Show that $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ is a square root of i.

100. Show that $\frac{\sqrt{3}}{2} + \frac{1}{2}i$ is a cube root of i.

101. Show that $-2 + i$ is a solution of the equation $x^2 + 4x + 5 = 0$.

102. Show that $-3 + 4i$ is a solution of the equation $x^2 + 6x + 25 = 0$.

1.4 Quadratic Equations

- Solving a Quadratic Equation
- Completing the Square
- The Quadratic Formula
- Solving for a Specified Variable
- The Discriminant

A *quadratic equation* is defined as follows.

Quadratic Equation in One Variable

An equation that can be written in the form

$$ax^2 + bx + c = 0,$$

where a, b, and c are real numbers with $a \neq 0$, is a **quadratic equation.** The given form is called **standard form.**

A quadratic equation is a **second-degree equation**—that is, an equation with a squared variable term and no terms of greater degree.

$$x^2 = 25, \quad 4x^2 + 4x - 5 = 0, \quad 3x^2 = 4x - 8$$ Quadratic equations

Solving a Quadratic Equation The factoring method of solving a quadratic equation depends on the **zero-factor property**.

Zero-Factor Property

If a and b are complex numbers with $ab = 0$, then $a = 0$ or $b = 0$ or both equal zero.

EXAMPLE 1 Using the Zero-Factor Property

Solve $6x^2 + 7x = 3$.

SOLUTION

Don't factor out x here.

$$\begin{aligned} 6x^2 + 7x &= 3 \\ 6x^2 + 7x - 3 &= 0 && \text{Standard form} \\ (3x - 1)(2x + 3) &= 0 && \text{Factor. (Section R.4)} \\ 3x - 1 = 0 \quad \text{or} \quad 2x + 3 &= 0 && \text{Zero-factor property} \\ 3x = 1 \quad \text{or} \quad 2x &= -3 && \text{Solve each equation. (Section 1.1)} \\ x = \frac{1}{3} \quad \text{or} \quad x &= -\frac{3}{2} \end{aligned}$$

CHECK $6x^2 + 7x = 3$ Original equation

$$\begin{aligned} 6\left(\frac{1}{3}\right)^2 + 7\left(\frac{1}{3}\right) &\stackrel{?}{=} 3 && \text{Let } x = \tfrac{1}{3}. \\ \frac{6}{9} + \frac{7}{3} &\stackrel{?}{=} 3 \\ 3 &= 3 \ \checkmark \ \text{True} \end{aligned}$$

$$\begin{aligned} 6\left(-\frac{3}{2}\right)^2 + 7\left(-\frac{3}{2}\right) &\stackrel{?}{=} 3 && \text{Let } x = -\tfrac{3}{2}. \\ \frac{54}{4} - \frac{21}{2} &\stackrel{?}{=} 3 \\ 3 &= 3 \ \checkmark \ \text{True} \end{aligned}$$

Both values check, since true statements result. The solution set is $\left\{\frac{1}{3}, -\frac{3}{2}\right\}$.

✓ ***Now Try Exercise 15.***

A quadratic equation of the form $x^2 = k$ can also be solved by factoring.

$$\begin{aligned} x^2 &= k \\ x^2 - k &= 0 && \text{Subtract } k. \\ (x - \sqrt{k})(x + \sqrt{k}) &= 0 && \text{Factor.} \\ x - \sqrt{k} = 0 \quad \text{or} \quad x + \sqrt{k} &= 0 && \text{Zero-factor property} \\ x = \sqrt{k} \quad \text{or} \quad x &= -\sqrt{k} && \text{Solve each equation.} \end{aligned}$$

This proves the **square root property**.

Square Root Property

If $x^2 = k$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$.

That is, the solution set of $x^2 = k$ is

$$\{\sqrt{k}, -\sqrt{k}\}, \quad \textit{which may be abbreviated} \quad \{\pm\sqrt{k}\}.$$

EXAMPLE 8 **Solving for a Quadratic Variable in a Formula**

Solve for the specified variable. Use ± when taking square roots.

(a) $\mathcal{A} = \frac{\pi d^2}{4}$, for d **(b)** $rt^2 - st = k \ (r \neq 0)$, for t

SOLUTION

(a) $$\mathcal{A} = \frac{\pi d^2}{4}$$

Goal: Isolate d, the specified variable.

$$4\mathcal{A} = \pi d^2 \qquad \text{Multiply each side by 4.}$$

$$\frac{4\mathcal{A}}{\pi} = d^2 \qquad \text{Divide each side by } \pi.$$

$$d = \pm\sqrt{\frac{4\mathcal{A}}{\pi}} \qquad \text{Square root property}$$

See the Note following this example.

$$d = \frac{\pm\sqrt{4\mathcal{A}}}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{\pi}} \qquad \text{Multiply by } \frac{\sqrt{\pi}}{\sqrt{\pi}} \text{ (Section R.7)}$$

$$d = \frac{\pm\sqrt{4\mathcal{A}\pi}}{\pi} \qquad \text{Multiply numerators. Multiply denominators.}$$

$$d = \frac{\pm 2\sqrt{\mathcal{A}\pi}}{\pi} \qquad \text{Simplify the radical.}$$

(b) Because $rt^2 - st = k$ has terms with t^2 and t, use the quadratic formula.

$$rt^2 - st - k = 0 \qquad \text{Write in standard form.}$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Quadratic formula}$$

$$t = \frac{-(-s) \pm \sqrt{(-s)^2 - 4(r)(-k)}}{2(r)} \qquad \text{Here, } a = r,\ b = -s, \text{ and } c = -k$$

$$t = \frac{s \pm \sqrt{s^2 + 4rk}}{2r} \qquad \text{Simplify.}$$

✔ ***Now Try Exercises 71 and 77.***

NOTE In Example 8, we took both positive and negative square roots. However, if the variable represents time or length in an application, we consider only the *positive* square root.

The Discriminant The quantity under the radical in the quadratic formula, $b^2 - 4ac$, is called the **discriminant.**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \longleftarrow \text{Discriminant}$$

When the numbers a, b, and c are *integers* (but not necessarily otherwise), the value of the discriminant can be used to determine whether the solutions of a quadratic equation are rational, irrational, or nonreal complex numbers. The number and type of solutions based on the value of the discriminant are shown in the following table.

Discriminant	Number of Solutions	Type of Solutions
Positive, perfect square	Two	Rational
Positive, but not a perfect square	Two	Irrational
Zero	One (a double solution)	Rational
Negative	Two	Nonreal complex

← As seen in Example 5 (Positive, but not a perfect square)

← As seen in Example 6 (Negative)

CAUTION ***The restriction on a, b, and c is important.*** For example,

$$x^2 - \sqrt{5}x - 1 = 0 \quad \text{has discriminant} \quad b^2 - 4ac = 5 + 4 = 9,$$

which would indicate two rational solutions *if the coefficients were integers.* By the quadratic formula, the two solutions $\frac{\sqrt{5} \pm 3}{2}$ are *irrational* numbers.

EXAMPLE 9 Using the Discriminant

Determine the number of distinct solutions, and tell whether they are *rational, irrational,* or *nonreal complex* numbers.

(a) $5x^2 + 2x - 4 = 0$ (b) $x^2 - 10x = -25$ (c) $2x^2 - x + 1 = 0$

SOLUTION

(a) For $5x^2 + 2x - 4 = 0$, use $a = 5$, $b = 2$, and $c = -4$.

$$b^2 - 4ac = 2^2 - 4(5)(-4) = 84 \leftarrow \text{Discriminant}$$

The discriminant 84 is positive and not a perfect square, so there are two distinct irrational solutions.

(b) First, write the equation in standard form as $x^2 - 10x + 25 = 0$. Thus, $a = 1$, $b = -10$, and $c = 25$.

$$b^2 - 4ac = (-10)^2 - 4(1)(25) = 0 \leftarrow \text{Discriminant}$$

There is one distinct rational solution, a double solution.

(c) For $2x^2 - x + 1 = 0$, use $a = 2$, $b = -1$, and $c = 1$.

$$b^2 - 4ac = (-1)^2 - 4(2)(1) = -7 \leftarrow \text{Discriminant}$$

There are two distinct nonreal complex solutions. (They are complex conjugates.)

✔ *Now Try Exercises 83, 85, and 89.*

1.4 Exercises

Concept Check Match the equation in Column I with its solution(s) in Column II.

I

1. $x^2 = 25$
2. $x^2 = -25$
3. $x^2 + 5 = 0$
4. $x^2 - 5 = 0$
5. $x^2 = -20$
6. $x^2 = 20$
7. $x - 5 = 0$
8. $x + 5 = 0$

II

A. $\pm 5i$

B. $\pm 2\sqrt{5}$

C. $\pm i\sqrt{5}$

D. 5

E. $\pm\sqrt{5}$

F. -5

G. ± 5

H. $\pm 2i\sqrt{5}$

Concept Check *Use Choices A–D to answer each question in Exercises* **9–12.**

A. $3x^2 - 17x - 6 = 0$
B. $(2x + 5)^2 = 7$
C. $x^2 + x = 12$
D. $(3x - 1)(x - 7) = 0$

9. Which equation is set up for direct use of the zero-factor property? Solve it.

10. Which equation is set up for direct use of the square root property? Solve it.

11. Only one of the equations does not require Step 1 of the method for completing the square described in this section. Which one is it? Solve it.

12. Only one of the equations is set up so that the values of a, b, and c can be determined immediately. Which one is it? Solve it.

Solve each equation by the zero-factor property. See Example 1.

13. $x^2 - 5x + 6 = 0$
14. $x^2 + 2x - 8 = 0$
15. $5x^2 - 3x - 2 = 0$
16. $2x^2 - x - 15 = 0$
17. $-4x^2 + x = -3$
18. $-6x^2 + 7x = -10$
19. $x^2 - 100 = 0$
20. $x^2 - 64 = 0$
21. $4x^2 - 4x + 1 = 0$
22. $9x^2 - 12x + 4 = 0$
23. $25x^2 + 30x + 9 = 0$
24. $36x^2 + 60x + 25 = 0$

Solve each equation by the square root property. See Example 2.

25. $x^2 = 16$
26. $x^2 = 121$
27. $27 - x^2 = 0$
28. $48 - x^2 = 0$
29. $x^2 = -81$
30. $x^2 = -400$
31. $(3x - 1)^2 = 12$
32. $(4x + 1)^2 = 20$
33. $(x + 5)^2 = -3$
34. $(x - 4)^2 = -5$
35. $(5x - 3)^2 = -3$
36. $(-2x + 5)^2 = -8$

Solve each equation by completing the square. See Examples 3 and 4.

37. $x^2 - 4x + 3 = 0$
38. $x^2 - 7x + 12 = 0$
39. $2x^2 - x - 28 = 0$
40. $4x^2 - 3x - 10 = 0$
41. $x^2 - 2x - 2 = 0$
42. $x^2 - 10x + 18 = 0$
43. $2x^2 + x = 10$
44. $3x^2 + 2x = 5$
45. $-2x^2 + 4x + 3 = 0$
46. $-3x^2 + 6x + 5 = 0$
47. $-4x^2 + 8x = 7$
48. $-3x^2 + 9x = 7$

49. *Concept Check* Francisco claimed that the equation $x^2 - 8x = 0$ cannot be solved by the quadratic formula since there is no value for c. Is he correct?

50. *Concept Check* Francesca, Francisco's twin sister, claimed that the equation $x^2 - 19 = 0$ cannot be solved by the quadratic formula since there is no value for b. Is she correct?

Solve each equation using the quadratic formula. See Examples 5 and 6.

51. $x^2 - x - 1 = 0$
52. $x^2 - 3x - 2 = 0$
53. $x^2 - 6x = -7$
54. $x^2 - 4x = -1$
55. $x^2 = 2x - 5$
56. $x^2 = 2x - 10$
57. $-4x^2 = -12x + 11$
58. $-6x^2 = 3x + 2$
59. $\frac{1}{2}x^2 + \frac{1}{4}x - 3 = 0$
60. $\frac{2}{3}x^2 + \frac{1}{4}x = 3$
61. $0.2x^2 + 0.4x - 0.3 = 0$
62. $0.1x^2 - 0.1x = 0.3$
63. $(4x - 1)(x + 2) = 4x$
64. $(3x + 2)(x - 1) = 3x$
65. $(x - 9)(x - 1) = -16$

66. Explain why the following equations have the same solution set. (Do not actually solve.)

$$-2x^2 + 3x - 6 = 0 \quad \text{and} \quad 2x^2 - 3x + 6 = 0$$

Solve each cubic equation using factoring and the quadratic formula. See Example 7.

67. $x^3 - 8 = 0$
68. $x^3 - 27 = 0$
69. $x^3 + 27 = 0$
70. $x^3 + 64 = 0$

Solve each equation for the indicated variable. Assume no denominators are 0. See Example 8.

71. $s = \frac{1}{2}gt^2$, for t

72. $\mathcal{A} = \pi r^2$, for r

73. $F = \frac{kMv^2}{r}$, for v

74. $E = \frac{e^2k}{2r}$, for e

75. $r = r_0 + \frac{1}{2}at^2$, for t

76. $s = s_0 + gt^2 + k$, for t

77. $h = -16t^2 + v_0t + s_0$, for t

78. $S = 2\pi rh + 2\pi r^2$, for r

For each equation, (a) solve for x in terms of y, and (b) solve for y in terms of x. See Example 8.

79. $4x^2 - 2xy + 3y^2 = 2$

80. $3y^2 + 4xy - 9x^2 = -1$

81. $2x^2 + 4xy - 3y^2 = 2$

82. $5x^2 - 6xy + 2y^2 = 1$

Evaluate the discriminant for each equation. Then use it to predict the number of distinct solutions, and whether they are rational, irrational, *or* nonreal complex. *Do not solve the equation. See Example 9.*

83. $x^2 - 8x + 16 = 0$

84. $x^2 + 4x + 4 = 0$

85. $3x^2 + 5x + 2 = 0$

86. $8x^2 = -14x - 3$

87. $4x^2 = -6x + 3$

88. $2x^2 + 4x + 1 = 0$

89. $9x^2 + 11x + 4 = 0$

90. $3x^2 = 4x - 5$

91. $8x^2 - 72 = 0$

92. Show that the discriminant for the equation

$$\sqrt{2}x^2 + 5x - 3\sqrt{2} = 0$$

is 49. If this equation is completely solved, it can be shown that the solution set is $\left\{-3\sqrt{2}, \frac{\sqrt{2}}{2}\right\}$. We have a discriminant that is positive and a perfect square, yet the two solutions are irrational. Does this contradict the discussion in this section? Explain.

93. Is it possible for the solution set of a quadratic equation with integer coefficients to consist of a single irrational number? Explain.

94. Is it possible for the solution set of a quadratic equation with real coefficients to consist of one real number and one nonreal complex number? Explain.

Find the values of a, b, and c for which the quadratic equation

$$ax^2 + bx + c = 0$$

has the given numbers as solutions. (Hint: Use the zero-factor property in reverse.)

95. $4, 5$

96. $-3, 2$

97. $1 + \sqrt{2}, 1 - \sqrt{2}$

98. $i, -i$

Chapter 1 Quiz (Sections 1.1–1.4)

1. Solve the linear equation $3(x - 5) + 2 = 1 - (4 + 2x)$.

2. Determine whether each equation is an *identity*, a *conditional equation*, or a *contradiction*. Give the solution set.

(a) $4x - 5 = -2(3 - 2x) + 3$

(b) $5x - 9 = 5(-2 + x) + 1$

(c) $5x - 4 = 3(6 - x)$

3. Solve the equation $ay + 2x = y + 5x$ for y. Assume $a \neq 1$.

4. *Earning Interest* Johnny Ramistella deposits some money at 2.5% annual interest and twice as much at 3.0%. Find the amount deposited at each rate if his total annual interest income is $850.

5. *(Modeling) Minimum Hourly Wage* One model for the minimum hourly wage in the United States for the period 1978–2009 is

$$y = 0.126x - 246.25,$$

where x represents the year and y represents the wage, in dollars. (*Source:* Bureau of Labor Statistics.) The actual 1999 minimum wage was $5.15. What does this model predict as the wage? What is the difference between the actual wage and the predicted wage?

6. Write $\frac{-4+\sqrt{-24}}{8}$ in standard form $a + bi$.

7. Write the quotient $\frac{7-2i}{2+4i}$ in standard form $a + bi$.

Solve each equation.

8. $3x^2 - x = -1$

9. $x^2 - 29 = 0$

10. $\mathcal{A} = \frac{1}{2}r^2\theta$, for r

1.5 Applications and Modeling with Quadratic Equations

- Geometry Problems
- Using the Pythagorean Theorem
- Height of a Projected Object
- Modeling with Quadratic Equations

Geometry Problems To solve these applications, we continue to use the problem-solving strategy discussed in Section 1.2.

EXAMPLE 1 **Solving a Problem Involving Volume**

A piece of machinery produces rectangular sheets of metal such that the length is three times the width. Equal-sized squares measuring 5 in. on a side can be cut from the corners so that the resulting piece of metal can be shaped into an open box by folding up the flaps. If specifications call for the volume of the box to be 1435 in.^{3}, find the dimensions of the original piece of metal.

SOLUTION

Step 1 **Read** the problem. We must find the dimensions of the original piece of metal.

Step 2 **Assign a variable.** We know that the length is three times the width.

Let $x =$ the width (in inches) and thus, $3x =$ the length.

The box is formed by cutting $5 + 5 = 10$ in. from both the length and the width. See **Figure 5**. **Figure 6** indicates that the width of the bottom of the box is $x - 10$, the length of the bottom of the box is $3x - 10$, and the height is 5 in. (the length of the side of each cut-out square).

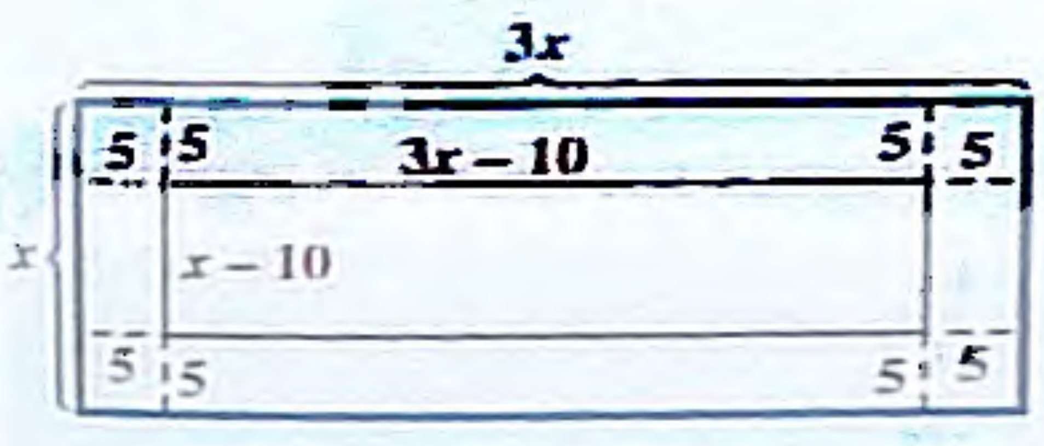


Figure 5

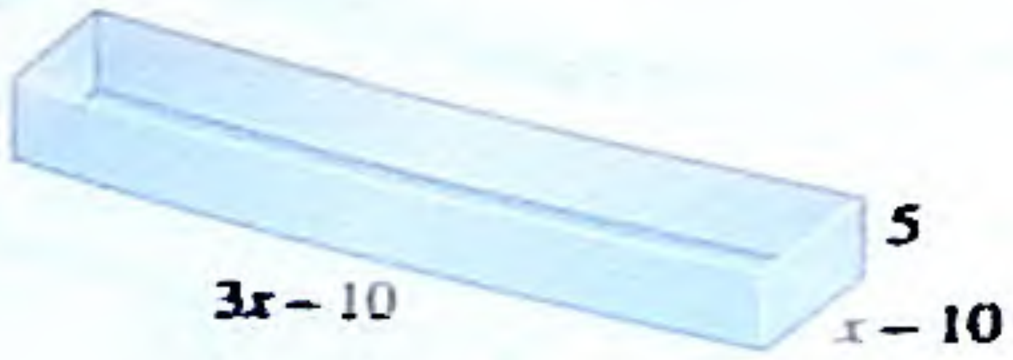


Figure 6

Step 3 **Write an equation.** The formula for volume of a box is $V = lwh$.

$$\begin{array}{ccc} \text{Volume} & = & \text{length} \times \text{width} \times \text{height} \\ \downarrow & & \downarrow \qquad\quad \downarrow \qquad\quad \downarrow \\ 1435 & = & (3x - 10)(x - 10)(5) \end{array}$$

(Note that the dimensions of the box must be positive numbers, so $3x - 10$ and $x - 10$ must be greater than 0, which implies $x > \frac{10}{3}$ and $x > 10$. These are both satisfied when $x > 10$.)

Step 4 **Solve the equation.**

$$\begin{aligned} 1435 &= 15x^2 - 200x + 500 && \text{Multiply (Section R.3)} \\ 0 &= 15x^2 - 200x - 935 && \text{Subtract 1435 from each side.} \\ 0 &= 3x^2 - 40x - 187 && \text{Divide each side by 5.} \\ 0 &= (3x + 11)(x - 17) && \text{Factor. (Section R.4)} \end{aligned}$$

$$3x + 11 = 0 \quad \text{or} \quad x - 17 = 0 \qquad \text{Zero-factor property (Section 1.4)}$$

The width cannot be negative.

$$x = -\frac{11}{3} \quad \text{or} \quad x = 17 \qquad \text{Solve each equation. (Section 1.1)}$$

Step 5 **State the answer.** Only 17 satisfies the restriction $x > 10$. Thus, the dimensions of the original piece should be 17 in. by $3(17) = 51$ in.

Step 6 **Check.** The length of the bottom of the box is $51 - 2(5) = 41$ in. The width is $17 - 2(5) = 7$ in. The height is 5 in. (the amount cut on each corner), so the volume of the box is

$$V = lwh = 41 \times 7 \times 5 = 1435 \text{ in.}^3, \quad \text{as required.}$$

✓ ***Now Try Exercise 23.***

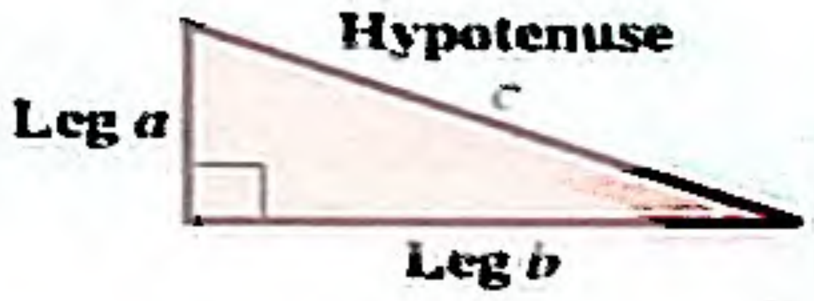
PROBLEM-SOLVING HINT We may get a solution that does not satisfy the physical constraints of a problem. As seen in **Example 1,** a dimension of a box cannot be a negative number such as $-\frac{11}{3}$. Therefore, we reject this outcome.

Using the Pythagorean Theorem Example 2 requires the use of the **Pythagorean theorem** for right triangles. Recall that the **legs** of a right triangle form the right angle, and the **hypotenuse** is the side opposite the right angle.

Pythagorean Theorem

In a right triangle, the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse.

$$a^2 + b^2 = c^2$$



EXAMPLE 2 Applying the Pythagorean Theorem

A piece of property has the shape of a right triangle. The longer leg is 20 m longer than twice the length of the shorter leg. The hypotenuse is 10 m longer than the length of the longer leg. Find the lengths of the sides of the triangular lot.

SOLUTION

Step 1 **Read** the problem. We must find the lengths of the three sides.

Step 2 **Assign a variable.**

Let $x =$ the length of the shorter leg (in meters).

Then $2x + 20 =$ the length of the longer leg, and

$(2x + 20) + 10$, or $2x + 30 =$ the length of the hypotenuse.

See **Figure 7.**

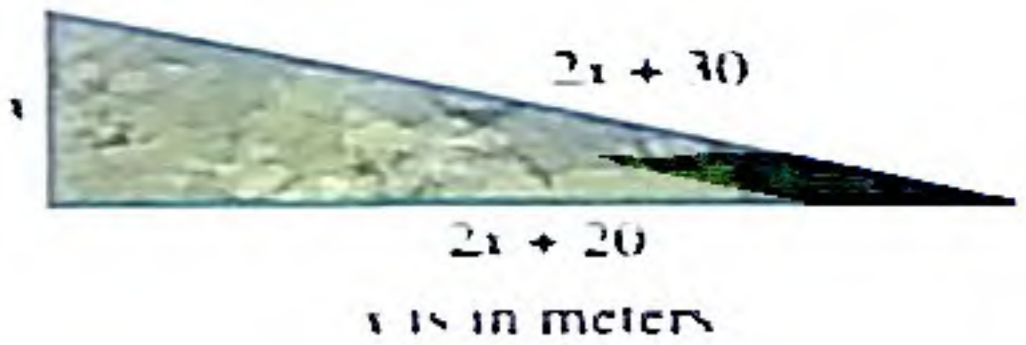


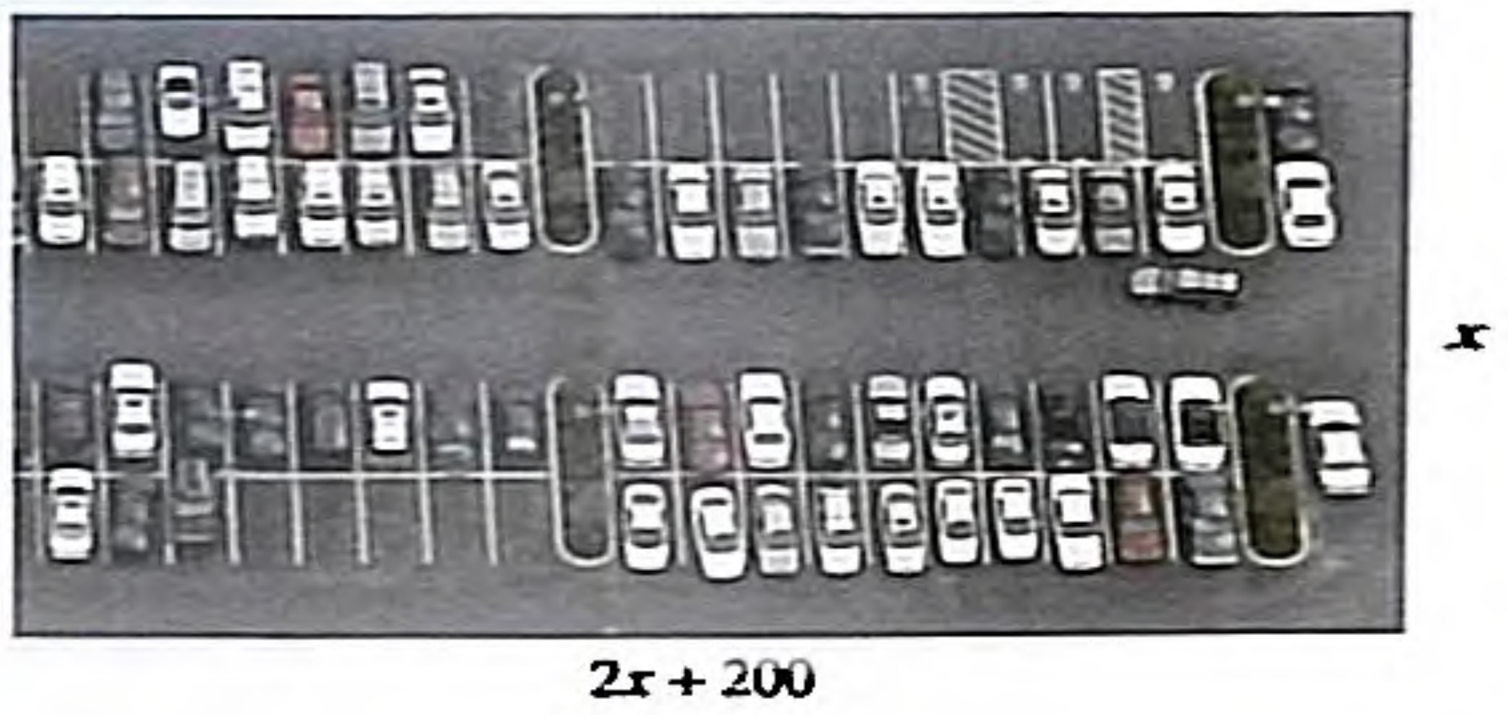
Figure 7

The year 2011 corresponds to $x = 11.7$. Round down to the year 2011 because 11.7 yr from 2000 occurs during 2011. There is no value on the bar graph to compare this to, because the last data value is for the year 2010. Always view results that are *beyond* the data in your model with skepticism, and realistically consider whether the model will continue as given. The model *predicts* that ridership will be 1.9 million again in the year 2011.

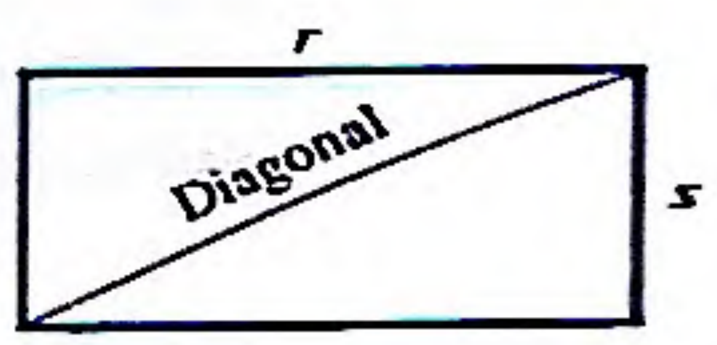
Now Try Exercise 47.

1.5 Exercises

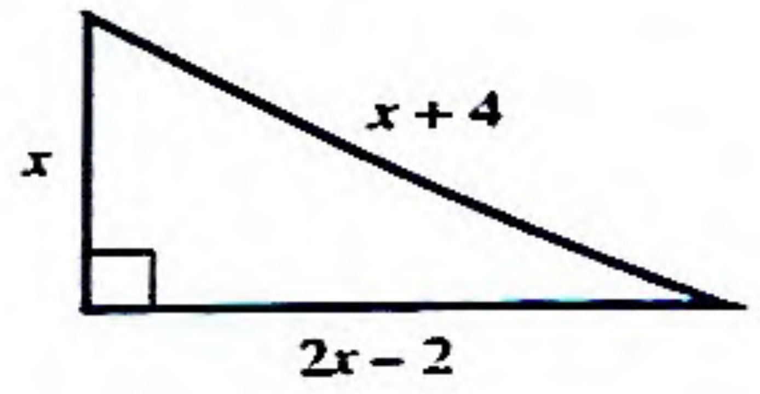
Concept Check *Answer each question.*



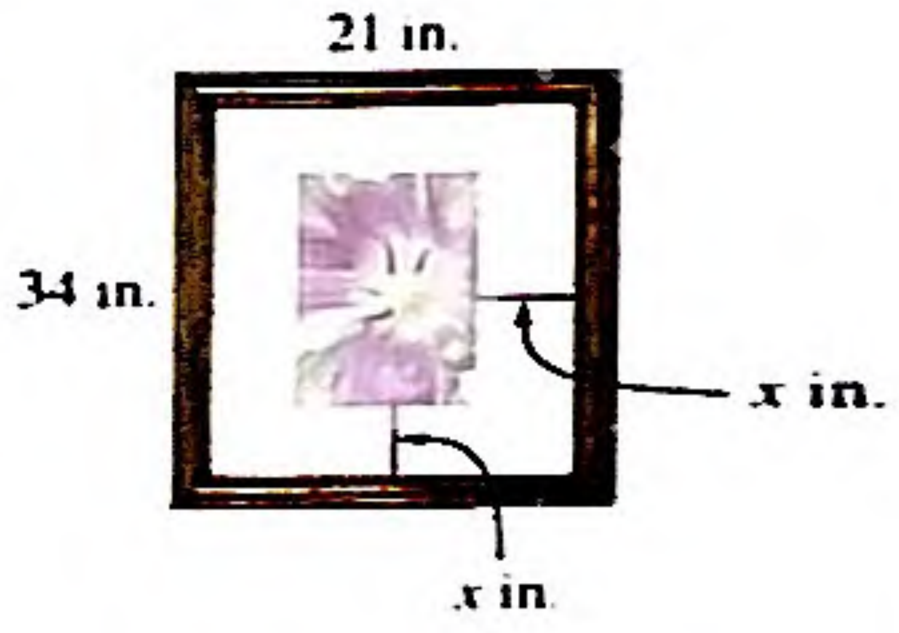
1. ***Area of a Parking Lot*** For the rectangular parking area of the shopping center shown, which one of the following equations says that the area is 40,000 yd²?
   - **A.** $x(2x + 200) = 40{,}000$
   - **B.** $2x + 2(2x + 200) = 40{,}000$
   - **C.** $x + (2x + 200) = 40{,}000$
   - **D.** $x^2 + (2x + 200)^2 = 40{,}000^2$



2. ***Diagonal of a Rectangle*** If a rectangle is r feet long and s feet wide, which one of the following expressions is the length of its diagonal in terms of r and s?
   - **A.** $\sqrt{rs}$
   - **B.** $r + s$
   - **C.** $\sqrt{r^2 + s^2}$
   - **D.** $r^2 + s^2$



3. ***Sides of a Right Triangle*** To solve for the lengths of the right triangle sides, which equation is correct?
   - **A.** $x^2 = (2x - 2)^2 + (x + 4)^2$
   - **B.** $x^2 + (x + 4)^2 = (2x - 2)^2$
   - **C.** $x^2 = (2x - 2)^2 - (x + 4)^2$
   - **D.** $x^2 + (2x - 2)^2 = (x + 4)^2$



4. ***Area of a Picture*** The mat around the picture shown measures x inches across. Which one of the following equations says that the area of the picture itself is 600 in.²?
   - **A.** $2(34 - 2x) + 2(21 - 2x) = 600$
   - **B.** $(34 - 2x)(21 - 2x) = 600$
   - **C.** $(34 - x)(21 - x) = 600$
   - **D.** $x(34)(21) = 600$

To prepare for the applications that come later, work the following basic problems that lead to quadratic equations.

Unknown Numbers *In Exercises 5–14, use the following facts.*

If x represents an integer, then $x + 1$ represents the next consecutive integer.

If x represents an even integer, then $x + 2$ represents the next consecutive even integer.

If x represents an odd integer, then $x + 2$ represents the next consecutive odd integer.

5. Find two consecutive integers whose product is 56.
6. Find two consecutive integers whose product is 110.

7. Find two consecutive even integers whose product is 168.

8. Find two consecutive even integers whose product is 224.

9. Find two consecutive odd integers whose product is 63.

10. Find two consecutive odd integers whose product is 143.

11. The sum of the squares of two consecutive odd integers is 202. Find the integers.

12. The sum of the squares of two consecutive even integers is 52. Find the integers.

13. The difference of the squares of two positive consecutive even integers is 84. Find the integers.

14. The difference of the squares of two positive consecutive odd integers is 32. Find the integers.

15. ***Dimensions of a Right Triangle*** The lengths of the sides of a right triangle are consecutive even integers. Find these lengths. (*Hint:* Use the Pythagorean theorem.)

16. ***Dimensions of a Right Triangle*** The lengths of the sides of a right triangle are consecutive positive integers. Find these lengths. (*Hint:* Use the Pythagorean theorem.)

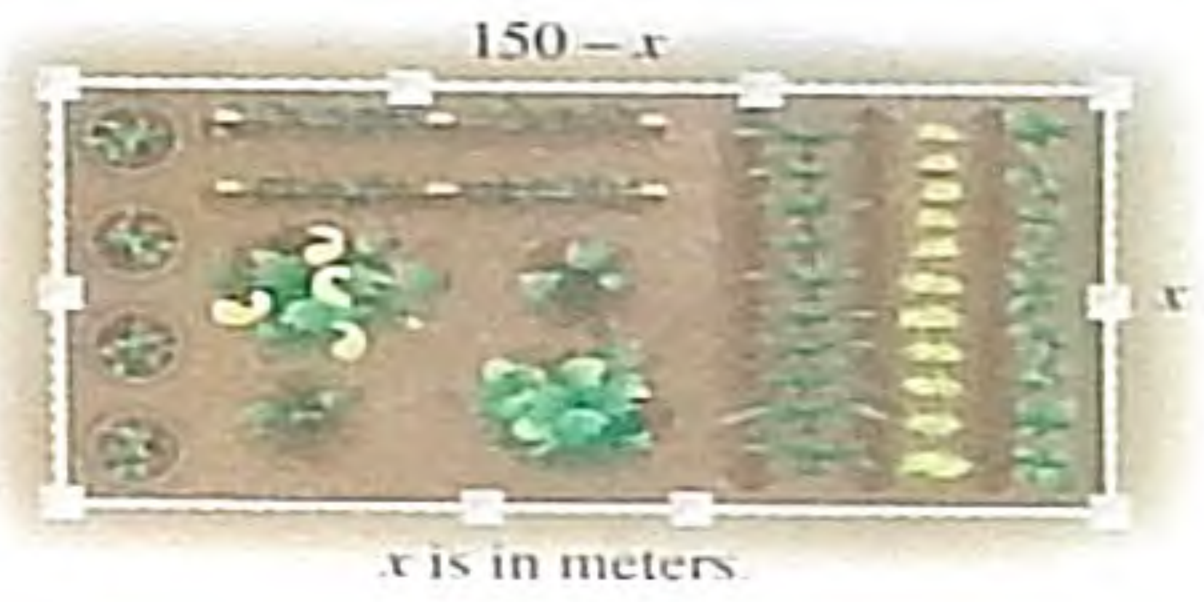
17. ***Dimensions of a Square*** The length of each side of a square is 3 in. more than the length of each side of a smaller square. The sum of the areas of the squares is 149 in.^{2}. Find the lengths of the sides of the two squares.

18. ***Dimensions of a Square*** The length of each side of a square is 5 in. more than the length of each side of a smaller square. The difference of the areas of the squares is 95 in.^{2}. Find the lengths of the sides of the two squares.

Solve each problem. See Example 1.

19. ***Dimensions of a Parking Lot*** A parking lot has a rectangular area of 40,000 yd^2. The length is 200 yd more than twice the width. Find the dimensions of the lot. (See Exercise 1.)

20. ***Dimensions of a Garden*** An ecology center wants to set up an experimental garden using 300 m of fencing to enclose a rectangular area of 5000 m^2. Find the dimensions of the garden.



21. ***Dimensions of a Rug*** Zachary Daniels wants to buy a rug for a room that is 12 ft wide and 15 ft long. He wants to leave a uniform strip of floor around the rug. He can afford to buy 108 ft^2 of carpeting. What dimensions should the rug have?

22. ***Width of a Flower Border*** A landscape architect has included a rectangular flower bed measuring 9 ft by 5 ft in her plans for a new building. She wants to use two colors of flowers in the bed, one in the center and the other for a border of the same width on all four sides. If she has enough plants to cover 24 ft^2 for the border, how wide can the border be?

23. ***Volume of a Box*** A rectangular piece of metal is 10 in. longer than it is wide. Squares with sides 2 in. long are cut from the four corners, and the flaps are folded upward to form an open box. If the volume of the box is 832 in.^3, what were the original dimensions of the piece of metal?

24. ***Volume of a Box*** In Exercise 23, suppose that the piece of metal has length twice the width, and 4-in. squares are cut from the corners. If the volume of the box is 1536 in.^3, what were the original dimensions of the piece of metal?

25. ***Manufacturing to Specifications*** A manufacturing firm wants to package its product in a cylindrical container 3 ft high with surface area 8π ft^2. What should the radius of the circular top and bottom of the container be? (*Hint:* The surface area consists of the circular top and bottom and a rectangle that represents the side cut open vertically and unrolled.)

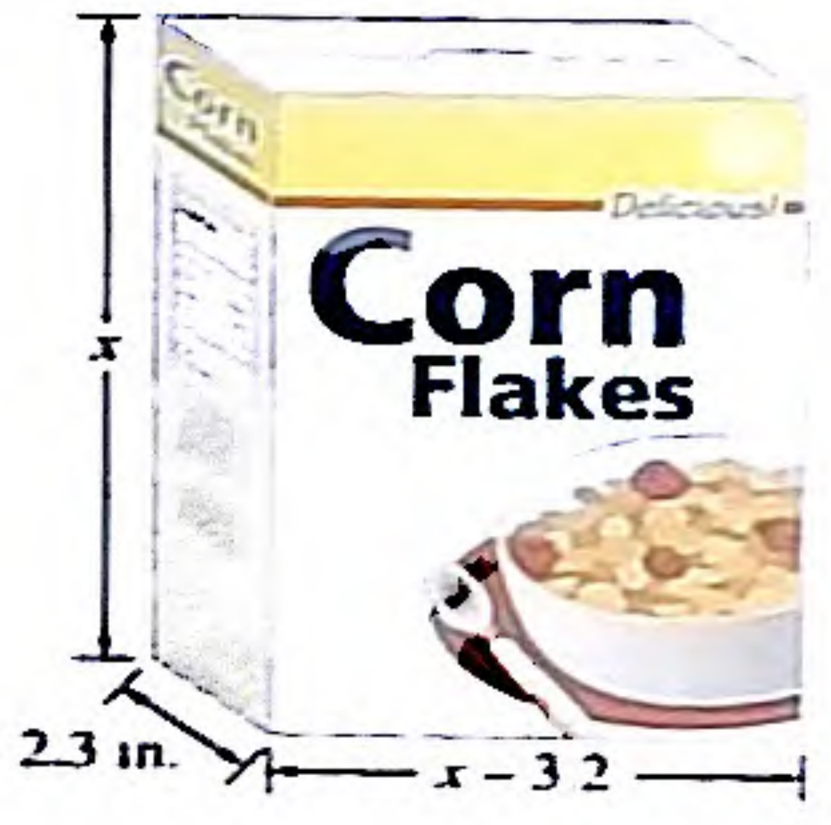
26. ***Manufacturing to Specifications*** In Exercise 25, what radius would produce a container with a volume of π times the radius? (*Hint:* The volume is the area of the circular base times the height.)

27. ***Dimensions of a Square*** What is the length of the side of a square if its area and perimeter are numerically equal?

28. ***Dimensions of a Rectangle*** A rectangle has an area that is numerically twice its perimeter. If the length is twice the width, what are its dimensions?

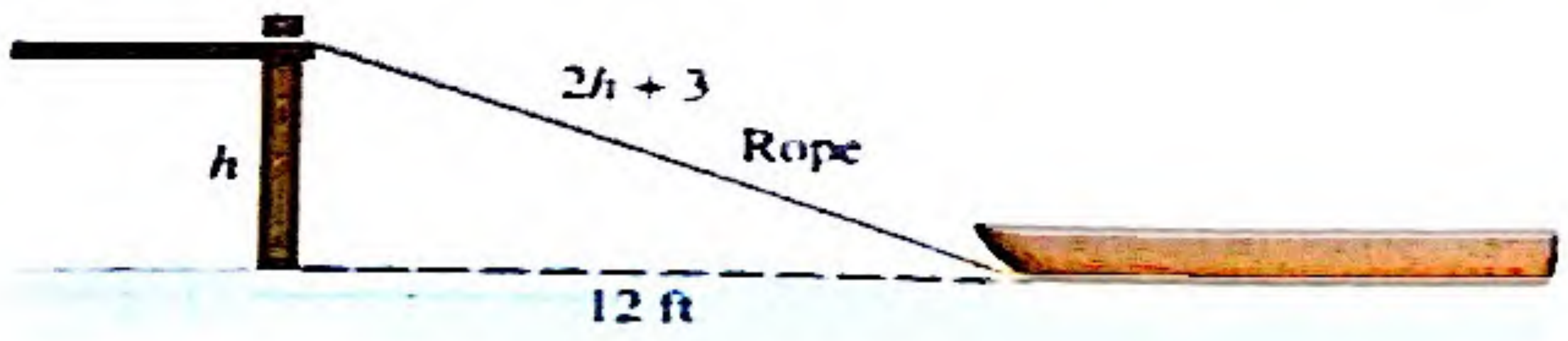
29. ***Radius of a Can*** A can of Blue Runner Red Kidney Beans has surface area 371 cm^2. Its height is 12 cm. What is the radius of the circular top? Round to the nearest hundredth.

30. ***Dimensions of a Cereal Box*** The volume of a 10-oz box of corn flakes is 180.4 in.^3. The width of the box is 3.2 in. less than the length, and its depth is 2.3 in. Find the length and width of the box to the nearest tenth.

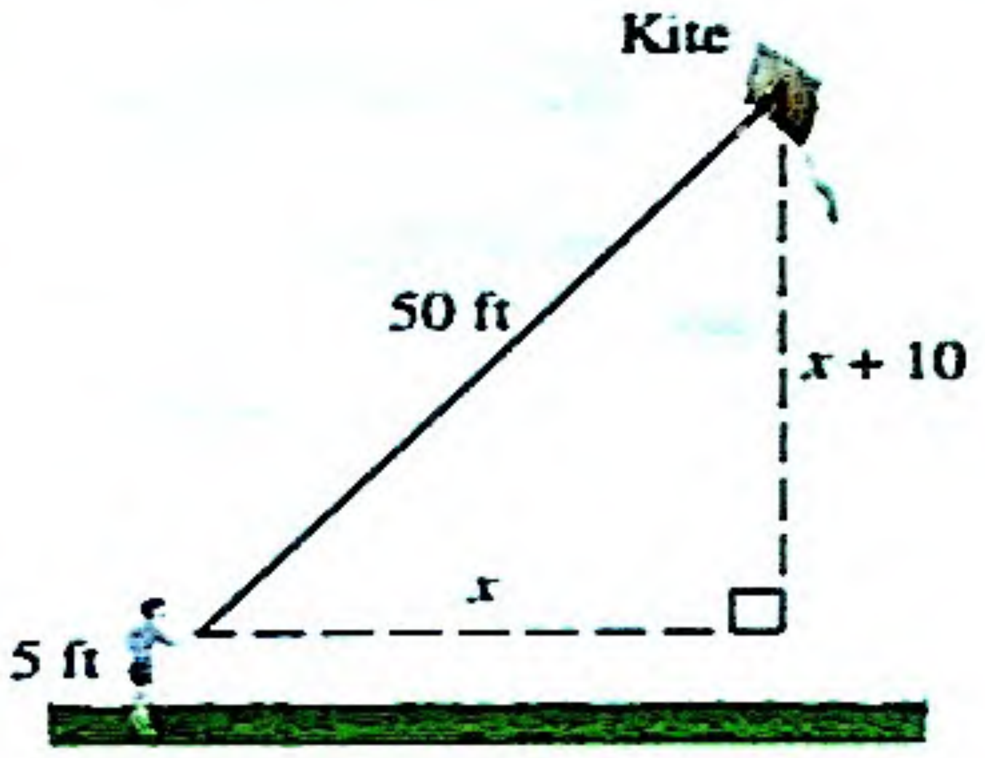


Solve each problem. See Example 2.

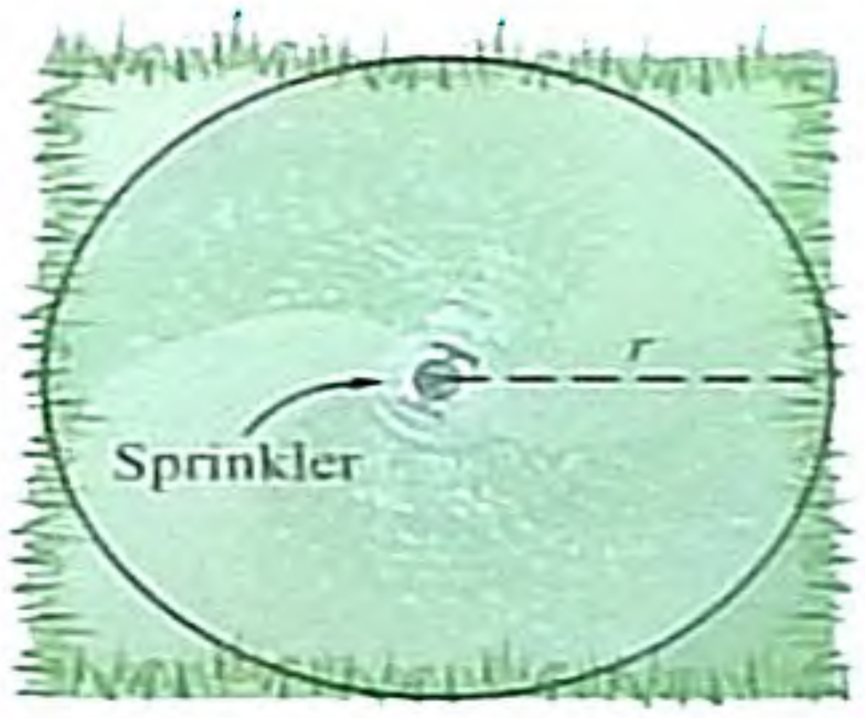
31. ***Height of a Dock*** A boat is being pulled into a dock with a rope attached to the boat at water level. When the boat is 12 ft from the dock, the length of the rope from the boat to the dock is 3 ft longer than twice the height of the dock above the water. Find the height of the dock.



32. ***Height of a Kite*** Grady is flying a kite on 50 ft of string. Its vertical distance from his hand is 10 ft more than the horizontal distance from his hand. Assuming that the string is being held 5 ft above ground level, find its horizontal distance from Grady and its vertical distance from the ground.

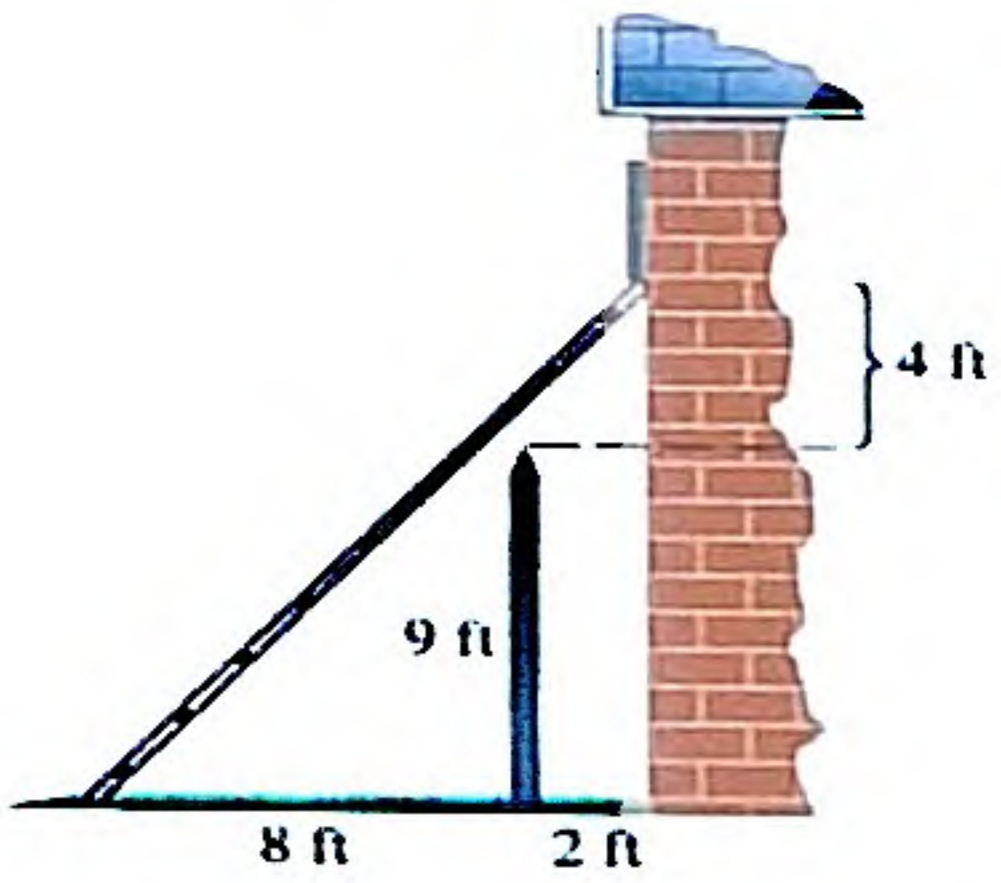


33. ***Radius Covered by a Circular Lawn Sprinkler*** A square lawn has area 800 ft^2. A sprinkler placed at the center of the lawn sprays water in a circular pattern as shown in the figure. What is the radius of the circle?



34. ***Dimensions of a Solar Panel Frame*** Molly has a solar panel with a width of 26 in. To get the proper inclination for her climate, she needs a right triangular support frame that has one leg twice as long as the other. To the nearest tenth of an inch, what dimensions should the frame have?

35. ***Length of a Ladder*** A building is 2 ft from a 9-ft fence that surrounds the property. A worker wants to wash a window in the building 13 ft from the ground. He plans to place a ladder over the fence so it rests against the building. (See the figure.) He decides he should place the ladder 8 ft from the fence for stability. To the nearest tenth of a foot, how long a ladder will he need?



36. ***Range of Receivers*** Tanner Jones and Sheldon Furst have received communications receivers for Christmas. If they leave from the same point at the same time, Tanner walking north at 2.5 mph and Sheldon walking east at 3 mph, how long will they be able to talk to each other if the range of the communications receivers is 4 mi? Round your answer to the nearest minute.

37. ***Length of a Walkway*** A nature conservancy group decides to construct a raised wooden walkway through a wetland area. To enclose the most interesting part of the wetlands, the walkway will have the shape of a right triangle with one leg 700 yd longer than the other and the hypotenuse 100 yd longer than the longer leg. Find the total length of the walkway.

38. ***Broken Bamboo*** Problems involving the Pythagorean theorem have appeared in mathematics for thousands of years. This one is taken from the ancient Chinese work, *Arithmetic in Nine Sections:*

> *There is a bamboo 10 ft high, the upper end of which, being broken, reaches the ground 3 ft from the stem. Find the height of the break.*

(Modeling) *Solve each problem. See Examples 3 and 4.*

Height of a Projectile *A projectile is launched from ground level with an initial velocity of v_0 feet per second. Neglecting air resistance, its height in feet t seconds after launch is given by*

$$s = -16t^2 + v_0 t.$$

In Exercises 39–42, find the time(s) that the projectile will (a) reach a height of 80 ft *and (b) return to the ground for the given value of v_0. Round answers to the nearest hundredth if necessary.*

39. $v_0 = 96$ 40. $v_0 = 128$ 41. $v_0 = 32$ 42. $v_0 = 16$

43. ***Height of a Projected Ball*** An astronaut on the moon throws a baseball upward. The astronaut is 6 ft, 6 in. tall, and the initial velocity of the ball is 30 ft per sec. The height s of the ball in feet is given by the equation

$$s = -2.7t^2 + 30t + 6.5,$$

where t is the number of seconds after the ball was thrown.

(a) After how many seconds is the ball 12 ft above the moon's surface? Round to the nearest hundredth.

(b) How many seconds will it take for the ball to return to the surface? Round to the nearest hundredth.

44. The ball in Exercise 43 will never reach a height of 100 ft. How can this be determined algebraically?

45. ***Carbon Monoxide Exposure*** Carbon monoxide (CO) combines with the hemoglobin of the blood to form carboxyhemoglobin (COHb), which reduces transport of oxygen to tissues. Smokers routinely have a 4% to 6% COHb level in their blood, which can cause symptoms such as blood flow alterations, visual impairment, and poorer vigilance ability. The quadratic model

$$T = 0.00787x^2 - 1.528x + 75.89$$

approximates the exposure time in hours necessary to reach this 4% to 6% level, where $50 \leq x \leq 100$ is the amount of carbon monoxide present in the air in parts per million (ppm). (*Source: Indoor Air Quality Environmental Information Handbook: Combustion Sources.*)

(a) A kerosene heater or a room full of smokers is capable of producing 50 ppm of carbon monoxide. How long would it take for a nonsmoking person to start feeling the above symptoms?

(b) Find the carbon monoxide concentration necessary for a person to reach the 4% to 6% COHb level in 3 hr. Round to the nearest tenth

46. ***Carbon Monoxide Exposure*** Refer to **Exercise 45.** High concentrations of carbon monoxide (CO) can cause coma and death. The time required for a person to reach a COHb level capable of causing a coma can be approximated by the quadratic model

$$T = 0.0002x^2 - 0.316x + 127.9,$$

where T is the exposure time in hours necessary to reach this level and $500 \leq x \leq 800$ is the amount of carbon monoxide present in the air in parts per million (ppm). (*Source: Indoor Air Quality Environmental Information Handbook: Combustion Sources.*)

(a) What is the exposure time when $x = 600$ ppm?

(b) Estimate the concentration of CO necessary to produce a coma in 4 hr

47. ***Methane Gas Emissions*** The table gives methane gas emissions from all sources in the United States, in millions of metric tons. The quadratic model

$$y = 1.493x^2 + 7.279x + 684.4$$

approximates the emissions for these years. In the model, x represents the number of years since 2004, so $x = 0$ represents 2004, $x = 1$ represents 2005, and so on.

Year	Millions of Metric Tons of Methane
2004	686.6
2005	691.8
2006	706.3
2007	722.7
2008	737.4

Source: U.S. Energy Information Administration.

(a) According to the model, what would be the emissions in 2010? Round to the nearest tenth of a million metric tons.

(b) Find the year beyond 2004 for which this model predicts that the emissions reached 700 million metric tons.

48. ***Cost of Public Colleges*** The average cost, in dollars, for tuition and fees for in-state students at four-year public colleges over the period 2000–2010 can be modeled by the equation

$$y = 3.026x^2 + 377.7x + 3449,$$

where $x = 0$ corresponds to 2000, $x = 1$ corresponds to 2001, and so on. Based on this model, for what year after 2000 was the average cost \$7605? (*Source:* The College Board, *Annual Survey of Colleges*.)

49. ***Internet Publishing and Broadcasting*** Estimated revenue from Internet publishing and broadcasting in the United States during the years 2004 through 2008 can be modeled by the equation

$$y = 318.4x^2 + 1612x + 8596,$$

where $x = 0$ corresponds to the year 2004, $x = 1$ corresponds to 2005, and so on, and y is in millions of dollars. Approximate the revenue from Internet publishing and broadcasting in 2006 to the nearest million. (*Source:* U.S. Census Bureau.)

50. ***Cable TV Subscribers*** The number of U.S. households subscribing to cable TV for the period 2000 through 2010 can be modeled by the equation

$$y = -0.0746x^2 + 3.146x + 79.52,$$

where $x = 0$ corresponds to 2000, $x = 1$ corresponds to 2001, and so on, and y is in millions. Based on this model, approximately how many U.S. households, to the nearest tenth of a million, subscribed to cable TV in 2008? (*Source:* Nielsen Media Research.)

Relating Concepts

For individual or collaborative investigation *(Exercises 51–54)*

If p units of an item are sold for x dollars per unit, the revenue is $R = px$. Use this idea to analyze the following problem, working Exercises 51–54 in order.

Number of Apartments Rented *The manager of an 80-unit apartment complex knows from experience that at a rent of \$300, all the units will be full. On the average, one additional unit will remain vacant for each \$20 increase in rent over \$300. Furthermore, the manager must keep at least 30 units rented due to other financial considerations. Currently, the revenue from the complex is \$35,000. How many apartments are rented?*

(continued)

51. Suppose that x represents the number of \$20 increases over \$300. Represent the number of apartment units that will be rented in terms of x.

52. Represent the rent per unit in terms of x.

53. Use the answers in **Exercises 51 and 52** to write an equation that defines the revenue generated when there are x increases of \$20 over \$300.

54. According to the problem, the revenue currently generated is \$35,000. Substitute this value for revenue into the equation from **Exercise 53.** Solve for x to answer the question in the problem.

Solve each problem. ***(See Relating Concepts Exercises 51–54.)***

55. *Number of Airline Passengers* The cost of a charter flight to Miami is \$225 each for 75 passengers, with a refund of \$5 per passenger for each passenger in excess of 75. How many passengers must take the flight to produce a revenue of \$16,000?

56. *Number of Bus Passengers* A charter bus company charges a fare of \$40 per person, plus \$2 per person for each unsold seat on the bus. If the bus holds 100 passengers and x represents the number of unsold seats, how many passengers must ride the bus to produce revenue of \$5950? (*Note:* Because of the company's commitment to efficient fuel use, the charter will not run unless filled to at least half-capacity.)

57. ***Harvesting a Cherry Orchard*** The manager of a cherry orchard wants to schedule the annual harvest. If the cherries are picked now, the average yield per tree will be 100 lb, and the cherries can be sold for 40 cents per pound. Past experience shows that the yield per tree will increase about 5 lb per week, while the price will decrease about 2 cents per pound per week. How many weeks should the manager wait to get an average revenue of \$38.40 per tree?

58. ***Recycling Aluminum Cans*** A local group of scouts has been collecting old aluminum cans for recycling. The group has already collected 12,000 lb of cans, for which they could currently receive \$4 per hundred pounds. The group can continue to collect cans at the rate of 400 lb per day. However, a glut in the old-can market has caused the recycling company to announce that it will lower its price, starting immediately, by \$0.10 per hundred pounds per day. The scouts can make only one trip to the recycling center. How many days should they wait in order to get \$490 for their cans?

1.6 Other Types of Equations and Applications

- Rational Equations
- Work Rate Problems
- Equations with Radicals
- Equations with Rational Exponents
- Equations Quadratic in Form

Rational Equations

A **rational equation** is an equation that has a rational expression for one or more terms. To solve a rational equation, multiply each side by the least common denominator (LCD) of the terms of the equation to eliminate fractions, and then solve the resulting equation.

A value of the variable that appears to be a solution after each side of a rational equation is multiplied by a variable expression (the LCD) is called a **proposed solution.** Because a rational expression is not defined when its denominator is 0, **proposed solutions for which any denominator equals 0 are excluded from the solution set.**

Be sure to check all proposed solutions in the original equation.

EXAMPLE 1 **Solving Rational Equations That Lead to Linear Equations**

Solve each equation.

(a) $\dfrac{3x-1}{3}-\dfrac{2x}{x-1}=x$ **(b)** $\dfrac{x}{x-2}=\dfrac{2}{x-2}+2$

SOLUTION

(a) The least common denominator is $3(x-1)$, which is equal to 0 if $x=1$. Therefore, 1 cannot possibly be a solution of this equation.

$$\begin{aligned}
\frac{3x-1}{3}-\frac{2x}{x-1}&=x\\
3(x-1)\left(\frac{3x-1}{3}\right)-3(x-1)\left(\frac{2x}{x-1}\right)&=3(x-1)x && \text{Multiply by the LCD } 3(x-1)\text{, where } x\neq 1 \text{ (Section R.5)}\\
(x-1)(3x-1)-3(2x)&=3x(x-1) && \text{Divide out common factors.}\\
3x^2-4x+1-6x&=3x^2-3x && \text{Multiply. (Section R.3)}\\
1-10x&=-3x && \text{Subtract } 3x^2\text{. Combine like terms.}\\
1&=7x && \text{Solve the linear equation. (Section 1.1)}\\
x&=\frac{1}{7} && \text{Proposed solution}
\end{aligned}$$

The proposed solution $\frac{1}{7}$ meets the requirement that $x\neq 1$ and does not cause any denominator to equal 0. Substitute to check for correct algebra.

CHECK

$$\begin{aligned}
\frac{3x-1}{3}-\frac{2x}{x-1}&=x && \text{Original equation}\\
\frac{3\left(\frac{1}{7}\right)-1}{3}-\frac{2\left(\frac{1}{7}\right)}{\frac{1}{7}-1}&\stackrel{?}{=}\frac{1}{7} && \text{Let } x=\tfrac{1}{7}.\\
-\frac{4}{21}-\left(-\frac{1}{3}\right)&\stackrel{?}{=}\frac{1}{7} && \text{Simplify the complex fractions. (Section R.5)}\\
\frac{1}{7}&=\frac{1}{7}\ \checkmark && \text{True}
\end{aligned}$$

The solution set is $\left\{\frac{1}{7}\right\}$.

(b)

$$\begin{aligned}
\frac{x}{x-2}&=\frac{2}{x-2}+2\\
(x-2)\left(\frac{x}{x-2}\right)&=(x-2)\left(\frac{2}{x-2}\right)+(x-2)2 && \text{Multiply by the LCD, } x-2\text{, where } x\neq 2\\
x&=2+2(x-2) && \text{Divide out common factors.}\\
x&=2+2x-4 && \text{Distributive property (Section R.2)}\\
-x&=-2 && \text{Solve the linear equation.}\\
x&=2 && \text{Proposed solution}
\end{aligned}$$

The proposed solution is 2. However, the variable is restricted to real numbers except 2. If $x=2$, then not only does it cause a zero denominator, but also multiplying by $x-2$ in the first step is multiplying both sides by 0, which is not valid. Thus, the solution set is $\emptyset$.

✔ *Now Try Exercises 7 and 9.*

Equation Quadratic in Form

An equation is said to be **quadratic in form** if it can be written as

$$au^2 + bu + c = 0,$$

where $a \neq 0$ and u is some algebraic expression.

EXAMPLE 8 Solving Equations Quadratic in Form

Solve each equation.

(a) $(x+1)^{2/3} - (x+1)^{1/3} - 2 = 0$ **(b)** $6x^{-2} + x^{-1} = 2$

SOLUTION

(a)

$$\begin{aligned}
(x+1)^{2/3} - (x+1)^{1/3} - 2 &= 0 && \text{Since } (x+1)^{2/3} = [(x+1)^{1/3}]^2, \\
u^2 - u - 2 &= 0 && \text{let } u = (x+1)^{1/3}. \\
(u-2)(u+1) &= 0 && \text{Factor.}
\end{aligned}$$

$$\begin{aligned}
u - 2 &= 0 \quad \text{or} \quad & u + 1 &= 0 && \text{Zero-factor property} \\
u &= 2 \quad \text{or} \quad & u &= -1 && \text{Solve each equation.} \\
(x+1)^{1/3} &= 2 \quad \text{or} \quad & (x+1)^{1/3} &= -1 && \text{Replace } u \text{ with } (x+1)^{1/3}. \\
[(x+1)^{1/3}]^3 &= 2^3 \quad \text{or} \quad & [(x+1)^{1/3}]^3 &= (-1)^3 && \text{Cube each side.} \\
x + 1 &= 8 \quad \text{or} \quad & x + 1 &= -1 && \text{Apply the exponents.} \\
x &= 7 \quad \text{or} \quad & x &= -2 && \text{Proposed solutions}
\end{aligned}$$

Don't forget this step

CHECK

$$(x+1)^{2/3} - (x+1)^{1/3} - 2 = 0 \qquad \text{Original equation}$$

$$\begin{aligned}
(7+1)^{2/3} - (7+1)^{1/3} - 2 &\stackrel{?}{=} 0 \quad \text{Let } x = 7. \\
8^{2/3} - 8^{1/3} - 2 &\stackrel{?}{=} 0 \\
4 - 2 - 2 &\stackrel{?}{=} 0 \\
0 &= 0 \ \checkmark \ \text{True}
\end{aligned}$$

$$\begin{aligned}
(-2+1)^{2/3} - (-2+1)^{1/3} - 2 &\stackrel{?}{=} 0 \quad \text{Let } x = -2. \\
(-1)^{2/3} - (-1)^{1/3} - 2 &\stackrel{?}{=} 0 \\
1 + 1 - 2 &\stackrel{?}{=} 0 \\
0 &= 0 \ \checkmark \ \text{True}
\end{aligned}$$

Both proposed solutions check, so the solution set is $\{-2, 7\}$.

(b)

$$\begin{aligned}
6x^{-2} + x^{-1} &= 2 \\
6x^{-2} + x^{-1} - 2 &= 0 && \text{Subtract 2 from each side.} \\
6u^2 + u - 2 &= 0 && \text{Let } u = x^{-1}. \text{ Then } u^2 = x^{-2}. \\
(3u+2)(2u-1) &= 0 && \text{Factor.}
\end{aligned}$$

$$\begin{aligned}
3u + 2 &= 0 \quad \text{or} \quad & 2u - 1 &= 0 && \text{Zero-factor property} \\
u &= -\frac{2}{3} \quad \text{or} \quad & u &= \frac{1}{2} && \text{Solve each equation.} \\
x^{-1} &= -\frac{2}{3} \quad \text{or} \quad & x^{-1} &= \frac{1}{2} && \text{Replace } u \text{ with } x^{-1}. \\
x &= -\frac{3}{2} \quad \text{or} \quad & x &= 2 && x^{-1} \text{ is the reciprocal of } x. \text{ (Section R.6)}
\end{aligned}$$

Don't stop here. Remember to substitute for u.

Both proposed solutions check, so the solution set is $\{-\frac{3}{2}, 2\}$.

✔ *Now Try Exercises 83 and 87.*

CAUTION *When using a substitution variable in solving an equation that is quadratic in form, do not forget the step that gives the solution in terms of the original variable.*

EXAMPLE 9 Solving an Equation Quadratic in Form

Solve $12x^4 - 11x^2 + 2 = 0$.

SOLUTION

$$
\begin{aligned}
12x^4 - 11x^2 + 2 &= 0 \\
12(x^2)^2 - 11x^2 + 2 &= 0 && x^4 = (x^2)^2 \\
12u^2 - 11u + 2 &= 0 && \text{Let } u = x^2. \text{ Then } u^2 = x^4. \\
(3u - 2)(4u - 1) &= 0 && \text{Solve the quadratic equation.}
\end{aligned}
$$

$$
\begin{aligned}
3u - 2 = 0 \quad &\text{or} \quad 4u - 1 = 0 && \text{Zero-factor property} \\
u = \frac{2}{3} \quad &\text{or} \quad u = \frac{1}{4} && \text{Solve each equation.} \\
x^2 = \frac{2}{3} \quad &\text{or} \quad x^2 = \frac{1}{4} && \text{Replace } u \text{ with } x^2. \\
x = \pm\sqrt{\frac{2}{3}} \quad &\text{or} \quad x = \pm\sqrt{\frac{1}{4}} && \text{Square root property (Section 1.4)} \\
x = \frac{\pm\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} \quad &\text{or} \quad x = \pm\frac{1}{2} && \text{Simplify radicals. (Section R.7)} \\
x = \pm\frac{\sqrt{6}}{3} \quad &
\end{aligned}
$$

Check that the solution set is $\left\{\pm\frac{\sqrt{6}}{3}, \pm\frac{1}{2}\right\}$.

Now Try Exercise 77.

NOTE Some equations that are quadratic in form are simple enough to avoid using the substitution variable technique. For example, to solve

$$12x^4 - 11x^2 + 2 = 0, \qquad \text{Equation from Example 9}$$

we could factor $12x^4 - 11x^2 + 2$ directly as $(3x^2 - 2)(4x^2 - 1)$, set each factor equal to zero, and then solve the resulting two quadratic equations. ***Which method to use is a matter of personal preference.***

1.6 Exercises

Decide what values of the variable cannot possibly be solutions for each equation. Do not solve. See Examples 1 and 2.

1. $\frac{5}{2x+3} - \frac{1}{x-6} = 0$

2. $\frac{2}{x+1} + \frac{3}{5x-2} = 0$

3. $\frac{3}{x-2} + \frac{1}{x+1} = \frac{3}{x^2-x-2}$

4. $\frac{2}{x+3} - \frac{5}{x-1} = \frac{-5}{x^2+2x-3}$

5. $\frac{1}{4x} - \frac{2}{x} = 3$

6. $\frac{5}{2x} + \frac{2}{x} = 6$

Solve each equation. See Example 1.

7. $\frac{2x+5}{2} - \frac{3x}{x-2} = x$

8. $\frac{4x+3}{4} - \frac{2x}{x+1} = x$

9. $\frac{x}{x-3} = \frac{3}{x-3} + 3$

10. $\frac{x}{x-4} = \frac{4}{x-4} + 4$

11. $\frac{-2}{x-3} + \frac{3}{x+3} = \frac{-12}{x^2-9}$

12. $\frac{3}{x-2} + \frac{1}{x+2} = \frac{12}{x^2-4}$

13. $\frac{4}{x^2+x-6} - \frac{1}{x^2-4} = \frac{2}{x^2+5x+6}$

14. $\frac{3}{x^2+x-2} - \frac{1}{x^2-1} = \frac{7}{2x^2+6x+4}$

Solve each equation. See Example 2.

15. $\frac{2x+1}{x-2} + \frac{3}{x} = \frac{-6}{x^2-2x}$

16. $\frac{4x+3}{x+1} + \frac{2}{x} = \frac{1}{x^2+x}$

17. $\frac{x}{x-1} - \frac{1}{x+1} = \frac{2}{x^2-1}$

18. $\frac{-x}{x+1} - \frac{1}{x-1} = \frac{-2}{x^2-1}$

19. $\frac{5}{x^2} - \frac{43}{x} = 18$

20. $\frac{7}{x^2} + \frac{19}{x} = 6$

21. $2 = \frac{3}{2x-1} + \frac{-1}{(2x-1)^2}$

22. $6 = \frac{7}{2x-3} + \frac{3}{(2x-3)^2}$

23. $\frac{2x-5}{x} = \frac{x-2}{3}$

24. $\frac{x+4}{2x} = \frac{x-1}{3}$

25. $\frac{2x}{x-2} = 5 + \frac{4x^2}{x-2}$

26. $\frac{3x^2}{x-1} + 2 = \frac{x}{x-1}$

Solve each problem. See Example 3.

27. ***Painting a House*** **(This problem appears in the 1994 movie *Little Big League*.) If Joe can paint a house in 3 hr, and Sam can paint the same house in 5 hr, how long does it take them to do it together?**

28. ***Painting a House*** **Repeat Exercise 27, but assume that Joe takes 6 hr working alone, and Sam takes 8 hr working alone.**

29. ***Pollution in a River*** **Two chemical plants are polluting a river. If plant A produces a predetermined maximum amount of pollutant twice as fast as plant B, and together they produce the maximum pollutant in 26 hr, how long will it take plant B alone?**

	Rate	Time	Part of Job Accomplished
Pollution from A	$\frac{1}{x}$	26	$\frac{1}{x}(26)$
Pollution from B		26	

30. ***Filling a Settling Pond*** **A sewage treatment plant has two inlet pipes to its settling pond. One pipe can fill the pond 3 times as fast as the other pipe, and together they can fill the pond in 12 hr. How long will it take the faster pipe to fill the pond alone?**

31. ***Filling a Pool*** **An inlet pipe can fill Blake's pool in 5 hr, while an outlet pipe can empty it in 8 hr. In his haste to surf the Internet, Blake left both pipes open. How long did it take to fill the pool?**

32. ***Filling a Pool*** Suppose Blake discovered his error (see **Exercise 31**) after an hour-long surf. If he then closed the outlet pipe, how much more time would be needed to fill the pool?

33. ***Filling a Sink*** With both taps open, Robert can fill his kitchen sink in 5 min. When full, the sink drains in 10 min. How long will it take to fill the sink if Robert forgets to put in the stopper?

34. ***Filling a Sink*** If Robert (see **Exercise 33**) remembers to put in the stopper after 1 min, how much longer will it take to fill the sink?

Solve each equation. ***See Examples 4–6.***

35. $x - \sqrt{2x + 3} = 0$

36. $x - \sqrt{3x + 18} = 0$

37. $\sqrt{3x + 7} = 3x + 5$

38. $\sqrt{4x + 13} = 2x - 1$

39. $\sqrt{4x + 5} - 6 = 2x - 11$

40. $\sqrt{6x + 7} - 9 = x - 7$

41. $\sqrt{4x} - x + 3 = 0$

42. $\sqrt{2x} - x + 4 = 0$

43. $\sqrt{x} - \sqrt{x - 5} = 1$

44. $\sqrt{x} - \sqrt{x - 12} = 2$

45. $\sqrt{x + 7} + 3 = \sqrt{x - 4}$

46. $\sqrt{x + 5} + 2 = \sqrt{x - 1}$

47. $\sqrt{2x + 5} - \sqrt{x + 2} = 1$

48. $\sqrt{4x + 1} - \sqrt{x - 1} = 2$

49. $\sqrt{3x} = \sqrt{5x + 1} - 1$

50. $\sqrt{2x} = \sqrt{3x + 12} - 2$

51. $\sqrt{x + 2} = 1 - \sqrt{3x + 7}$

52. $\sqrt{2x - 5} = 2 + \sqrt{x - 2}$

53. $\sqrt{2\sqrt{7x + 2}} = \sqrt{3x + 2}$

54. $\sqrt{3\sqrt{2x + 3}} = \sqrt{5x - 6}$

55. $3 - \sqrt{x} = \sqrt{2\sqrt{x} - 3}$

56. $\sqrt{x} + 2 = \sqrt{4 + 7\sqrt{x}}$

57. $\sqrt[3]{4x + 3} = \sqrt[3]{2x - 1}$

58. $\sqrt[3]{2x} = \sqrt[3]{5x + 2}$

59. $\sqrt[3]{5x^2 - 6x + 2} - \sqrt[3]{x} = 0$

60. $\sqrt[3]{3x^2 - 9x + 8} = \sqrt[3]{x}$

61. $\sqrt[4]{x - 15} = 2$

62. $\sqrt[4]{3x + 1} = 1$

63. $\sqrt[4]{x^2 + 2x} = \sqrt[4]{3}$

64. $\sqrt[4]{x^2 + 6x} = 2$

Solve each equation. ***See Example 7.***

65. $x^{3/2} = 125$

66. $x^{5/4} = 32$

67. $(x^2 + 24x)^{1/4} = 3$

68. $(3x^2 + 52x)^{1/4} = 4$

69. $(x - 3)^{2/5} = 4$

70. $(x + 200)^{2/3} = 36$

71. $(2x + 5)^{1/3} - (6x - 1)^{1/3} = 0$

72. $(3x + 7)^{1/3} - (4x + 2)^{1/3} = 0$

73. $(2x - 1)^{2/3} = x^{1/3}$

74. $(x - 3)^{2/5} = (4x)^{1/5}$

75. $x^{2/3} = 2x^{1/3}$

76. $3x^{3/4} = x^{1/2}$

Solve each equation. ***See Examples 8 and 9.***

77. $2x^4 - 7x^2 + 5 = 0$

78. $4x^4 - 8x^2 + 3 = 0$

79. $x^4 + 2x^2 - 15 = 0$

80. $3x^4 + 10x^2 - 25 = 0$

81. $(x - 1)^{2/3} + (x - 1)^{1/3} - 12 = 0$

82. $(2x - 1)^{2/3} + 2(2x - 1)^{1/3} - 3 = 0$

83. $(x + 1)^{2/5} - 3(x + 1)^{1/5} + 2 = 0$

84. $(x + 5)^{2/3} + (x + 5)^{1/3} - 20 = 0$

85. $6(x + 2)^4 - 11(x + 2)^2 = -4$

86. $8(x - 4)^4 - 10(x - 4)^2 = -3$

87. $10x^{-2} + 33x^{-1} - 7 = 0$

88. $7x^{-2} - 10x^{-1} - 8 = 0$

89. $x^{-2/3} + x^{-1/3} - 6 = 0$

90. $2x^{-2/5} - x^{-1/5} - 1 = 0$

91. $16x^{-4} - 65x^{-2} + 4 = 0$

92. $625x^{-4} - 125x^{-2} + 4 = 0$

Relating Concepts

For individual or collaborative investigation *(Exercises 93–96)*

In this section we introduced methods of solving equations quadratic in form by substitution and solving equations involving radicals by raising each side of the equation to a power. Suppose we wish to solve

$$x - \sqrt{x} - 12 = 0.$$

We can solve this equation using either of the two methods. ***Work Exercises 93–96 in order*** *to see how both methods apply.*

93. Let $u = \sqrt{x}$ and solve the equation by substitution. What is the value of u that does not lead to a solution of the equation?

94. Solve the equation by isolating $\sqrt{x}$ on one side and then squaring. What is the value of x that does not satisfy the equation?

95. Which one of the methods used in **Exercises 93 and 94** do you prefer? Why?

96. Solve $3x - 2\sqrt{x} - 8 = 0$ using one of the two methods described.

Solve each equation for the indicated variable. Assume all denominators are nonzero.

97. $d = k\sqrt{h}$, for h

98. $x^{2/3} + y^{2/3} = a^{2/3}$, for y

99. $m^{3/4} + n^{3/4} = 1$, for m

100. $\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$, for R

101. $\frac{E}{e} = \frac{R + r}{r}$, for e

102. $a^2 + b^2 = c^2$, for b

Summary Exercises on Solving Equations

This section of miscellaneous equations provides practice in solving all the types introduced in this chapter so far. Solve each equation.

1. $4x - 3 = 2x + 3$

2. $5 - (6x + 3) = 2(2 - 2x)$

3. $x(x + 6) = 9$

4. $x^2 = 8x - 12$

5. $\sqrt{x + 2} + 5 = \sqrt{x + 15}$

6. $\frac{5}{x + 3} - \frac{6}{x - 2} = \frac{3}{x^2 + x - 6}$

7. $\frac{3x + 4}{3} - \frac{2x}{x - 3} = x$

8. $\frac{x}{2} + \frac{4}{3}x = x + 5$

9. $5 - \frac{2}{x} + \frac{1}{x^2} = 0$

10. $(2x + 1)^2 = 9$

11. $x^{-2/5} - 2x^{-1/5} - 15 = 0$

12. $\sqrt{x + 2} + 1 = \sqrt{2x + 6}$

13. $x^4 - 3x^2 - 4 = 0$

14. $1.2x + 0.3 = 0.7x - 0.9$

15. $\sqrt[6]{2x + 1} = \sqrt[6]{9}$

16. $3x^2 - 2x = -1$

17. $3[2x - (6 - 2x) + 1] = 5x$

18. $\sqrt{x} + 1 = \sqrt{11 - \sqrt{x}}$

19. $(14 - 2x)^{2/3} = 4$

20. $2x^{-1} - x^{-2} = 1$

21. $\frac{3}{x - 3} = \frac{3}{x - 3}$

22. $a^2 + b^2 = c^2$, for a

1.7 Inequalities

- Linear Inequalities
- Three-Part Inequalities
- Quadratic Inequalities
- Rational Inequalities

An **inequality** says that one expression is greater than, greater than or equal to, less than, or less than or equal to another (**Section R.2**). As with equations, a value of the variable for which the inequality is true is a solution of the inequality, and the set of all solutions is the solution set of the inequality. Two inequalities with the same solution set are equivalent.

Inequalities are solved with the properties of inequality, which are similar to the properties of equality in **Section 1.1**.

Properties of Inequality

Let a, b, and c represent real numbers.

1. If $a < b$, then $a + c < b + c$.

2. If $a < b$ and if $c > 0$, then $ac < bc$.

3. If $a < b$ and if $c < 0$, then $ac > bc$.

Replacing $<$ with $>$, $\leq$, or $\geq$ results in similar properties. (Restrictions on c remain the same.)

NOTE Multiplication may be replaced by division in Properties 2 and 3. ***Always remember to reverse the direction of the inequality symbol when multiplying or dividing by a negative number.***

Linear Inequalities The definition of a *linear inequality* is similar to the definition of a linear equation.

Linear Inequality in One Variable

A linear inequality in one variable is an inequality that can be written in the form

$$ax + b > 0,$$

where a and b are real numbers, with $a \neq 0$. (Any of the symbols $\geq$, $<$, and $\leq$ may also be used.)

EXAMPLE 1 Solving a Linear Inequality

Solve $-3x + 5 > -7$.

SOLUTION

$$-3x + 5 > -7$$

$$-3x + 5 - 5 > -7 - 5 \quad \text{Subtract 5.}$$

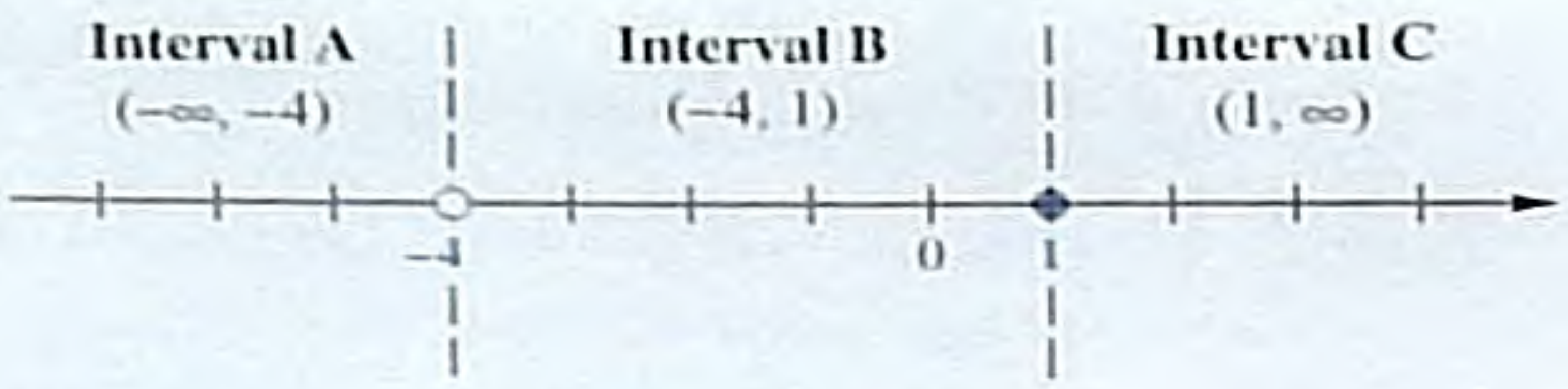
$$-3x > -12 \quad \text{Combine like terms.}$$

$$\frac{-3x}{-3} < \frac{-12}{-3}$$

Divide by -3. Reverse the direction of the inequality symbol when multiplying or dividing by a negative number.

Don't forget to reverse the inequality symbol here

$$x < 4$$



Use a solid circle on 1, since the symbol is ≥. The value −4 cannot be in the solution set since it causes the denominator to equal 0. Use an open circle on −4.

Figure 17

Step 3 Choose test values.

Interval	Test Value	Is $\frac{5}{x+4} \geq 1$ True or False?
A: $(-\infty, -4)$	-5	$\frac{5}{-5+4} \overset{?}{\geq} 1$ $-5 \geq 1$ False
B: $(-4, 1)$	0	$\frac{5}{0+4} \overset{?}{\geq} 1$ $\frac{5}{4} \geq 1$ True
C: $(1, \infty)$	2	$\frac{5}{2+4} \overset{?}{\geq} 1$ $\frac{5}{6} \geq 1$ False

The values in the interval $(-4, 1)$ satisfy the original inequality. The value 1 makes the nonstrict inequality true, so it must be included in the solution set. Since -4 makes the denominator 0, it must be excluded. The solution set is $(-4, 1]$.

Now Try Exercise 73.

CAUTION ***Be careful with the endpoints of the intervals when solving rational inequalities.***

EXAMPLE 9 Solving a Rational Inequality

Solve $\frac{2x-1}{3x+4} < 5$.

SOLUTION

$$\frac{2x-1}{3x+4} - 5 < 0 \quad \text{Subtract 5.}$$

$$\frac{2x-1}{3x+4} - \frac{5(3x+4)}{3x+4} < 0 \quad \text{Common denominator is } 3x+4$$

$$\frac{2x-1-5(3x+4)}{3x+4} < 0 \quad \text{Write as a single fraction.}$$

Be careful with signs.

$$\frac{2x-1-15x-20}{3x+4} < 0 \quad \text{Distributive property (Section R.2)}$$

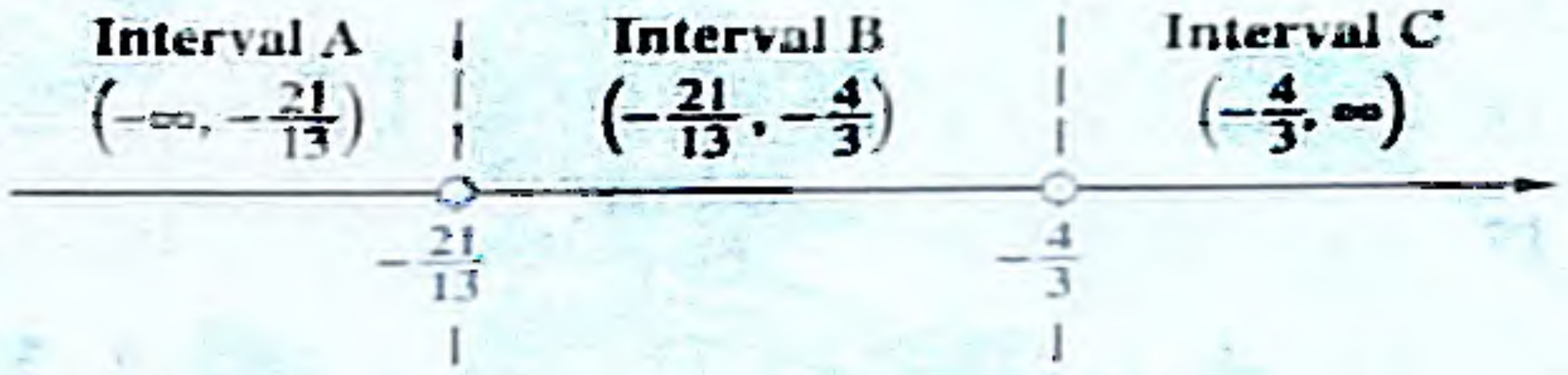
$$\frac{-13x-21}{3x+4} < 0 \quad \text{Combine like terms in the numerator.}$$

Set the numerator and denominator equal to 0 and solve the resulting equations to find the values of x where sign changes may occur.

$$-13x - 21 = 0 \quad \text{or} \quad 3x + 4 = 0$$

$$x = -\frac{21}{13} \quad \text{or} \quad x = -\frac{4}{3}$$

Use these values to form intervals on the number line, as seen in **Figure 18**.



Use an open circle at $-\frac{21}{13}$ because of the strict inequality, and use an open circle at $-\frac{4}{3}$ since it causes the denominator to equal 0.

Figure 18

Now choose test values from the intervals in **Figure 18**. Verify that

-2 from Interval A makes the inequality true.

-1.5 from Interval B makes the inequality false.

0 from Interval C makes the inequality true.

Because of the $<$ symbol, neither endpoint satisfies the inequality, so the solution set is

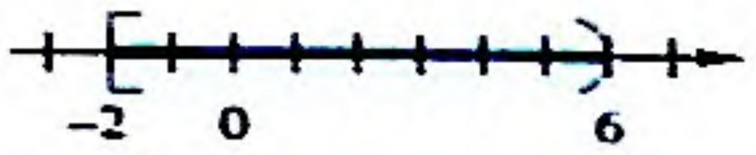
$$\left(-\infty, -\frac{21}{13}\right) \cup \left(-\frac{4}{3}, \infty\right).$$

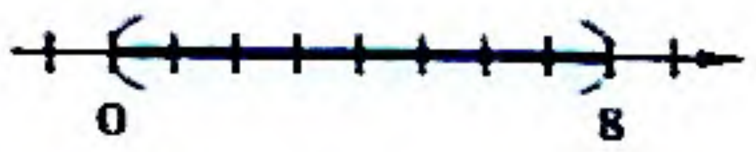
✔ ***Now Try Exercise 85.***

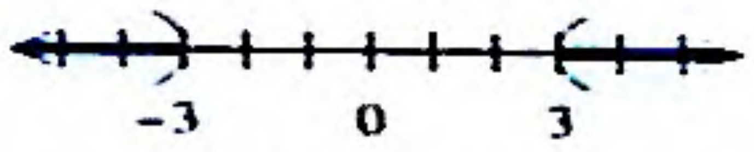
1.7 Exercises

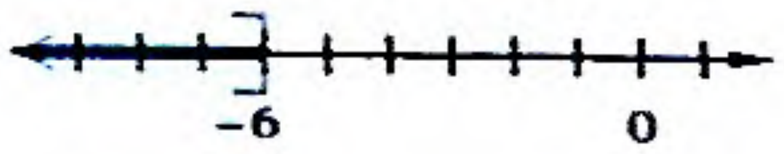
Concept Check *Match the inequality in each exercise in Column I with its equivalent interval notation in Column II.*

I

1. $x < -6$
2. $x \leq 6$
3. $-2 < x \leq 6$
4. $x^2 \geq 0$
5. $x \geq -6$
6. $6 \leq x$
7. 

8. 

9. 

10. 

II

A. $(-2, 6]$

B. $[-2, 6)$

C. $(-\infty, -6]$

D. $[6, \infty)$

E. $(-\infty, -3) \cup (3, \infty)$

F. $(-\infty, -6)$

G. $(0, 8)$

H. $(-\infty, \infty)$

I. $[-6, \infty)$

J. $(-\infty, 6]$

11. Explain how to determine whether to use a parenthesis or a square bracket when graphing the solution set of a linear inequality.

12. *Concept Check* The three-part inequality $a < x < b$ means "a is less than x and x is less than b." Which one of the following inequalities is not satisfied by some real number x?

A. $-3 < x < 10$ **B.** $0 < x < 6$

C. $-3 < x < -1$ **D.** $-8 < x < -10$

Solve each inequality. Write each solution set in interval notation. See Examples 1 and 2.

13. $-2x + 8 \le 16$

14. $-3x - 8 \le 7$

15. $-2x - 2 \le 1 + x$

16. $-4x + 3 \ge -2 + x$

17. $3(x + 5) + 1 \ge 5 + 3x$

18. $6x - (2x + 3) \ge 4x - 5$

19. $8x - 3x + 2 < 2(x + 7)$

20. $2 - 4x + 5(x - 1) < -6(x - 2)$

21. $\frac{4x + 7}{-3} \le 2x + 5$

22. $\frac{2x - 5}{-8} \le 1 - x$

23. $\frac{1}{3}x + \frac{2}{5}x - \frac{1}{2}(x + 3) \le \frac{1}{10}$

24. $-\frac{2}{3}x - \frac{1}{6}x + \frac{2}{3}(x + 1) \le \frac{4}{3}$

Break-Even Interval *Find all intervals where each product will at least break even. See Example 3.*

25. The cost to produce x units of picture frames is $C = 50x + 5000$, while the revenue is $R = 60x$.

26. The cost to produce x units of baseball caps is $C = 100x + 6000$, while the revenue is $R = 500x$.

27. The cost to produce x units of coffee cups is $C = 105x + 900$, while the revenue is $R = 85x$.

28. The cost to produce x units of briefcases is $C = 70x + 500$, while the revenue is $R = 60x$.

Solve each inequality. Write each solution set in interval notation. See Example 4.

29. $-5 < 5 + 2x < 11$

30. $-7 < 2 + 3x < 5$

31. $10 \le 2x + 4 \le 16$

32. $-6 \le 6x + 3 \le 21$

33. $-11 > -3x + 1 > -17$

34. $2 > -6x + 3 > -3$

35. $-4 \le \frac{x + 1}{2} \le 5$

36. $-5 \le \frac{x - 3}{3} \le 1$

37. $-3 \le \frac{x - 4}{-5} < 4$

38. $1 \le \frac{4x - 5}{-2} < 9$

Solve each quadratic inequality. Write each solution set in interval notation. See Examples 5 and 6.

39. $x^2 - x - 6 > 0$

40. $x^2 - 7x + 10 > 0$

41. $2x^2 - 9x \le 18$

42. $3x^2 + x \le 4$

43. $-x^2 - 4x - 6 \le -3$

44. $-x^2 - 6x - 16 > -8$

45. $x(x - 1) \le 6$

46. $x(x + 1) < 12$

47. $x^2 \le 9$

48. $x^2 > 16$

49. $x^2 + 5x + 7 < 0$

50. $4x^2 + 3x + 1 \le 0$

51. $x^2 - 2x \le 1$

52. $x^2 + 4x > -1$

53. *Concept Check* Which one of the following inequalities has solution set $(-\infty, \infty)$?

A. $(x - 3)^2 \ge 0$

B. $(5x - 6)^2 \le 0$

C. $(6x + 4)^2 > 0$

D. $(8x + 7)^2 < 0$

54. *Concept Check* Which one of the inequalities in Exercise 53 has solution set $\varnothing$?

Relating Concepts

For individual or collaborative investigation *(Exercises 55–58)*

Inequalities that involve more than two factors, such as

$$(3x - 4)(x + 2)(x + 6) \le 0,$$

can be solved using an extension of the method shown in ***Examples 5 and 6. Work Exercises 55–58 in order,*** *to see how the method is extended.*

55. Use the zero-factor property to solve $(3x - 4)(x + 2)(x + 6) = 0$.

56. Plot the three solutions in **Exercise 55** on a number line, using closed circles because of the nonstrict inequality, $\le$.

57. The number line from **Exercise 56** should show four intervals formed by the three points. For each interval, choose a number from the interval and decide whether it satisfies the original inequality.

58. On a single number line, graph the intervals that satisfy the inequality, including endpoints. This is the graph of the solution set of the inequality. Write the solution set in interval notation.

Use the technique described in ***Relating Concepts Exercises 55–58*** *to solve each inequality. Write each solution set in interval notation.*

59. $(2x - 3)(x + 2)(x - 3) \ge 0$

60. $(x + 5)(3x - 4)(x + 2) \ge 0$

61. $4x - x^3 \ge 0$

62. $16x - x^3 \ge 0$

63. $(x + 1)^2(x - 3) < 0$

64. $(x - 5)^2(x + 1) < 0$

65. $x^3 + 4x^2 - 9x \ge 36$

66. $x^3 + 3x^2 - 16x \le 48$

67. $x^2(x + 4)^2 \ge 0$

68. $-x^2(2x - 3)^2 \le 0$

Solve each rational inequality. Write each solution set in interval notation. See ***Examples 8 and 9.***

69. $\frac{x - 3}{x + 5} \le 0$

70. $\frac{x + 1}{x - 4} > 0$

71. $\frac{1 - x}{x + 2} < -1$

72. $\frac{6 - x}{x + 2} > 1$

73. $\frac{3}{x - 6} \le 2$

74. $\frac{3}{x - 2} < 1$

75. $\frac{-4}{1 - x} < 5$

76. $\frac{-6}{3x - 5} \le 2$

77. $\frac{10}{3 + 2x} \le 5$

78. $\frac{1}{x + 2} \ge 3$

79. $\frac{7}{x + 2} \ge \frac{1}{x + 2}$

80. $\frac{5}{x + 1} > \frac{12}{x + 1}$

81. $\frac{3}{2x - 1} > \frac{-4}{x}$

82. $\frac{-5}{3x + 2} \ge \frac{5}{x}$

83. $\frac{4}{2 - x} \ge \frac{3}{1 - x}$

84. $\frac{4}{x + 1} < \frac{2}{x + 3}$

85. $\frac{x + 3}{x - 5} \le 1$

86. $\frac{x + 2}{3 + 2x} \le 5$

Solve each rational inequality. Write each solution set in interval notation.

87. $\frac{2x - 3}{x^2 + 1} \ge 0$

88. $\frac{9x - 8}{4x^2 + 25} < 0$

89. $\frac{(5 - 3x)^2}{(2x - 5)^3} > 0$

90. $\frac{(5x - 3)^3}{(25 - 8x)^2} \le 0$

91. $\frac{(2x - 3)(3x + 8)}{(x - 6)^3} \ge 0$

92. $\frac{(9x - 11)(2x + 7)}{(3x - 8)^3} > 0$

(Modeling) Solve each problem. For Exercises 95 and 96, see Example 7.

93. *Box Office Receipts* U.S. movie box office receipts, in billions of dollars, are shown in 5-year increments from 1989 to 2009. (*Source:* www.boxofficemojo.com)

Year	Receipts
1989	5.033
1994	5.396
1999	7.413
2004	9.381
2009	10.595

These receipts R are reasonably approximated by the linear model

$$R = 0.3022x + 4.542,$$

where $x = 0$ corresponds to 1989, $x = 5$ corresponds to 1994, and so on. Using the model, calculate the year in which the receipts first exceed each amount.

(a) \$6.5 billion (b) \$10 billion

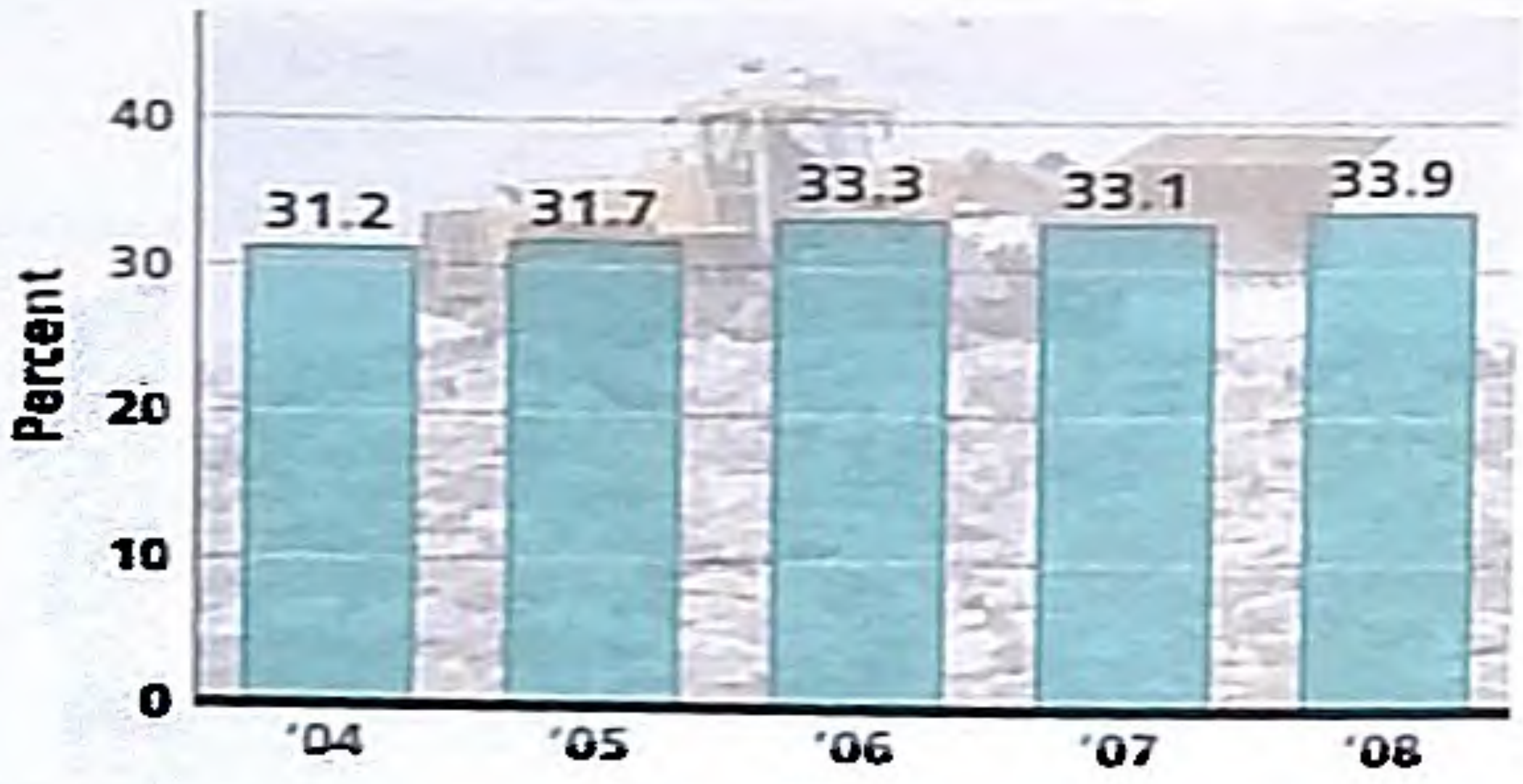
94. *Recovery of Solid Waste* The percent W of municipal solid waste recovered is shown in the bar graph. The linear model

$$W = 0.68x + 30.6,$$

where $x = 1$ represents 2004, $x = 2$ represents 2005, and so on, fits the data reasonably well.

(a) Based on this model, when did the percent of waste recovered first exceed 32%?
(b) In what years was it between 33% and 34%?

Municipal Solid Waste Recovered



Source: U.S. Environmental Protection Agency

95. *Height of a Projectile* A projectile is fired straight up from ground level. After t seconds, its height above the ground is s feet, where

$$s = -16t^2 + 220t.$$

For what time period is the projectile at least 624 ft above the ground?

96. *Height of a Baseball* A baseball is hit so that its height, s, in feet after t seconds is

$$s = -16t^2 + 44t + 4.$$

For what time period is the ball at least 32 ft above the ground?

97. *Velocity of an Object* Suppose the velocity, v, of an object is given by

$$v = 2t^2 - 5t - 12,$$

where t is time in seconds. (Here t can be positive or negative.) Find the intervals where the velocity is negative.

98. *Velocity of an Object* The velocity of an object, v, after t seconds is given by

$$v = 3t^2 - 18t + 24.$$

Find the interval where the velocity is negative.

99. A student attempted to solve the inequality

$$\frac{2x-3}{x+2} \leq 0$$

by multiplying each side by $x + 2$ to get

$$2x - 3 \leq 0$$

$$x \leq \frac{3}{2}.$$

He wrote the solution set as $\left(-\infty, \frac{3}{2}\right]$. Is his solution correct? Explain.

100. A student solved the inequality $x^2 \leq 144$ by taking the square root of each side to get $x \leq 12$. She wrote the solution set as $(-\infty, 12]$. Is her solution correct? Explain.

1.8 Absolute Value Equations and Inequalities

- Basic Concepts
- Absolute Value Equations
- Absolute Value Inequalities
- Special Cases
- Absolute Value Models for Distance and Tolerance

Basic Concepts Recall from Section R.2 that the absolute value of a number a, written $|a|$, gives the distance from a to 0 on a number line. By this definition, the equation $|x| = 3$ can be solved by finding all real numbers at a distance of 3 units from 0. As shown in **Figure 19**, two numbers satisfy this equation, 3 and −3, so the solution set is $\{-3, 3\}$.

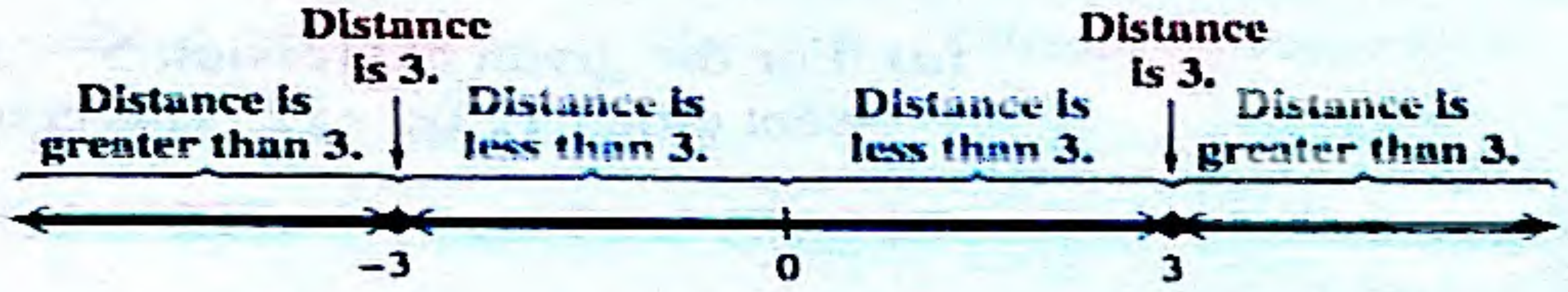


Figure 19

Similarly, $|x| < 3$ is satisfied by all real numbers whose distances from 0 are less than 3—that is, the interval

$$-3 < x < 3, \quad \text{or} \quad (-3, 3).$$

See **Figure 19**. Finally, $|x| > 3$ is satisfied by all real numbers whose distances from 0 are greater than 3. These numbers are less than −3 or greater than 3, so the solution set is

$$(-\infty, -3) \cup (3, \infty).$$

Notice in **Figure 19** that the union of the solution sets of $|x| = 3$, $|x| < 3$, and $|x| > 3$ is the set of real numbers.

These observations support the cases for solving absolute value equations and inequalities summarized in the table on the next page. If the equation or inequality fits the form of Case 1, 2, or 3, change it to its equivalent form and solve. The solution set and its graph will look similar to those shown.

LOOKING AHEAD TO CALCULUS

The precise definition of a limit in calculus requires writing absolute value inequalities.

A standard problem in calculus is to find the "interval of convergence" of something called a **power series**, by solving an inequality of the form

$$|x - a| < r.$$

This inequality says that x can be any number within r units of a on the number line, so its solution set is indeed an interval—namely the interval $(a - r, a + r)$

Special Cases The three cases given in this section require the constant k to be positive. *When $k \leq 0$, use the fact that the absolute value of any expression must be nonnegative, and consider the truth of the statement.*

EXAMPLE 4 Solving Special Cases

Solve each equation or inequality.

(a) $|2 - 5x| \geq -4$ **(b)** $|4x - 7| < -3$ **(c)** $|5x + 15| = 0$

SOLUTION

(a) Since the absolute value of a number is always nonnegative, the inequality

$$|2 - 5x| \geq -4 \text{ is always true.}$$

The solution set includes all real numbers, written $(-\infty, \infty)$.

(b) There is no number whose absolute value is less than -3 (or less than *any* negative number).

$$\text{The solution set of } |4x - 7| < -3 \text{ is } \emptyset.$$

(c) The absolute value of a number will be 0 only if that number is 0. Therefore, $|5x + 15| = 0$ is equivalent to

$$5x + 15 = 0, \quad \text{which has solution set } \{-3\}.$$

CHECK Substitute -3 into the original equation.

$$|5x + 15| = 0 \quad \text{Original equation}$$

$$|5(-3) + 15| \stackrel{?}{=} 0 \quad \text{Let } x = -3.$$

$$0 = 0 \ ✓ \quad \text{True}$$

✓ *Now Try Exercises 55, 57, and 59.*

Absolute Value Models for Distance and Tolerance If a and b represent two real numbers, then the absolute value of their difference,

$$\text{either } |a - b| \text{ or } |b - a|, \quad \text{(Section R.2)}$$

represents the distance between them.

EXAMPLE 5 Using Absolute Value Inequalities to Describe Distances

Write each statement using an absolute value inequality.

(a) k is no less than 5 units from 8. **(b)** n is within 0.001 unit of 6.

SOLUTION

(a) Since the distance from k to 8, written $|k - 8|$ or $|8 - k|$, is no less than 5, the distance is greater than or equal to 5. This can be written as

$$|k - 8| \geq 5, \quad \text{or, equivalently,} \quad |8 - k| \geq 5. \quad \text{Either form is acceptable}$$

(b) This statement indicates that the distance between n and 6 is less than 0.001.

$$|n - 6| < 0.001, \quad \text{or, equivalently,} \quad |6 - n| < 0.001$$

✓ *Now Try Exercises 83 and 85.*

EXAMPLE 6 Using Absolute Value to Model Tolerance

In quality control and other applications, we often wish to keep the difference between two quantities within some predetermined amount, called the tolerance. Suppose $y = 2x + 1$ and we want y to be within 0.01 unit of 4. For what values of x will this be true?

SOLUTION

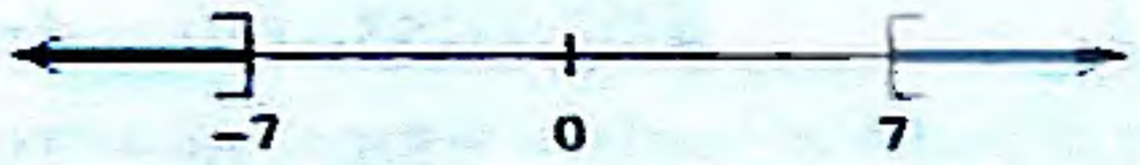
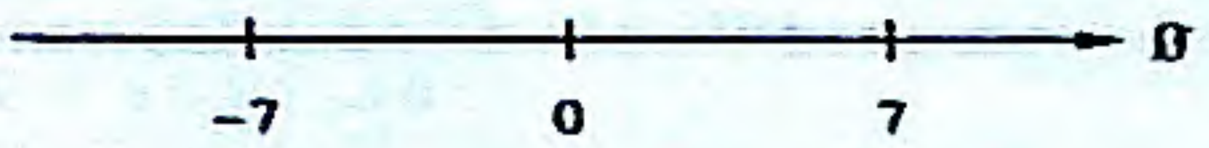
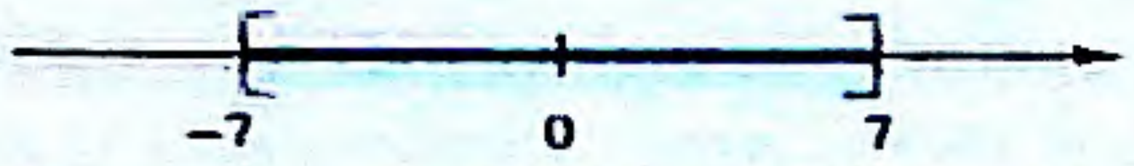
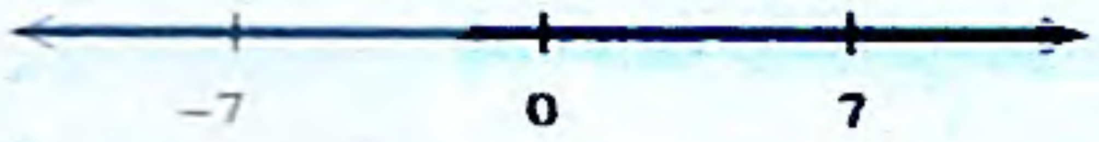
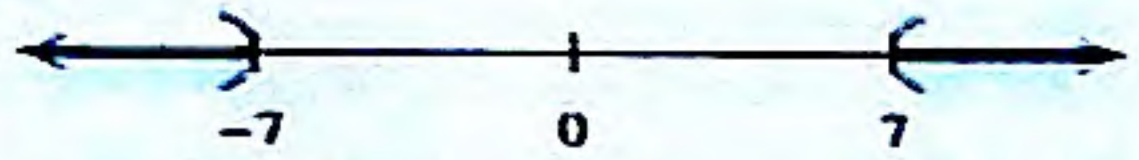
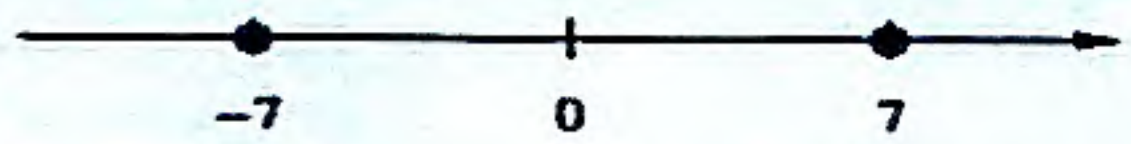
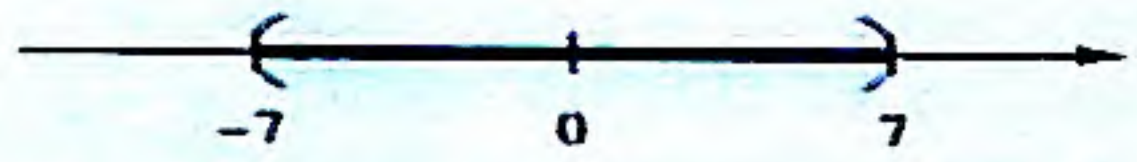
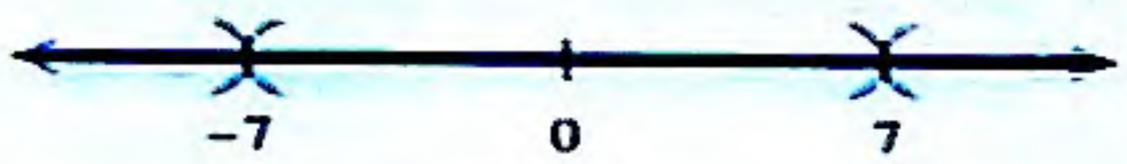
$$\begin{aligned} |y - 4| &< 0.01 && \text{Write an absolute value inequality.} \\ |2x + 1 - 4| &< 0.01 && \text{Substitute } 2x + 1 \text{ for } y. \\ |2x - 3| &< 0.01 && \text{Combine like terms.} \\ -0.01 < 2x - 3 &< 0.01 && \text{Case 2} \\ 2.99 < 2x &< 3.01 && \text{Add 3 to each part.} \\ 1.495 < x &< 1.505 && \text{Divide each part by 2.} \end{aligned}$$

Reversing these steps shows that keeping x in the interval $(1.495, 1.505)$ ensures that the difference between y and 4 is within 0.01 unit.

Now Try Exercise 89.

1.8 Exercises

Concept Check *Match each equation or inequality in Column I with the graph of its solution set in Column II.*

I	II
1. $\|x\| = 7$	A. 
2. $\|x\| = -7$	B. 
3. $\|x\| > -7$	C. 
4. $\|x\| > 7$	D. 
5. $\|x\| < 7$	E. 
6. $\|x\| \geq 7$	F. 
7. $\|x\| \leq 7$	G. 
8. $\|x\| \neq 7$	H. 

Solve each equation. See Example 1.

9. $|3x - 1| = 2$

10. $|4x + 2| = 5$

11. $|5 - 3x| = 3$

12. $|7 - 3x| = 3$

13. $\left|\frac{x - 4}{2}\right| = 5$

14. $\left|\frac{x + 2}{2}\right| = 7$

15. $\left|\frac{5}{x - 3}\right| = 10$

16. $\left|\frac{3}{2x - 1}\right| = 4$

17. $\left|\frac{6x + 1}{x - 1}\right| = 3$

18. $\left|\frac{2x + 3}{3x - 4}\right| = 1$

19. $|2x - 3| = |5x + 4|$

20. $|x + 1| = |1 - 3x|$

21. $|4 - 3x| = |2 - 3x|$

22. $|3 - 2x| = |5 - 2x|$

23. $|5x - 2| = |2 - 5x|$

24. The equation $|5x - 6| = 3x$ cannot have a negative solution. Why?

25. The equation $|7x + 3| = -5x$ cannot have a positive solution. Why?

26. *Concept Check* Determine the solution set of each equation by inspection.

(a) $-|x| = |x|$ (b) $|-x| = |x|$ (c) $|x^2| = |x|$ (d) $-|x| = 9$

Solve each inequality. Give the solution set using interval notation. See Example 2.

27. $|2x + 5| < 3$
28. $|3x - 4| < 2$
29. $|2x + 5| \geq 3$
30. $|3x - 4| \geq 2$
31. $\left|\frac{1}{2} - x\right| < 2$
32. $\left|\frac{3}{5} + x\right| < 1$
33. $4|x - 3| > 12$
34. $5|x + 1| > 10$
35. $|5 - 3x| > 7$
36. $|7 - 3x| > 4$
37. $|5 - 3x| \leq 7$
38. $|7 - 3x| \leq 4$
39. $\left|\frac{2}{3}x + \frac{1}{2}\right| \leq \frac{1}{6}$
40. $\left|\frac{5}{3} - \frac{1}{2}x\right| > \frac{2}{9}$
41. $|0.01x + 1| < 0.01$

42. Explain why the equation $|x| = \sqrt{x^2}$ has infinitely many solutions.

Solve each equation or inequality. See Examples 3 and 4.

43. $|4x + 3| - 2 = -1$
44. $|8 - 3x| - 3 = -2$
45. $|6 - 2x| + 1 = 3$
46. $|4 - 4x| + 2 = 4$
47. $|3x + 1| - 1 < 2$
48. $|5x + 2| - 2 < 3$
49. $\left|5x + \frac{1}{2}\right| - 2 < 5$
50. $\left|2x + \frac{1}{3}\right| + 1 < 4$
51. $|10 - 4x| + 1 \geq 5$
52. $|12 - 6x| + 3 \geq 9$
53. $|3x - 7| + 1 < -2$
54. $|-5x + 7| - 4 < -6$

Solve each equation or inequality. See Example 4.

55. $|10 - 4x| \geq -4$
56. $|12 - 9x| \geq -12$
57. $|6 - 3x| < -11$
58. $|18 - 3x| < -13$
59. $|8x + 5| = 0$
60. $|7 + 2x| = 0$
61. $|4.3x + 9.8| < 0$
62. $|1.5x - 14| < 0$
63. $|2x + 1| \leq 0$
64. $|3x + 2| \leq 0$
65. $|3x + 2| > 0$
66. $|4x + 3| > 0$

Relating Concepts

For individual or collaborative investigation *(Exercises 67–70)*

To see how to solve an equation that involves ***the absolute value of a quadratic*** *polynomial, such as* $|x^2 - x| = 6$, *work* ***Exercises 67–70 in order.***

67. For $x^2 - x$ to have an absolute value equal to 6, what are the two possible values that it may be? (*Hint:* One is positive and the other is negative.)

68. Write an equation stating that $x^2 - x$ is equal to the positive value you found in Exercise 67, and solve it using factoring.

69. Write an equation stating that $x^2 - x$ is equal to the negative value you found in Exercise 67, and solve it using the quadratic formula. (*Hint:* The solutions are not real numbers.)

70. Give the complete solution set of $|x^2 - x| = 6$, using the results from Exercises 68 and 69.

Use the method described in Relating Concepts Exercises 67–70, if applicable, and properties of absolute value to solve each equation or inequality. (Hint: Exercises 77 and 78 can be solved by inspection.)

71. $|3x^2 + x| = 14$
72. $|2x^2 - 3x| = 5$
73. $|4x^2 - 23x - 6| = 0$

74. $|6x^3 + 23x^2 + 7x| = 0$ **75.** $|x^2 + 1| - |2x| = 0$ **76.** $\left|\frac{x^2 + 2}{x}\right| - \frac{11}{3} = 0$

77. $|x^4 + 2x^2 + 1| < 0$ **78.** $|x^2 + 10| < 0$

79. $\left|\frac{x - 4}{3x + 1}\right| \geq 0$ **80.** $\left|\frac{9 - x}{7 + 8x}\right| \geq 0$

81. ***Concept Check*** Write an equation involving absolute value that says the distance between p and q is 2 units.

82. ***Concept Check*** Write an equation involving absolute value that says the distance between r and s is 6 units.

Write each statement as an absolute value equation or inequality. See Example 5.

83. m is no more than 2 units from 7. **84.** z is no less than 5 units from 4.

85. p is within 0.0001 unit of 9. **86.** k is within 0.0002 unit of 10.

87. r is no less than 1 unit from 29. **88.** q is no more than 8 units from 22.

89. ***Tolerance*** Suppose that $y = 5x + 1$ and we want y to be within 0.002 unit of 6. For what values of x will this be true?

90. ***Tolerance*** Repeat Exercise 89, but let $y = 10x + 2$.

(Modeling) Solve each problem. See Example 6.

91. ***Weights of Babies*** Dr. Tydings has found that, over the years, 95% of the babies he has delivered weighed x pounds, where

$$|x - 8.2| \leq 1.5.$$

What range of weights corresponds to this inequality?

92. ***Temperatures on Mars*** The temperatures on the surface of Mars in degrees Celsius approximately satisfy the inequality $|C + 84| \leq 56$. What range of temperatures corresponds to this inequality?

93. ***Conversion of Methanol to Gasoline*** The industrial process that is used to convert methanol to gasoline is carried out at a temperature range of 680°F to 780°F. Using F as the variable, write an absolute value inequality that corresponds to this range.

94. ***Wind Power Extraction Tests*** When a model kite was flown in crosswinds in tests to determine its limits of power extraction, it attained speeds of 98 to 148 ft per sec in winds of 16 to 26 ft per sec. Using x as the variable in each case, write absolute value inequalities that correspond to these ranges.

(Modeling) Carbon Dioxide Emissions When humans breathe, carbon dioxide is emitted. In one study, the emission rates of carbon dioxide by college students were measured during both lectures and exams. The average individual rate R_L (in grams per hour) during a lecture class satisfied the inequality

$$|R_L - 26.75| \leq 1.42,$$

whereas during an exam the rate R_E satisfied the inequality

$$|R_E - 38.75| \leq 2.17.$$

(*Source:* Wang, T. C., *ASHRAE Trans.*, 81 (Part 1), 32.)

Use this information in Exercises 95–96.

95. Find the range of values for R_L and R_E.

96. The class had 225 students. If T_L and T_E represent the total amounts of carbon dioxide in grams emitted during a one-hour lecture and exam, respectively, write inequalities that model the ranges for T_L and T_E.

Concepts	Examples
1.8 Absolute Value Equations and Inequalities	
Solving Absolute Value Equations and Inequalities For each equation or inequality in Cases 1-3, assume that $k > 0$.	
Case 1: To solve $\|x\| = k$, use the equivalent form $x = k \quad \text{or} \quad x = -k.$	Solve. $\|5x - 2\| = 3$ $5x - 2 = 3 \quad \text{or} \quad 5x - 2 = -3$ $5x = 5 \quad \text{or} \quad 5x = -1$ $x = 1 \quad \text{or} \quad x = -\frac{1}{5}$ Solution set: $\{-\frac{1}{5}, 1\}$
Case 2: To solve $\|x\| < k$, use the equivalent form $-k < x < k.$	**Solve.** $\|5x - 2\| < 3$ $-3 < 5x - 2 < 3$ $-1 < 5x < 5$ $-\frac{1}{5} < x < 1$ **Solution set:** $(-\frac{1}{5}, 1)$
Case 3: To solve $\|x\| > k$, use the equivalent form $x < -k \quad \text{or} \quad x > k.$	**Solve.** $\|5x - 2\| \geq 3$ $5x - 2 \leq -3 \quad \text{or} \quad 5x - 2 \geq 3$ $5x \leq -1 \quad \text{or} \quad 5x \geq 5$ $x \leq -\frac{1}{5} \quad \text{or} \quad x \geq 1$ **Solution set:** $(-\infty, -\frac{1}{5}] \cup [1, \infty)$

Chapter 1 Review Exercises

Solve each equation.

1. $2x + 8 = 3x + 2$

2. $\frac{1}{6}x - \frac{1}{12}(x - 1) = \frac{1}{2}$

3. $5x - 2(x + 4) = 3(2x + 1)$

4. $9x - 11(k + p) = x(a - 1)$, for x

5. $A = \frac{24f}{B(p + 1)}$, for f (approximate annual interest rate)

6. *Concept Check* Which of the following cannot be a correct equation to solve a geometry problem, if x represents the measure of a side of a rectangle? (*Hint:* Solve the equations and consider the solutions.)

A. $2x + 2(x + 2) = 20$

B. $2x + 2(5 + x) = -2$

C. $8(x + 2) + 4x = 16$

D. $2x + 2(x - 3) = 10$

7. *Concept Check* If x represents the number of pennies in a jar in an applied problem, which of the following equations cannot be a correct equation for finding x? (*Hint:* Solve the equations and consider the solutions.)

A. $5x + 3 = 11$
B. $12x + 6 = -4$
C. $100x = 50(x + 3)$
D. $6(x + 4) = x + 24$

8. *Airline Carry-On Baggage Size* Carry-on rules for domestic economy-class travel differ from one airline to another, as shown in the table.

Airline	Size (linear inches)
AirTran	55
Alaska/Horizon	51
American	45
Delta	45
Southwest	50
United	45
USAirways	45

Source: Individual airline websites.

To determine the number of linear inches for a carry-on, add the length, width, and height of the bag.

(a) One Samsonite rolling bag measures 9 in. by 12 in. by 21 in. Are there any airlines that would not allow it as a carry-on?
(b) A Lark wheeled bag measures 10 in. by 14 in. by 22 in. On which airlines does it qualify as a carry-on?

Solve each problem.

9. *Dimensions of a Square* If the length of each side of a square is decreased by 4 in., the perimeter of the new square is 10 in. more than half the perimeter of the original square. What are the dimensions of the original square?

10. *Distance from a Library* Becky Anderson can ride her bike to the university library in 20 min. The trip home, which is all uphill, takes her 30 min. If her rate is 8 mph faster on her trip there than her trip home, how far does she live from the library?

11. *Alcohol Mixture* Alan wishes to strengthen a mixture that is 10% alcohol to one that is 30% alcohol. How much pure alcohol should he add to 12 L of the 10% mixture?

12. *Loan Interest Rates* A realtor borrowed $90,000 to develop some property. He was able to borrow part of the money at 5.5% interest and the rest at 6%. The annual interest on the two loans amounts to $5125. How much was borrowed at each rate?

13. *Speed of an Excursion Boat* An excursion boat travels upriver to a landing and then returns to its starting point. The trip upriver takes 1.2 hr, and the trip back takes 0.9 hr. If the average speed on the return trip is 5 mph faster than on the trip upriver, what is the boat's speed upriver?

14. *Toxic Waste* Two chemical plants are releasing toxic waste into a holding tank. Plant I releases waste twice as fast as Plant II. Together they fill the tank in 3 hr. How long would it take the slower plant to fill the tank working alone?

15. *(Modeling) Lead Intake* As directed by the "Safe Drinking Water Act" of December 1974, the EPA proposed a maximum lead level in public drinking water of 0.05 mg per liter. This standard assumed an individual consumption of two liters of water per day.

(a) If EPA guidelines are followed, write an equation that models the maximum amount of lead A ingested in x years. Assume that there are 365.25 days in a year.
(b) If the average life expectancy is 72 yr, find the EPA maximum lead intake from water over a lifetime.

16. *(Modeling) Online Retail Sales* Projected e-commerce sales (in billions of dollars) for the years 2007–2012 can be modeled by the equation

$$y = 32.13x + 172.59,$$

where $x = 0$ corresponds to 2007, $x = 1$ corresponds to 2008, and so on. Based on this model, what would you expect the retail e-commerce sales to be in 2012? (*Source:* Forrester Research, Inc.)

17. ***(Modeling) Minimum Wage*** Some values of the U.S. minimum hourly wage, in dollars, for selected years from 1956 to 2009 are shown in the table. The linear model

$$y = 0.1132x + 0.4609$$

approximates the minimum wage during this time period, where x is the number of years after 1956 and y is the minimum wage in dollars.

Year	Minimum Wage	Year	Minimum Wage
1956	1.00	1996	4.75
1963	1.25	1997	5.15
1975	2.10	2007	5.85
1981	3.35	2008	6.55
1990	3.80	2009	7.25

Source: Bureau of Labor Statistics.

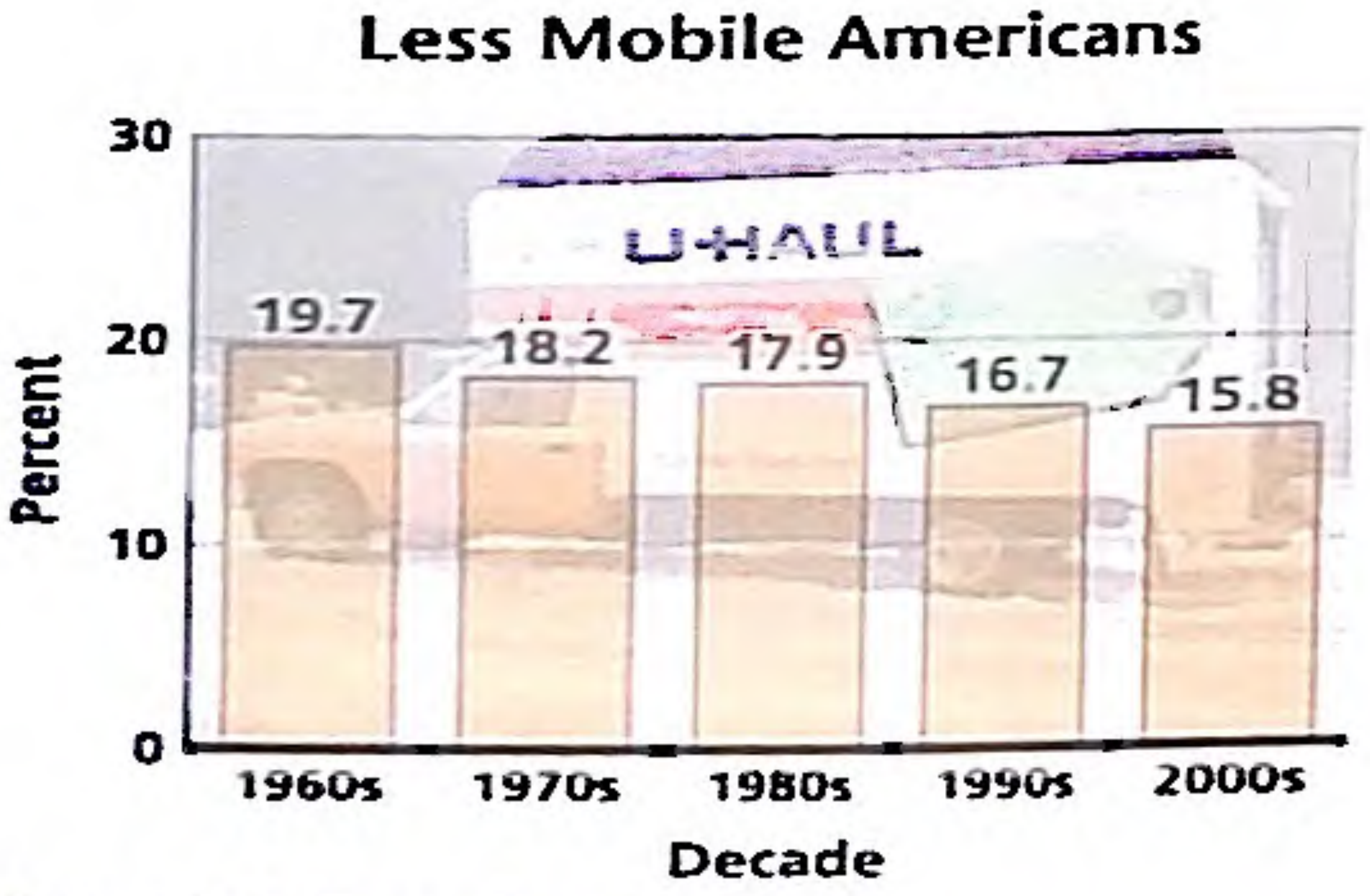
(a) Use the model to approximate the minimum wage in 1990. How does it compare to the data in the table?

(b) Use the model to approximate the year in which the minimum wage was $5.85 How does your answer compare to the data in the table?

18. ***(Modeling) Mobility of the U.S. Population*** The U.S. population, in millions, is given in the table. The figures are for the midyear of each decade from the 1960s through the 2000s. If we let the midyear shown in the table represent each decade, we can use the population figures of people moving with the data given in the graph

Year	Population
1965	194
1975	216
1985	238
1995	263
2005	296

Source: U.S. Census Bureau.



Source: U.S. Census Bureau

(a) Find the number of Americans, to the nearest tenth of a million, moving in each year given in the table.

(b) The percents given in the graph decrease each decade, while the populations given in the table increase each decade. From your answers to part (a), is the number of Americans moving each decade increasing or decreasing? Explain.

Perform each operation and write answers in standard form.

19. $(6 - i) + (7 - 2i)$

20. $(-11 + 2i) - (8 - 7i)$

21. $15i - (3 + 2i) - 11$

22. $-6 + 4i - (8i - 2)$

23. $(5 - i)(3 + 4i)$

24. $(-8 + 2i)(-1 + i)$

25. $(5 - 11i)(5 + 11i)$

26. $(4 - 3i)^2$

27. $-5i(3 - i)^2$

28. $4i(2 + 5i)(2 - i)$

29. $\dfrac{-12 - i}{-2 - 5i}$

30. $\dfrac{-7 + i}{-1 - i}$

Find each power of i.

31. i^{11}

32. i^{60}

33. i^{1001}

34. i^{110}

35. i^{-27}

36. $\dfrac{1}{i^{17}}$

Solve each equation.

37. $(x + 7)^2 = 5$

38. $(2 - 3x)^2 = 8$

39. $2x^2 + x - 15 = 0$

40. $12x^2 = 8x - 1$

41. $-2x^2 + 11x = -21$

42. $-x(3x + 2) = 5$

43. $(2x + 1)(x - 4) = x$

44. $\sqrt{2}x^2 - 4x + \sqrt{2} = 0$

45. $x^2 - \sqrt{5}x - 1 = 0$

46. $(x + 4)(x + 2) = 2x$

47. *Concept Check* Which one of the following equations has two real, distinct solutions? Do not actually solve.

A. $(3x - 4)^2 = -9$

B. $(4 - 7x)^2 = 0$

C. $(5x - 9)(5x - 9) = 0$

D. $(7x + 4)^2 = 11$

48. *Concept Check* Which equations in **Exercise 47** have only one distinct, real solution?

49. *Concept Check* Which one of the equations in **Exercise 47** has two nonreal complex solutions?

Evaluate the discriminant for each equation, and then use it to predict the number and type of solutions.

50. $8x^2 = -2x - 6$

51. $-6x^2 + 2x = -3$

52. $16x^2 + 3 = -26x$

53. $-8x^2 + 10x = 7$

54. $25x^2 + 110x + 121 = 0$

55. $x(9x + 6) = -1$

56. Explain how the discriminant of $ax^2 + bx + c = 0$ $(a \neq 0)$ is used to determine the number and type of solutions.

Solve each problem.

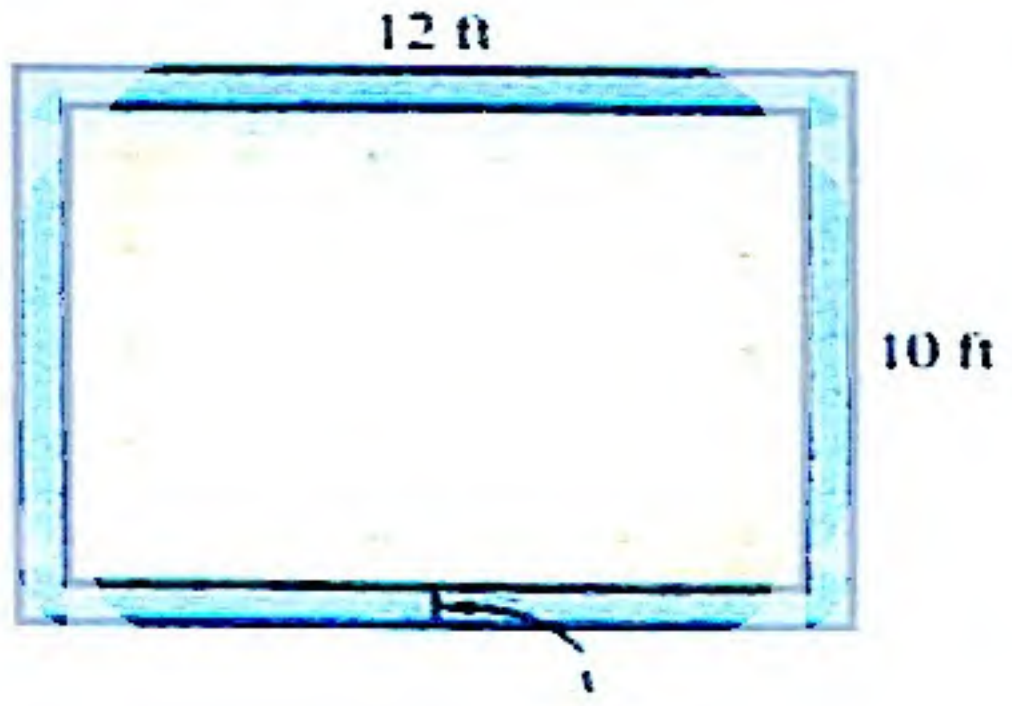
57. *(Modeling) Height of a Projectile* A projectile is fired straight up from ground level. After t seconds its height s, in feet above the ground, is given by

$$s = 220t - 16t^2.$$

At what times is the projectile exactly 750 ft above the ground?

58. *Dimensions of a Picture Frame* Zach Levy went into a frame-it-yourself shop. He wanted a frame 3 in. longer than it was wide. The frame he chose extended 1.5 in. beyond the picture on each side. Find the outside dimensions of the frame if the area of the unframed picture is 70 in.^{2}.

59. *Kitchen Flooring* Paula Story plans to replace the vinyl floor covering in her 10-ft by 12-ft kitchen. She wants to have a border of even width of a special material. She can afford only 21 ft^2 of this material. How wide a border can she have?



60. ***(Modeling) Airplane Landing Speed*** To determine the appropriate landing speed of a small airplane, the formula

$$D = 0.1s^2 - 3s + 22$$

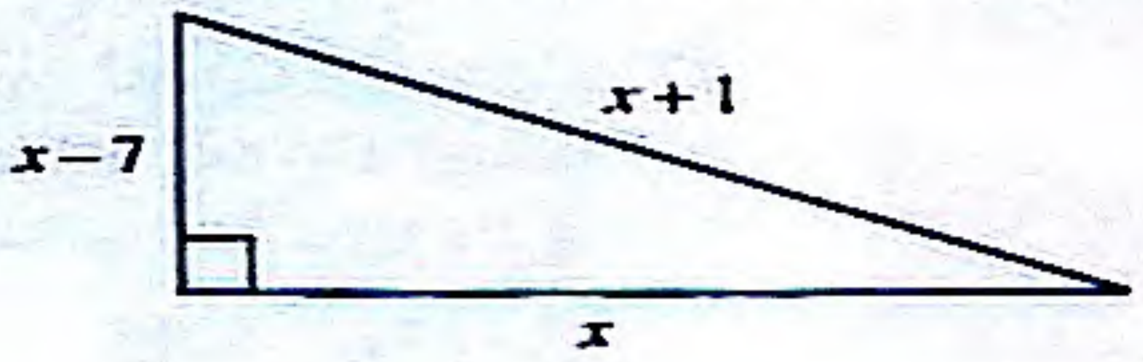
is used, where s is the initial landing speed in feet per second and D is the length of the runway in feet. If the landing speed is too fast, the pilot may run out of runway. If the speed is too slow, the plane may stall. If the runway is 800 ft long, what is the appropriate landing speed? Round to the nearest tenth.

61. ***(Modeling) U.S. Government Spending on Medical Care*** The amount spent in billions of dollars by the U.S. government on medical care during the period 1990–2010 can be approximated by the equation

$$y = 1.717x^2 + 0.8179x + 167.3,$$

where $x = 0$ corresponds to 1990, $x = 1$ corresponds to 1991, and so on. According to this model, about how much was spent by the U.S. government on medical care in 2005? Round to the nearest tenth of a billion. (*Source*: *U.S. Office of Management and Budget.*)

62. ***Dimensions of a Right Triangle*** The lengths of the sides of a right triangle are such that the shortest side is 7 in. shorter than the middle side, while the longest side (the hypotenuse) is 1 in. longer than the middle side. Find the lengths of the sides.



Solve each equation.

63. $4x^4 + 3x^2 - 1 = 0$

64. $x^2 - 2x^4 = 0$

65. $\frac{2}{x} - \frac{4}{3x} = 8 + \frac{3}{x}$

66. $2 - \frac{5}{x} = \frac{3}{x^2}$

67. $\frac{10}{4x - 4} = \frac{1}{1 - x}$

68. $\frac{13}{x^2 + 10} = \frac{2}{x}$

69. $\frac{x}{x + 2} + \frac{1}{x} + 3 = \frac{2}{x^2 + 2x}$

70. $\frac{2}{x + 2} + \frac{1}{x + 4} = \frac{4}{x^2 + 6x + 8}$

71. $(2x + 3)^{2/3} + (2x + 3)^{1/3} - 6 = 0$

72. $(x + 3)^{-2/3} - 2(x + 3)^{-1/3} = 3$

73. $\sqrt{4x - 2} = \sqrt{3x + 1}$

74. $\sqrt{2x + 3} = x + 2$

75. $\sqrt{x + 2} - x = 2$

76. $\sqrt{x} - \sqrt{x + 3} = -1$

77. $\sqrt{x + 3} - \sqrt{3x + 10} = 1$

78. $\sqrt{5x - 15} - \sqrt{x + 1} = 2$

79. $\sqrt{x^2 + 3x} - 2 = 0$

80. $\sqrt[5]{2x} = \sqrt[5]{3x + 2}$

81. $\sqrt[3]{6x + 2} - \sqrt[3]{4x} = 0$

82. $(x - 2)^{2/3} = x^{1/3}$

Solve each inequality. Write each solution set using interval notation.

83. $-9x + 3 < 4x + 10$

84. $11x \geq 2(x - 4)$

85. $-5x - 4 \geq 3(2x - 5)$

86. $7x - 2(x - 3) \leq 5(2 - x)$

87. $5 \leq 2x - 3 \leq 7$

88. $-8 > 3x - 5 > -12$

89. $x^2 + 3x - 4 \leq 0$

90. $x^2 + 4x - 21 > 0$

91. $6x^2 - 11x < 10$

92. $x^2 - 3x \geq 5$

93. $x^3 - 16x \leq 0$

94. $2x^3 - 3x^2 - 5x < 0$

95. $\frac{3x + 6}{x - 5} > 0$

96. $\frac{x + 7}{2x + 1} - 1 \leq 0$

97. $\dfrac{3x-2}{x} - 4 > 0$

98. $\dfrac{5x+2}{x} < -1$

99. $\dfrac{3}{x-1} \le \dfrac{5}{x+3}$

100. $\dfrac{3}{x+2} > \dfrac{2}{x-4}$

(Modeling) Solve each problem.

101. ***Ozone Concentration*** Guideline levels for indoor ozone are less than 50 parts per billion (ppb). In a scientific study, a Purafil air filter was used to reduce an initial ozone concentration of 140 ppb. The filter removed 43% of the ozone. (*Source:* Parmar and Grosjean, *Removal of Air Pollutants from Museum Display Cases*, Getty Conservation Institute, Marina del Rey, CA.)

(a) What is the ozone concentration after the Purafil air filter is used?

(b) What is the maximum initial concentration of ozone that this filter will reduce to an acceptable level?

102. ***Break-Even Interval*** A company produces earbuds. The revenue from the sale of x units of these earbuds is $R = 8x$. The cost to produce x units of earbuds is $C = 3x + 1500$. In what interval will the company at least break even?

103. ***Height of a Projectile*** A projectile is launched upward from the ground. Its height s in feet above the ground after t seconds is given by

$$s = 320t - 16t^2.$$

(a) After how many seconds in the air will it hit the ground?

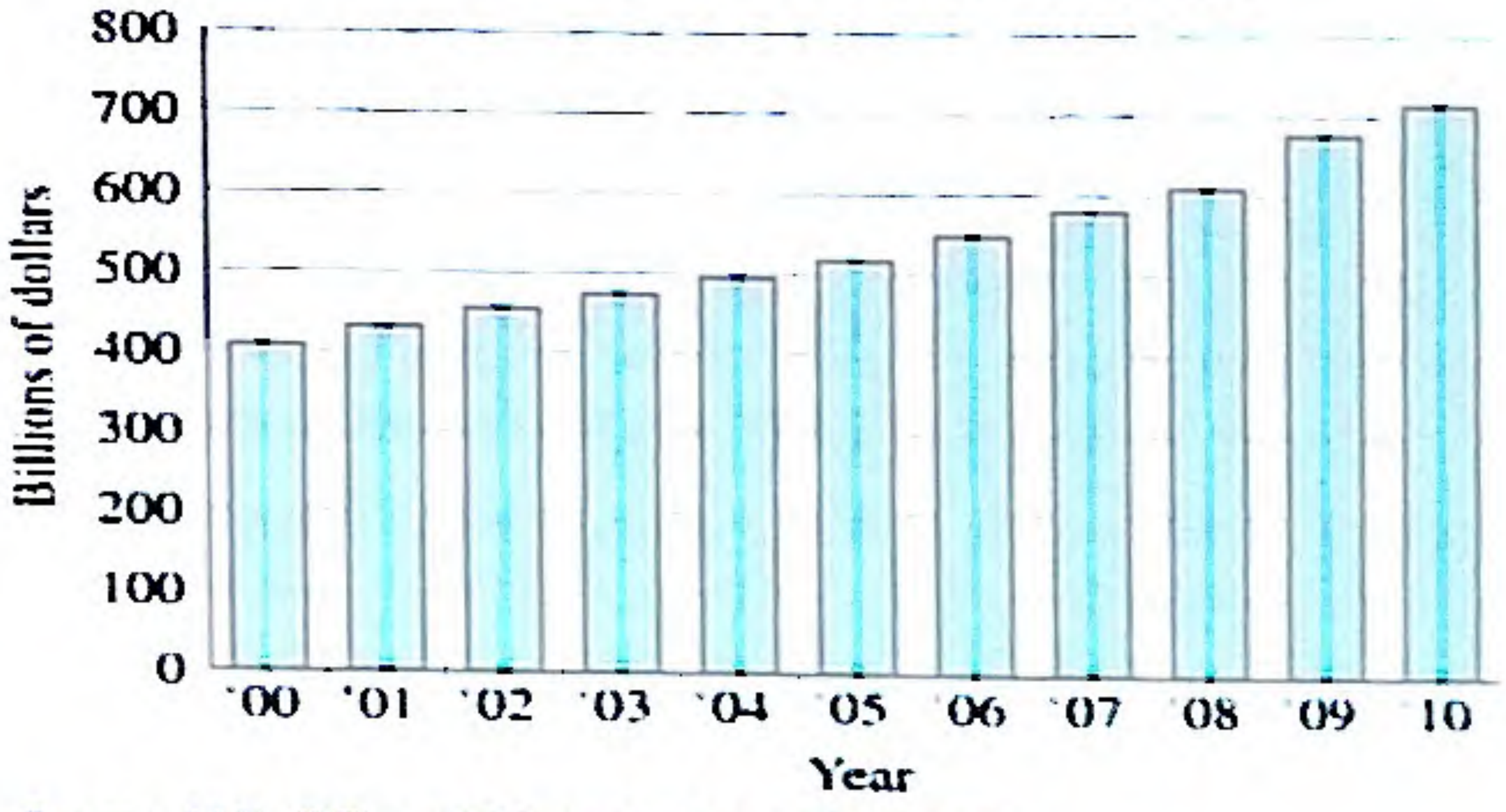
(b) During what time interval is the projectile more than 576 ft above the ground?

104. ***Social Security*** The total amount paid by the U.S. government to individuals for Social Security retirement and disability insurance benefits during the period 2000–2010 can be approximated by the linear model

$$y = 29.4x + 391.2,$$

where $x = 0$ corresponds to 2000, $x = 1$ corresponds to 2001, and so on. The variable y is in billions of dollars. Based on this model, during what year did the amount paid by the government first exceed $600 billion? Round your answer to the nearest year. Compare your answer to the bar graph.

Social Security Retirement and Disability Insurance Benefits



Source: U.S. Office of Management and Budget

105. Without actually solving the inequality, explain why 3 cannot be in the solution set of $\frac{14x+9}{x-3} < 0$

106. Without actually solving the inequality, explain why -4 must be in the solution set of $\frac{x+4}{2x+1} \ge 0$

Rewrite the numerical part of each statement as an inequality using the indicated variable.

107. ***Teacher Retirement*** A teacher in the Public School Retirement System of Missouri can retire at any age with at least 30 yr of credit. (*Source:* Public School Retirement System of Missouri.) Let c represent the number of years of credit that a retiree has earned.

108. ***Honor Roll*** To make the honor list for the 2010 fall semester, students at Missouri University of Science and Technology had to carry a minimum of 12 credit hours and have a grade point average of 3.2 or above. (*Source: Warren County Record.*) Let h represent the number of credit hours, and let g represent the grade point average of a student on the honor list.

109. ***Blood Donation*** Severe winter weather throughout much of the eastern half of the United States caused the cancellation of more than 14,000 blood and platelet donations through the American Red Cross. (*Source: O'Fallon Community News.*) Let x represent the number of donation cancellations.

110. ***Blood Donation*** Blood is perishable and has no substitute. Red blood cells have a shelf life of 42 days, and platelets have a shelf life of just 5 days. (*Source: O'Fallon Community News.*) Let d represent the number of days that red blood cells may be stored prior to use. Let p represent the number of days that platelets may be stored prior to use.

Solve each equation or inequality.

111. $|x + 4| = 7$

112. $|2 - x| - 3 = 0$

113. $\left|\frac{7}{2 - 3x}\right| - 9 = 0$

114. $\left|\frac{8x - 1}{3x + 2}\right| = 7$

115. $|5x - 1| = |2x + 3|$

116. $|x + 10| = |x - 11|$

117. $|2x + 9| \leq 3$

118. $|8 - 5x| \geq 2$

119. $|7x - 3| > 4$

120. $\left|\frac{1}{2}x + \frac{2}{3}\right| < 3$

121. $|3x + 7| - 5 = 0$

122. $|7x + 8| - 6 > -3$

123. $|4x - 12| \geq -3$

124. $|7 - 2x| \leq -9$

125. $|x^2 + 4x| \leq 0$

126. $|x^2 + 4x| > 0$

Write as an absolute value equation or inequality.

127. k is 12 units from 6 on the number line.

128. p is at least 3 units from 1 on the number line.

129. t is no less than 0.01 unit from 5

130. s is no more than 0.001 unit from 100

Chapter 1 Test

Solve each equation.

1. $3(x - 4) - 5(x + 2) = 2 - (x + 24)$

2. $\frac{2}{3}x + \frac{1}{2}(x - 4) = x - 4$

3. $6x^2 - 11x - 7 = 0$

4. $(3x + 1)^2 = 8$

5. $3x^2 + 2x = -2$

6. $\frac{12}{x^2 - 9} = \frac{2}{x - 3} - \frac{3}{x + 3}$

7. $\frac{4x}{x - 2} + \frac{3}{x} = \frac{-6}{x^2 - 2x}$

8. $\sqrt{3x + 4} + 5 = 2x + 1$

9. $\sqrt{-2x + 3} + \sqrt{x + 3} = 3$

10. $\sqrt[3]{3x - 8} = \sqrt[3]{9x + 4}$

11. $x^4 - 17x^2 + 16 = 0$

12. $(x + 3)^{2/3} + (x + 3)^{1/3} - 6 = 0$

13. $|4x + 3| = 7$

14. $|2x + 1| = |5 - x|$

15. ***Surface Area of a Rectangular Solid*** The formula for the surface area of a rectangular solid is

$$S = 2HW + 2LW + 2LH,$$

where S, H, W, and L represent surface area, height, width, and length, respectively. Solve this formula for W.

16. Perform each operation. Give the answer in standard form.

(a) $(9 - 3i) - (4 + 5i)$

(b) $(4 + 3i)(-5 + 3i)$

(c) $(8 + 3i)^2$

(d) $\frac{3 + 19i}{1 + 3i}$

17. Simplify each power of i.

(a) i^{42}

(b) i^{-31}

(c) $\frac{1}{i^{19}}$

Solve each problem.

18. ***(Modeling) Water Consumption for Snowmaking*** Ski resorts require large amounts of water in order to make snow. Snowmass Ski Area in Colorado plans to pump between 1120 and 1900 gal of water per minute at least 12 hr per day from Snowmass Creek between mid-October and late December. (*Source:* York Snow Incorporated.)

(a) Determine an equation that will calculate the *minimum* amount of water A (in gallons) pumped after x days during mid-October to late December.

(b) Find the minimum amount of water pumped in 30 days.

(c) Suppose the water being pumped from Snowmass Creek was used to fill swimming pools. The average backyard swimming pool holds 20,000 gal of water. Determine an equation that will give the minimum number of pools P that could be filled after x days. How many pools could be filled each day?

(d) In how many days could a minimum of 1000 pools be filled?

19. ***Dimensions of a Rectangle*** The perimeter of a rectangle is 620 m. The length is 20 m less than twice the width. What are the length and width?

20. ***Nut Mixture*** To make a special mix, the owner of a fruit and nut stand wants to combine cashews that sell for \$7.00 per lb with walnuts that sell for \$5.50 per lb to obtain 35 lb of a mixture that sells for \$6.50 per lb. How many pounds of each type of nut should be used in the mixture?

21. ***Speed of a Plane*** Mary Lynn left by plane to visit her mother in Louisiana, 420 km away. Fifteen minutes later, her mother left to meet her at the airport. She drove the 20 km to the airport at 40 km per hr, arriving just as the plane taxied in. What was the speed of the plane?

22. ***(Modeling) Height of a Projectile*** A projectile is launched straight up from ground level with an initial velocity of 96 ft per sec. Its height in feet, s, after t seconds is given by the equation

$$s = -16t^2 + 96t.$$

(a) At what time(s) will it reach a height of 80 ft?
(b) After how many seconds will it return to the ground?

23. ***(Modeling) U.S. Airline Passengers*** The number of fliers on scheduled flights of U.S. airline companies was approximately 548 million in 1995, 739 million in 2005, and 704 million in 2009. (*Source:* Air Transport Association of America.) Here are three possible models for these data, where $x = 0$ corresponds to the year 1995.

Year	A	B	C
0	562	547.8	544.5
10	688.8	738.6	726.9
14	739.5	703.9	708.5

A. $y = 12.68x + 562.0$
B. $y = -1.983x^2 + 38.91x + 547.8$
C. $y = -1.632x^2 + 34.56x + 544.5$

The table shows each equation evaluated at the years 1995, 2004, and 2009. Decide which equation most closely models the data for these years.

Solve each inequality. Give the answer using interval notation.

24. $-2(x - 1) - 12 < 2(x + 1)$

25. $-3 \leq \frac{1}{2}x + 2 \leq 3$

26. $2x^2 - x \geq 3$

27. $\frac{x + 1}{x - 3} < 5$

28. $|2x - 5| < 9$

29. $|2x + 1| - 11 \geq 0$

30. $|3x + 7| \leq 0$

2 Graphs and Functions

The mirror image of the left and right sides of this cat's face is an example of *symmetry,* a phenomenon found throughout nature and interpreted mathematically in this chapter.

2.1 Rectangular Coordinates and Graphs

2.2 Circles

2.3 Functions

2.4 Linear Functions

Chapter 2 Quiz

2.5 Equations of Lines and Linear Models

Summary Exercises on Graphs, Circles, Functions, and Equations

2.6 Graphs of Basic Functions

2.7 Graphing Techniques

Chapter 2 Quiz

2.8 Function Operations and Composition

2.1 Exercises

Concept Check *Decide whether each statement in Exercises 1–5 is* true *or* false. ***If the statement is false, tell why.***

1. The point $(-1, 3)$ lies in quadrant III of the rectangular coordinate system.
2. The distance from $P(x_1, y_1)$ to $Q(x_2, y_2)$ is given by

$$d(P, Q) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

3. The distance from the origin to the point (a, b) is $\sqrt{a^2 + b^2}$.
4. The midpoint of the segment joining (a, b) and $(3a, -3b)$ has coordinates $(2a, -b)$.
5. The graph of $y = 2x + 4$ has x-intercept -2 and y-intercept 4.
6. In your own words, list the steps for graphing an equation.

In Exercises 7–10, give three ordered pairs from each table. ***See Example 1.***

7.

x	y
2	−5
−1	7
3	−9
5	−17
6	−21

8.

x	y
3	3
−5	−21
8	18
4	6
0	−6

9. *Percent of High School Students Who Smoke*

Year	Percent
1997	36
1999	35
2001	29
2003	22
2005	23
2007	20

Source: Centers for Disease Control and Prevention

10. *Number of U.S. Viewers of the Super Bowl*

Year	Viewers (millions)
1998	90.0
2000	88.5
2002	86.8
2004	89.8
2006	90.7
2008	97.4
2010	106.5

Source: www.tvbythenumbers.com

For the points P and Q, find (a) the distance d(P, Q) and (b) the coordinates of the midpoint of the segment PQ. ***See Examples 2 and 5(a).***

11. $P(-5, -7)$, $Q(-13, 1)$

12. $P(-4, 3)$, $Q(2, -5)$

13. $P(8, 2)$, $Q(3, 5)$

14. $P(-8, 4)$, $Q(3, -5)$

15. $P(-6, -5)$, $Q(6, 10)$

16. $P(6, -2)$, $Q(4, 6)$

17. $P(3\sqrt{2}, 4\sqrt{5})$, $Q(\sqrt{2}, -\sqrt{5})$

18. $P(-\sqrt{7}, 8\sqrt{3})$, $Q(5\sqrt{7}, -\sqrt{3})$

Determine whether the three points are the vertices of a right triangle. ***See Example 3.***

19. $(-6, -4)$, $(0, -2)$, $(-10, 8)$

20. $(-2, -8)$, $(0, -4)$, $(-4, -7)$

21. $(-4, 1)$, $(1, 4)$, $(-6, -1)$

22. $(-2, -5)$, $(1, 7)$, $(3, 15)$

23. $(-4, 3)$, $(2, 5)$, $(-1, -6)$

24. $(-7, 4)$, $(6, -2)$, $(0, -15)$

Determine whether the three points are collinear. **See Example 4.**

25. $(0, -7)$, $(-3, 5)$, $(2, -15)$

26. $(-1, 4)$, $(-2, -1)$, $(1, 14)$

27. $(0, 9)$, $(-3, -7)$, $(2, 19)$

28. $(-1, -3)$, $(-5, 12)$, $(1, -11)$

29. $(-7, 4)$, $(6, -2)$, $(-1, 1)$

30. $(-4, 3)$, $(2, 5)$, $(-1, 4)$

Find the coordinates of the other endpoint of each segment, given its midpoint and one endpoint. **See Example 5(b).**

31. midpoint $(5, 8)$, endpoint $(13, 10)$

32. midpoint $(-7, 6)$, endpoint $(-9, 9)$

33. midpoint $(12, 6)$, endpoint $(19, 16)$

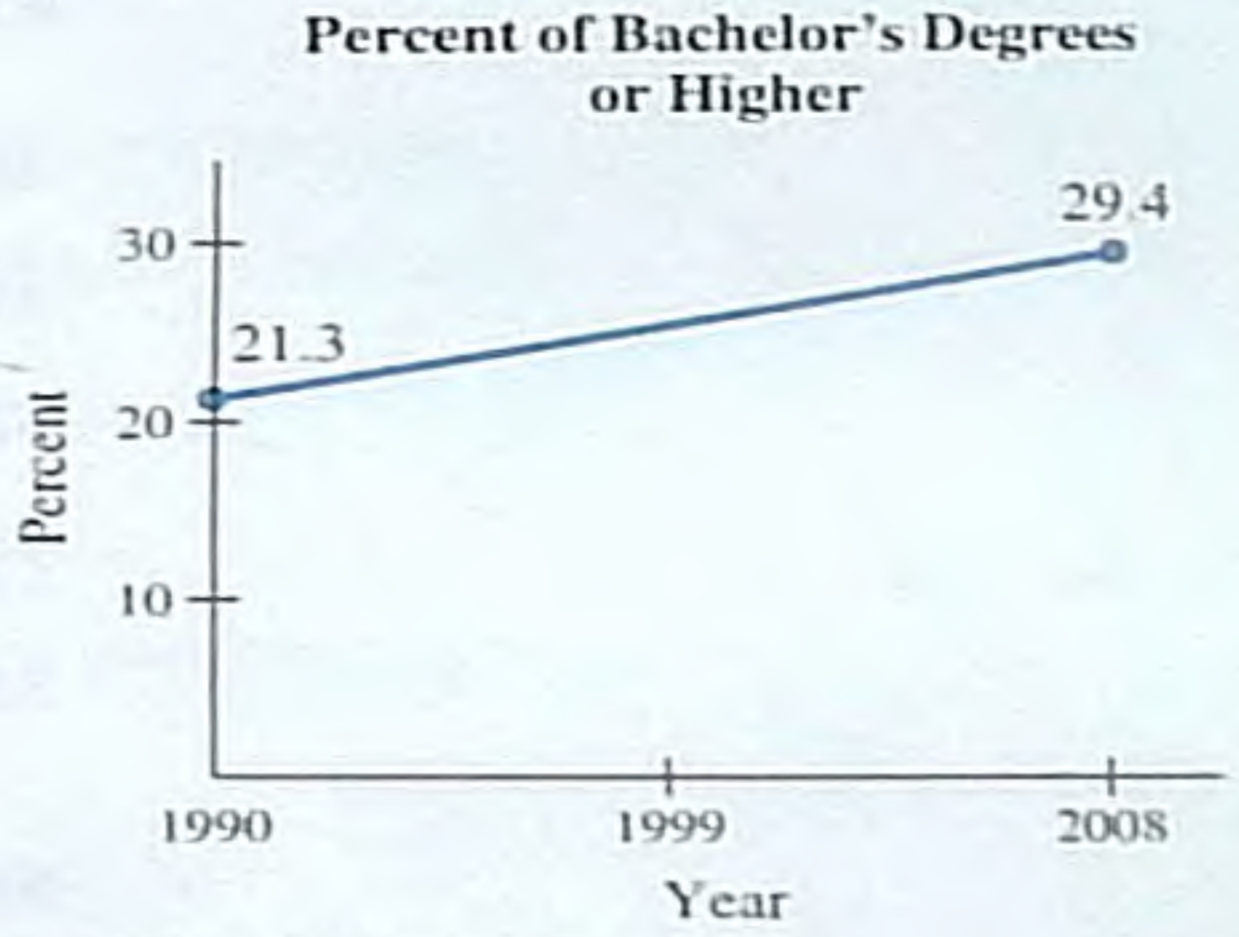
34. midpoint $(-9, 8)$, endpoint $(-16, 9)$

35. midpoint (a, b), endpoint (p, q)

36. midpoint $\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$, endpoint (b, d)

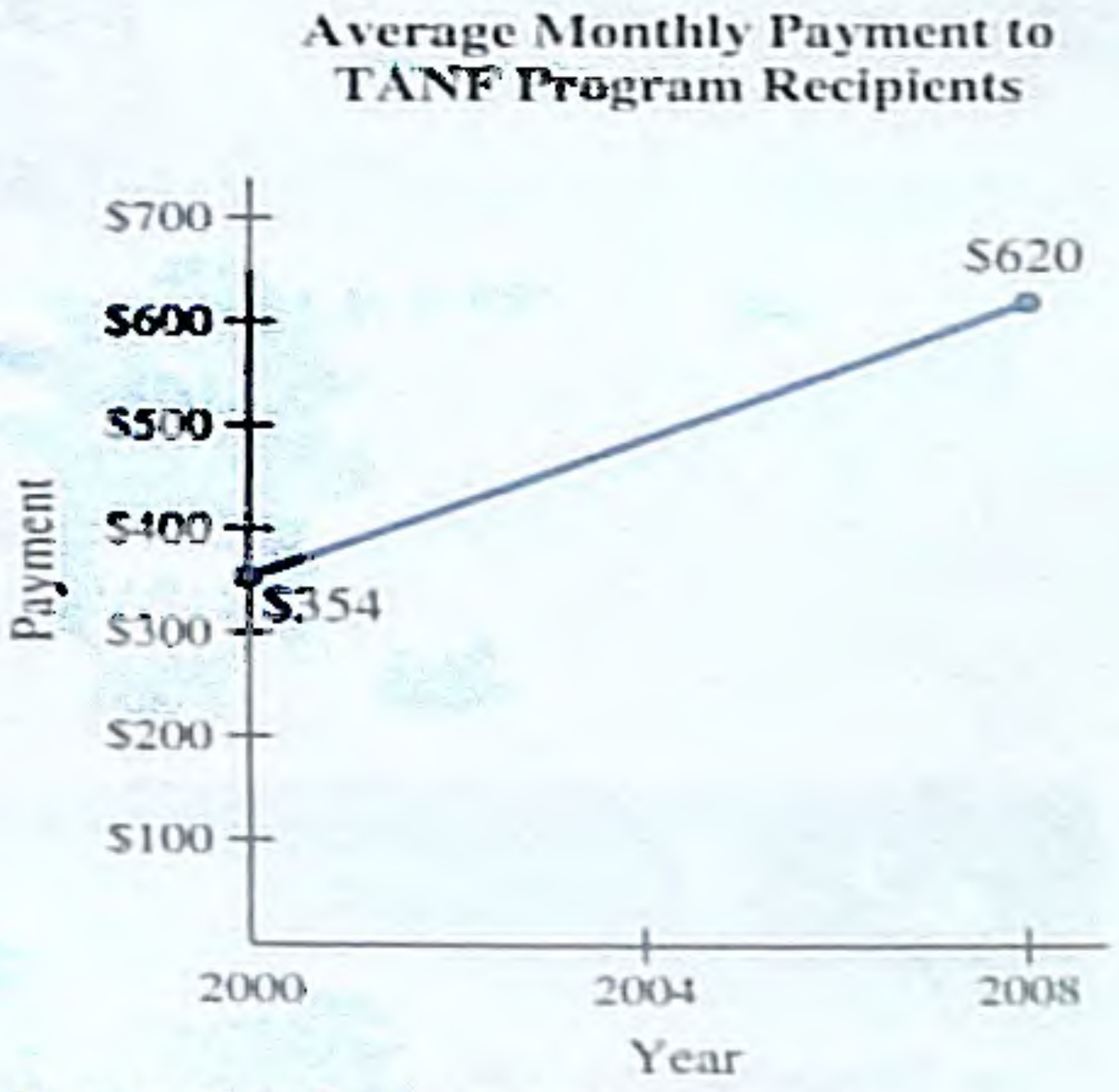
Solve each problem. **See Example 6.**

37. ***Bachelor's Degree Attainment*** The graph shows a straight line that approximates the percentage of Americans 25 years and older who had earned bachelor's degrees or higher for the years 1990–2008. Use the midpoint formula and the two given points to estimate the percent in 1999. Compare your answer with the actual percent of 25.2.



Source: U.S. Census Bureau.

38. ***Temporary Assistance for Needy Families (TANF)*** The graph shows an idealized linear relationship for the average monthly payment per recipient to needy families in the TANF program. Based on this information, what was the average payment to families in 2004?



Source: U.S. Department of Health and Human Services.

39. ***Poverty Level Income Cutoffs*** The table lists how poverty level income cutoffs (in dollars) for a family of four have changed over time. Use the midpoint formula to approximate the poverty level cutoff in 2006 to the nearest dollar.

Year	Income (in dollars)
1980	8414
1990	13,359
2000	17,604
2004	19,307
2008	22,025

Source: U.S. Census Bureau.

40. *Public College Enrollment* Enrollments in public colleges for recent years are shown in the table. Assuming a linear relationship, estimate the enrollments for (a) 2002 and (b) 2006. Give answers to the nearest tenth of thousands if applicable.

Year	Enrollment (in thousands)
2000	11,753
2004	12,980
2008	13,972

Source: U.S. Census Bureau.

41. Show that if M is the midpoint of the line segment with endpoints $P(x_1, y_1)$ and $Q(x_2, y_2)$, then $d(P, M) + d(M, Q) = d(P, Q)$ and $d(P, M) = d(M, Q)$.

42. Write the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ using a rational exponent.

For each equation, (a) give a table with at least three ordered pairs that are solutions, and (b) graph the equation. See Examples 7 and 8.

43. $y = \frac{1}{2}x - 2$
44. $y = -x + 3$
45. $2x + 3y = 5$
46. $3x - 2y = 6$
47. $y = x^2$
48. $y = x^2 + 2$
49. $y = \sqrt{x - 3}$
50. $y = \sqrt{x} - 3$
51. $y = |x - 2|$
52. $y = -|x + 4|$
53. $y = x^3$
54. $y = -x^3$

Concept Check *Answer the following.*

55. If a vertical line is drawn through the point $(4, 3)$, at what point will it intersect the x-axis?

56. If a horizontal line is drawn through the point $(4, 3)$, at what point will it intersect the y-axis?

57. If the point (a, b) is in the second quadrant, in what quadrant is $(a, -b)$? $(-a, b)$? $(-a, -b)$? (b, a)?

58. Show that the points $(-2, 2)$, $(13, 10)$, $(21, -5)$, and $(6, -13)$ are the vertices of a rhombus (all sides equal in length).

59. Are the points $A(1, 1)$, $B(5, 2)$, $C(3, 4)$, and $D(-1, 3)$ the vertices of a parallelogram (opposite sides equal in length)? of a rhombus (all sides equal in length)?

60. Find the coordinates of the points that divide the line segment joining $(4, 5)$ and $(10, 14)$ into three equal parts.

2.2 Circles

- Center-Radius Form
- General Form
- An Application

Center-Radius Form By definition, a **circle** is the set of all points in a plane that lie a given distance from a given point. The given distance is the **radius** of the circle, and the given point is the **center**.

We can find the equation of a circle from its definition by using the distance formula. Suppose that the point (h, k) is the center and the circle has radius r, where $r > 0$. Let (x, y) represent any point on the circle. See **Figure 13**.

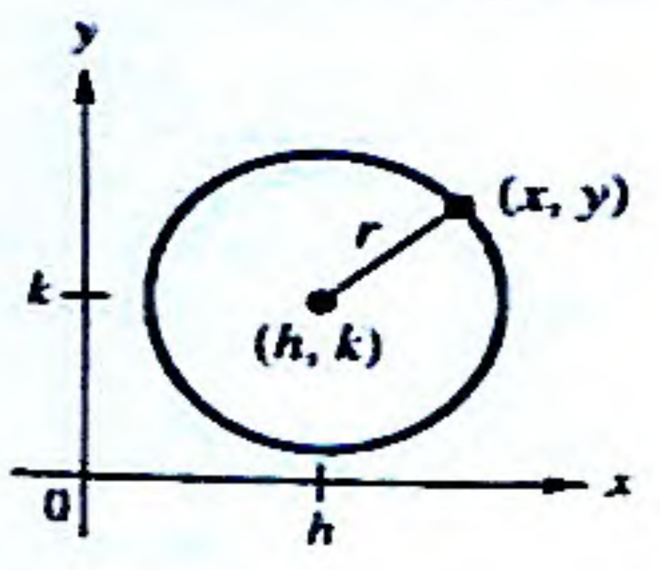


Figure 13

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = r \quad \text{Distance formula (Section 2.1)}$$

$$\sqrt{(x - h)^2 + (y - k)^2} = r \quad (h, k) \text{ [illegible] and [illegible]}$$

$$(x - h)^2 + (y - k)^2 = r^2 \quad \text{Square each side (Section 1.6)}$$

LOOKING AHEAD TO CALCULUS
The circle $x^2 + y^2 = 1$ is called the **unit circle.** It is important in interpreting the *trigonometric* or *circular* functions that appear in the study of calculus.

Center-Radius Form of the Equation of a Circle

A circle with center (h, k) and radius r has equation

$$(x - h)^2 + (y - k)^2 = r^2,$$

which is the **center-radius form** of the equation of the circle. As a special case, a circle with **center (0, 0) and radius** r **has the following equation.**

$$x^2 + y^2 = r^2$$

EXAMPLE 1 Finding the Center-Radius Form

Find the center-radius form of the equation of each circle described.

(a) center at $(-3, 4)$, radius 6 **(b)** center at $(0, 0)$, radius 3

SOLUTION

(a)

$$(x - h)^2 + (y - k)^2 = r^2 \quad \text{Center-radius form}$$

$$[x - (-3)]^2 + (y - 4)^2 = 6^2 \quad \text{Substitute. Let } (h, k) = (-3, 4) \text{ and } r = 6$$

Watch signs here.

$$(x + 3)^2 + (y - 4)^2 = 36 \quad \text{Simplify.}$$

(b) The center is the origin and $r = 3$.

$$x^2 + y^2 = r^2 \quad \text{Special case of the center-radius form}$$

$$x^2 + y^2 = 3^2 \quad \text{Let } r = 3.$$

$$x^2 + y^2 = 9 \quad \text{Apply the exponent.}$$

✔ *Now Try Exercises 1(a) and 7(a).*

EXAMPLE 2 Graphing Circles

Graph each circle discussed in **Example 1.**

(a) $(x + 3)^2 + (y - 4)^2 = 36$ **(b)** $x^2 + y^2 = 9$

SOLUTION

(a) Writing the given equation in center-radius form

$$[x - (-3)]^2 + (y - 4)^2 = 6^2$$

gives $(-3, 4)$ as the center and 6 as the radius. See **Figure 14.**

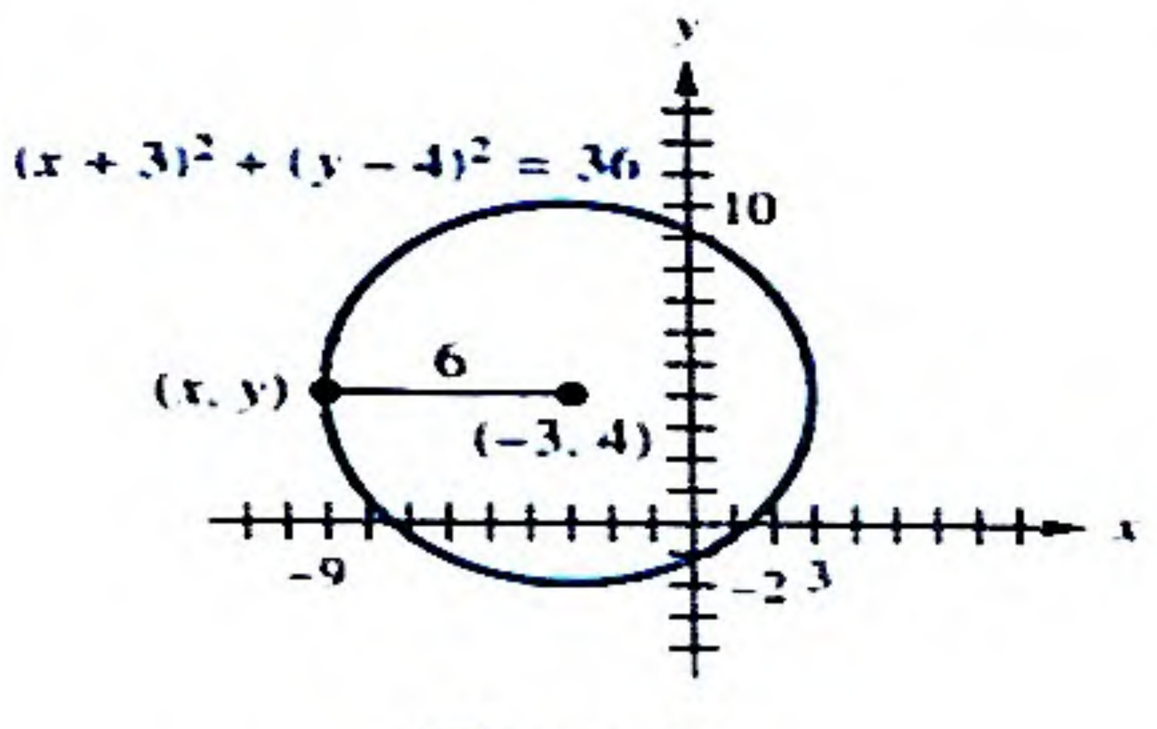


Figure 14

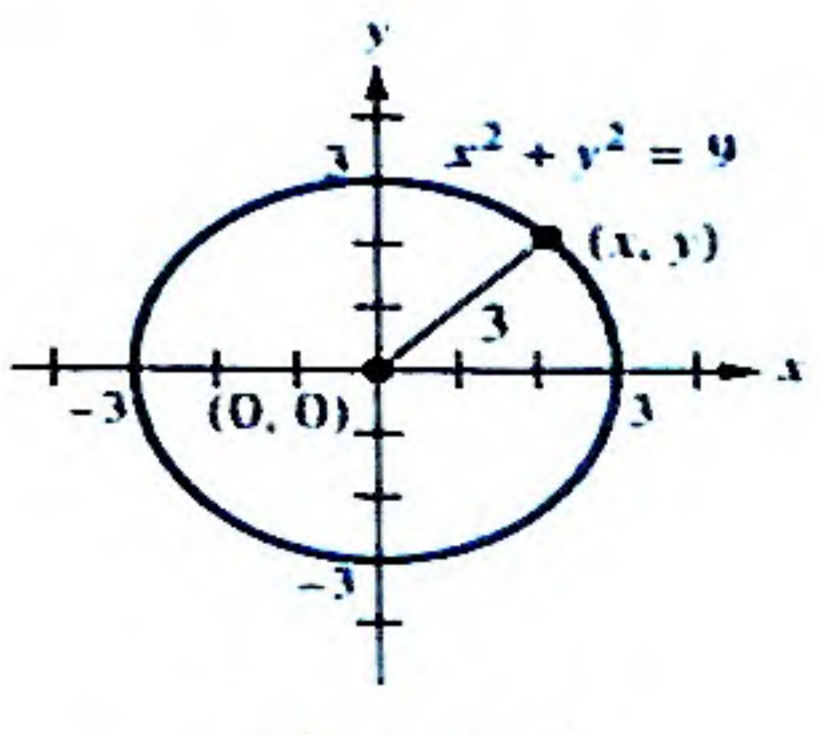


Figure 15

(b) The graph with center $(0, 0)$ and radius 3 is shown in **Figure 15.**

✔ *Now Try Exercises 1(b) and 7(b).*

EXAMPLE 5 **Determining Whether a Graph Is a Point or Nonexistent**

The graph of the equation $x^2 + 10x + y^2 - 4y + 33 = 0$ either is a **point** or is nonexistent. Which is it?

SOLUTION

$$x^2 + 10x + y^2 - 4y + 33 = 0$$

$$x^2 + 10x + y^2 - 4y = -33 \qquad \text{Subtract 33.}$$

Think: $$\left[\frac{1}{2}(10)\right]^2 = 25 \quad \text{and} \quad \left[\frac{1}{2}(-4)\right]^2 = 4 \qquad \text{Prepare to complete the square for both } x \text{ and } y.$$

$$(x^2 + 10x + 25) + (y^2 - 4y + 4) = -33 + 25 + 4 \qquad \text{Complete the square.}$$

$$(x + 5)^2 + (y - 2)^2 = -4 \qquad \text{Factor on the left, and add.}$$

Since $-4 < 0$, there are *no* ordered pairs (x, y), with x and y both real numbers, satisfying the equation. The graph of the given equation is nonexistent—it contains no points. (If the constant on the right side were 0, the graph would consist of the single point $(-5, 2)$.)

✔ *Now Try Exercise 25.*

An Application Seismologists can locate the epicenter of an earthquake by determining the intersection of three circles. The radii of these circles represent the distances from the epicenter to each of three receiving stations. The centers of the circles represent the receiving stations.

EXAMPLE 6 **Locating the Epicenter of an Earthquake**

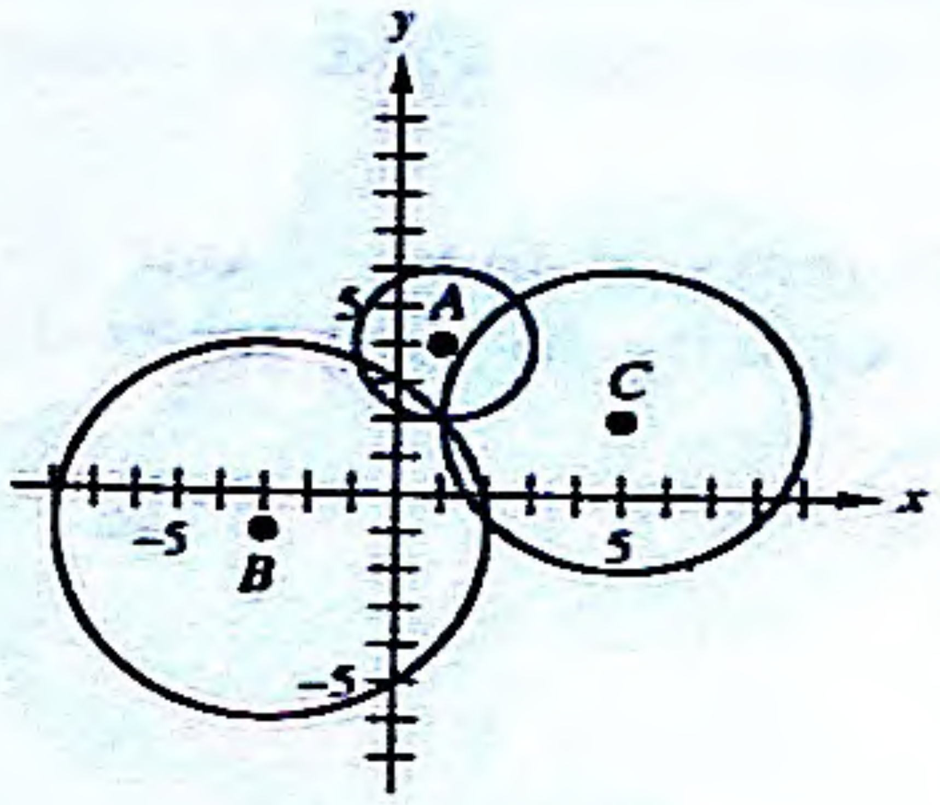


Figure 18

Suppose receiving stations A, B, and C are located on a coordinate plane at the points $(1, 4)$, $(-3, -1)$, and $(5, 2)$. Let the distances from the earthquake epicenter to these stations be 2 units, 5 units, and 4 units, respectively. Where on the coordinate plane is the epicenter located?

SOLUTION Graph the three circles as shown in **Figure 18**. From the graph it appears that the epicenter is located at $(1, 2)$. To check this algebraically, determine the equation for each circle and substitute $x = 1$ and $y = 2$.

Station A:	Station B:	Station C:
$(x-1)^2 + (y-4)^2 = 4$	$(x+3)^2 + (y+1)^2 = 25$	$(x-5)^2 + (y-2)^2 = 16$
$(1-1)^2 + (2-4)^2 \overset{?}{=} 4$	$(1+3)^2 + (2+1)^2 \overset{?}{=} 25$	$(1-5)^2 + (2-2)^2 \overset{?}{=} 16$
$0 + 4 \overset{?}{=} 4$	$16 + 9 \overset{?}{=} 25$	$16 + 0 \overset{?}{=} 16$
$4 = 4$	$25 = 25$	$16 = 16$

The point $(1, 2)$ lies on all three graphs. Thus, we can conclude that the epicenter of the earthquake is at $(1, 2)$.

✔ *Now Try Exercise 41.*

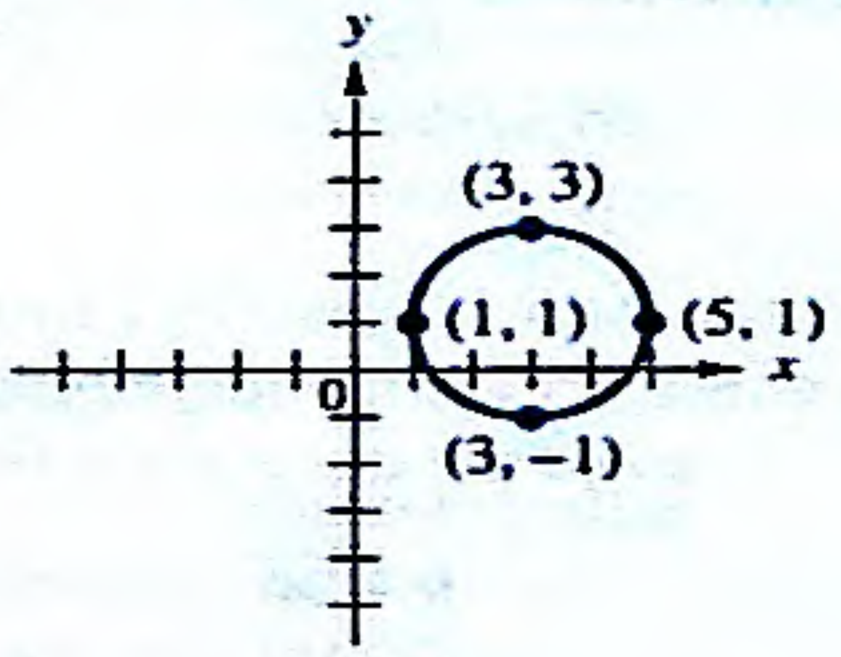
2.2 Exercises

In Exercises 1–12, ***(a)*** ***find the center-radius form of the equation of each circle, and (b) graph it. See Examples 1 and 2.***

1. center $(0, 0)$, radius 6
2. center $(0, 0)$, radius 9
3. center $(2, 0)$, radius 6
4. center $(3, 0)$, radius 3
5. center $(0, 4)$, radius 4
6. center $(0, -3)$, radius 7
7. center $(-2, 5)$, radius 4
8. center $(4, 3)$, radius 5
9. center $(5, -4)$, radius 7
10. center $(-3, -2)$, radius 6
11. center $(\sqrt{2}, \sqrt{2})$, radius $\sqrt{2}$
12. center $(-\sqrt{3}, -\sqrt{3})$, radius $\sqrt{3}$

Connecting Graphs with Equations *In Exercises* ***13–16, use each graph to determine the equation of the circle in (a) center-radius form and (b) general form.***

13.



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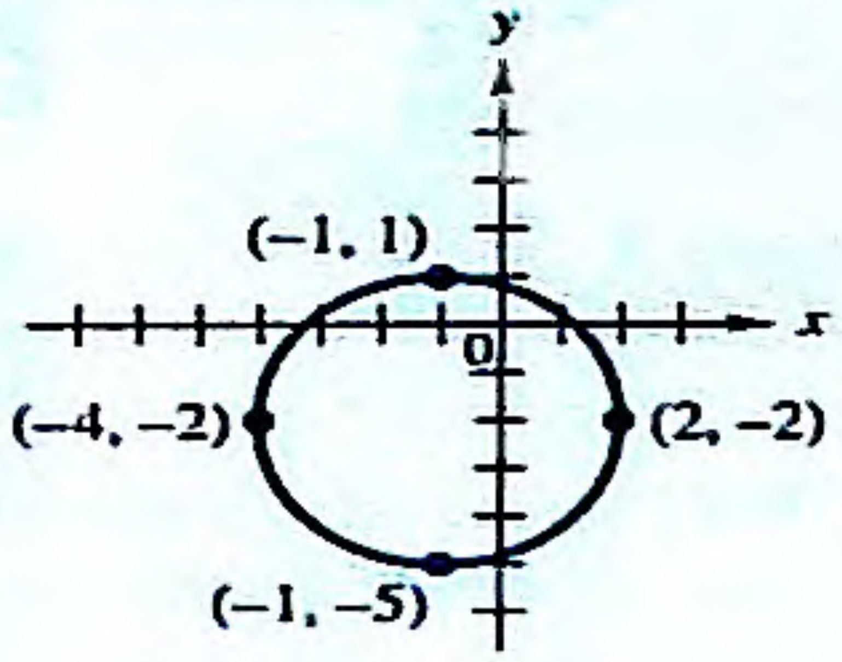

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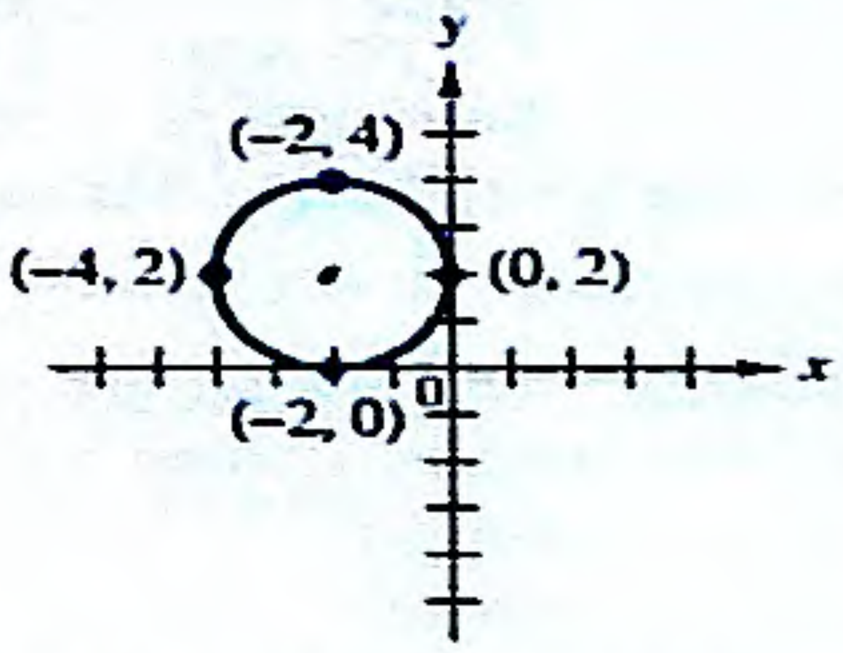
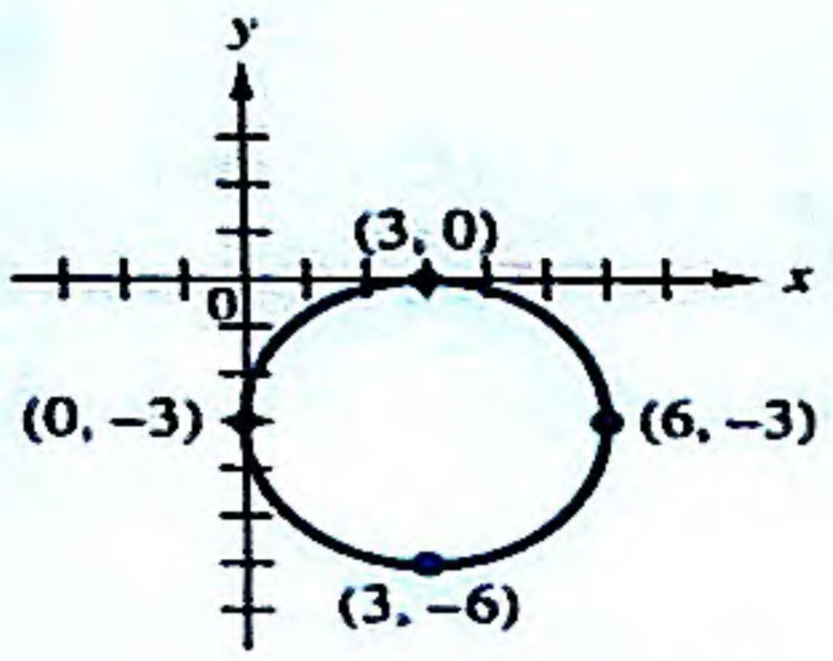
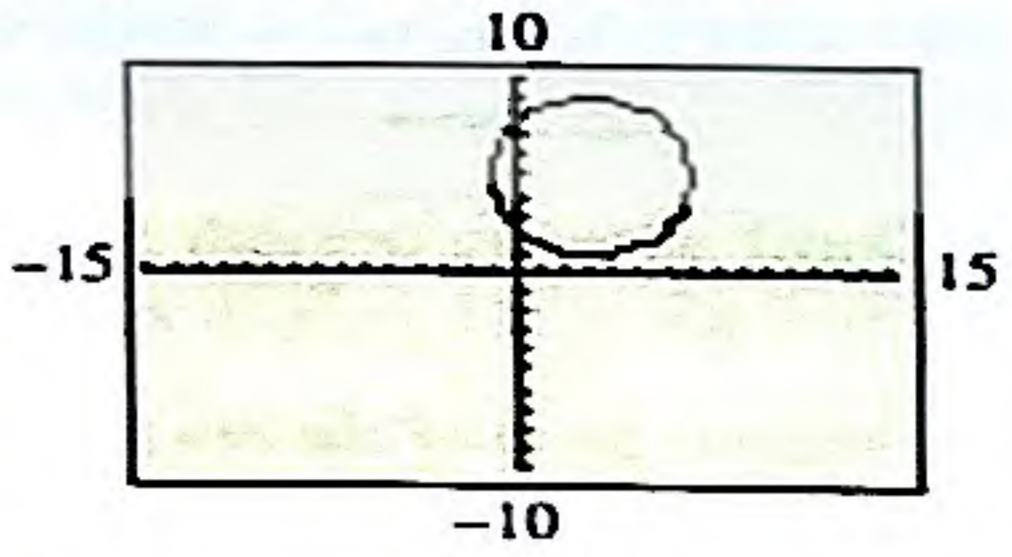


16.



17. ***Concept Check*** Which one of the two screens is the correct graph of the circle with center $(-3, 5)$ and radius 4?

A.



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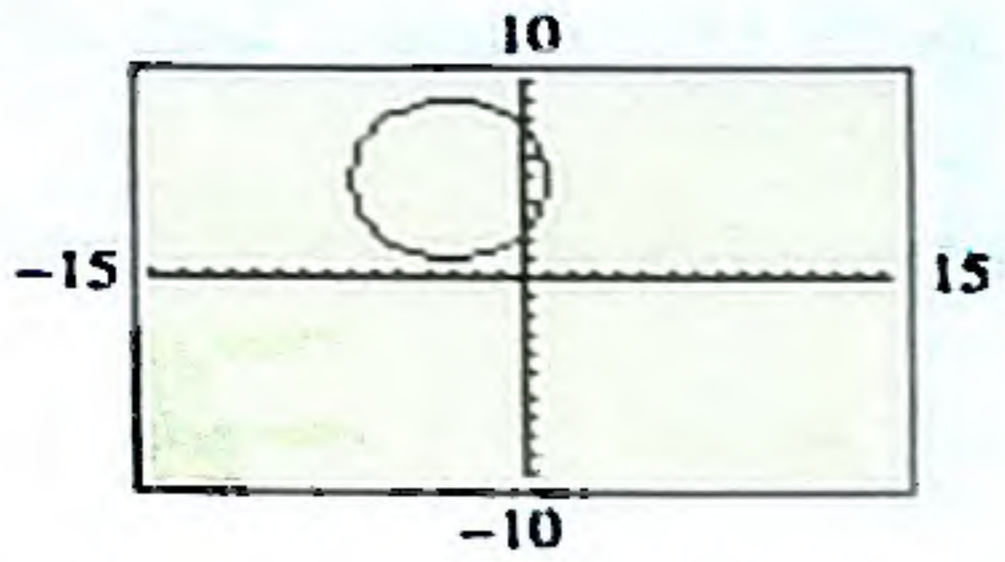


18. When the equation of a circle is written in the form

$$(x - h)^2 + (y - k)^2 = m,$$

how does the value of m indicate whether the graph is a circle, is a point, or is nonexistent?

Decide whether or not each equation has a circle as its graph. If it does, give the center and the radius. If it does not, describe the graph. ***See Examples 3–5.***

19. $x^2 + y^2 + 6x + 8y + 9 = 0$
20. $x^2 + y^2 + 8x - 6y + 16 = 0$
21. $x^2 + y^2 - 4x + 12y = -4$
22. $x^2 + y^2 - 12x + 10y = -25$
23. $4x^2 + 4y^2 + 4x - 16y - 19 = 0$
24. $9x^2 + 9y^2 + 12x - 18y - 23 = 0$

25. $x^2 + y^2 + 2x - 6y + 14 = 0$

26. $x^2 + y^2 + 4x - 8y + 32 = 0$

27. $x^2 + y^2 - 6x - 6y + 18 = 0$

28. $x^2 + y^2 + 4x + 4y + 8 = 0$

29. $9x^2 + 9y^2 - 6x + 6y - 23 = 0$

30. $4x^2 + 4y^2 + 4x - 4y - 7 = 0$

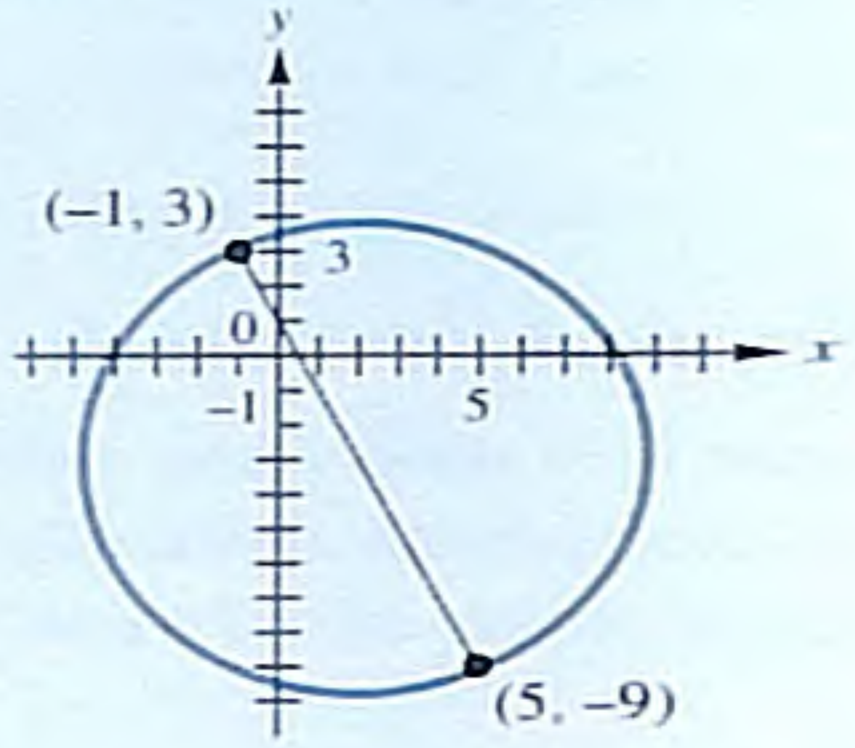
Relating Concepts

For individual or collaborative investigation *(Exercises 31–36)*

The distance formula, the midpoint formula, and the center-radius form of the equation of a circle are closely related in the following problem.

A circle has a diameter with endpoints $(-1, 3)$ and $(5, -9)$. Find the center-radius form of the equation of this circle.

***Work Exercises 31–36 in order** to see the relationships among these concepts.*



31. To find the center-radius form, we must find both the radius and the coordinates of the center. Find the coordinates of the center using the midpoint formula. (The center of the circle must be the midpoint of the diameter.)

32. There are several ways to find the radius of the circle. One way is to find the distance between the center and the point $(-1, 3)$. Use your result from **Exercise 31** and the distance formula to find the radius.

33. Another way to find the radius is to repeat **Exercise 32,** but use the point $(5, -9)$ rather than $(-1, 3)$. Do this to obtain the same answer you found in **Exercise 32.**

34. There is yet another way to find the radius. Because the radius is half the diameter, it can be found by finding half the length of the diameter. Using the endpoints of the diameter given in the problem, find the radius in this manner. You should once again obtain the same answer you found in **Exercise 32.**

35. Using the center found in **Exercise 31** and the radius found in **Exercises 32–34,** give the center-radius form of the equation of the circle.

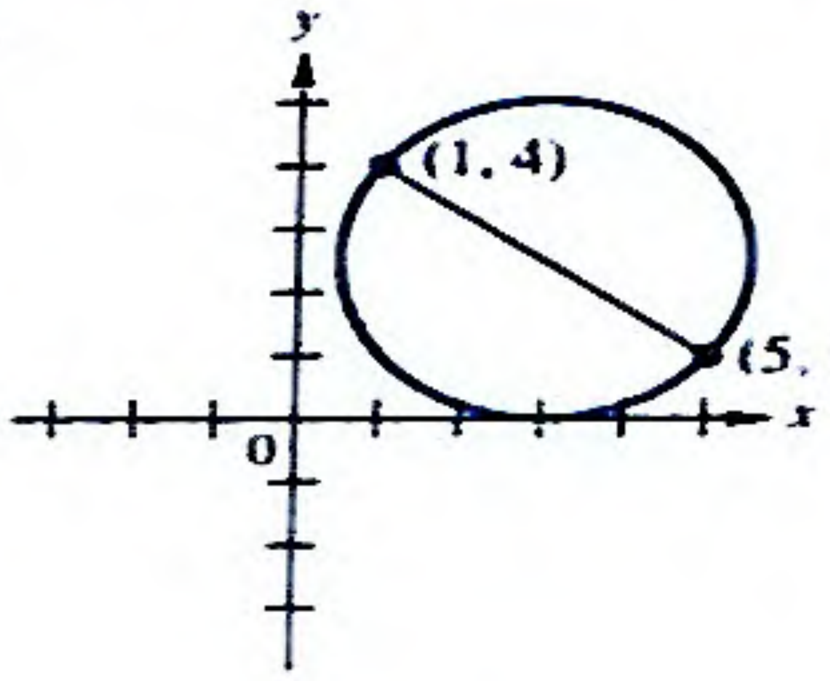
36. Use the method described in **Exercises 31–35** to find the center-radius form of the equation of the circle with diameter having endpoints $(3, -5)$ and $(-7, 3)$.

Find the center-radius form of the circle described or graphed. (See Relating Concepts Exercises 31–36.)

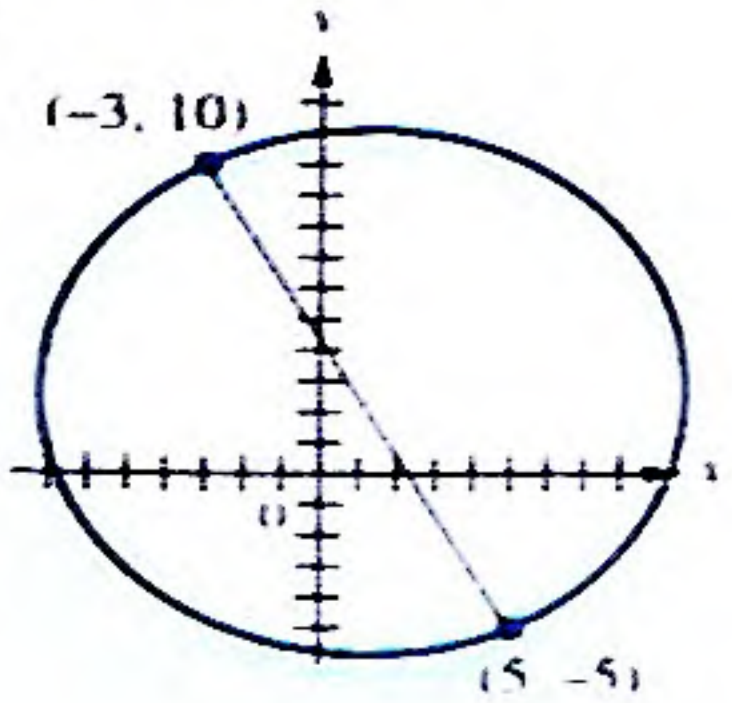
37. a circle having a diameter with endpoints $(-1, 2)$ and $(11, 7)$

38. a circle having a diameter with endpoints $(5, 4)$ and $(-3, -2)$

39.



40.



Epicenter of an Earthquake *Solve each problem. To visualize the situation, use graph paper and a pair of compasses to carefully draw the graphs of the circles. See* **Example 6.**

41. Suppose that receiving stations X, Y, and Z are located on a coordinate plane at the points $(7, 4)$, $(-9, -4)$, and $(-3, 9)$, respectively. The epicenter of an earthquake is determined to be 5 units from X, 13 units from Y, and 10 units from Z. Where on the coordinate plane is the epicenter located?

42. Suppose that receiving stations P, Q, and R are located on a coordinate plane at the points $(3, 1)$, $(5, -4)$, and $(-1, 4)$, respectively. The epicenter of an earthquake is determined to be $\sqrt{5}$ units from P, 6 units from Q, and $2\sqrt{10}$ units from R. Where on the coordinate plane is the epicenter located?

43. The locations of three receiving stations and the distances to the epicenter of an earthquake are contained in the following three equations: $(x - 2)^2 + (y - 1)^2 = 25$, $(x + 2)^2 + (y - 2)^2 = 16$, and $(x - 1)^2 + (y + 2)^2 = 9$. Determine the location of the epicenter.

44. The locations of three receiving stations and the distances to the epicenter of an earthquake are contained in the following three equations: $(x - 2)^2 + (y - 4)^2 = 25$, $(x - 1)^2 + (y + 3)^2 = 25$, and $(x + 3)^2 + (y + 6)^2 = 100$. Determine the location of the epicenter.

Concept Check *Work each of the following.*

45. Find the center-radius form of the equation of a circle with center $(3, 2)$ and tangent to the x-axis. (*Hint:* A line **tangent** to a circle touches it at exactly one point.)

46. Find the equation of a circle with center at $(-4, 3)$, passing through the point $(5, 8)$. Write it in center-radius form.

47. Find all points (x, y) with $x = y$ that are 4 units from $(1, 3)$.

48. Find all points satisfying $x + y = 0$ that are 8 units from $(-2, 3)$.

49. Find the coordinates of all points whose distance from $(1, 0)$ is $\sqrt{10}$ and whose distance from $(5, 4)$ is $\sqrt{10}$.

50. Find the equation of the circle of least radius that contains the points $(1, 4)$ and $(-3, 2)$ within or on its boundary.

51. Find all values of y such that the distance between $(3, y)$ and $(-2, 9)$ is 12.

52. Suppose that a circle is tangent to both axes, is in the third quadrant, and has radius $\sqrt{2}$. Find the center-radius form of its equation.

53. Find the shortest distance from the origin to the graph of the circle with equation $x^2 - 16x + y^2 - 14y + 88 = 0$.

54. Find the coordinates of the points of intersection of the line $y = 1$ and the circle centered at $(3, 0)$ with radius 2.

55. Phlash Phelps is the morning radio personality on SiriusXM Satellite Radio's *Sixties on Six* Decades channel. Phlash is an expert on U.S. geography and loves traveling around the country to strange, out-of-the-way locations. The photo shows Curt Gilchrist (standing) and Phlash (seated) visiting a small Arizona settlement called *Nothing*. (Nothing is so small that it's not named on current maps.) The sign indicates that Nothing is 50 mi from Wickenburg, AZ, 75 mi from Kingman, AZ, 105 mi from Phoenix, AZ, and 180 mi from Las Vegas, NV. Discuss how the concepts of **Example 6** can be used to locate Nothing, AZ, on a map of Arizona and southern Nevada.

It is not incorrect to say that $f(x) = x^2 + 4$ is increasing on $(0, \infty)$—there are infinitely many intervals over which it increases. However, we generally give the largest possible interval when determining where a function increases or decreases. (*Source:* Stewart J., *Calculus*, Fourth Edition, Brooks/Cole Publishing Company, p. 21.)

EXAMPLE 9 Determining Intervals over Which a Function Is Increasing, Decreasing, or Constant

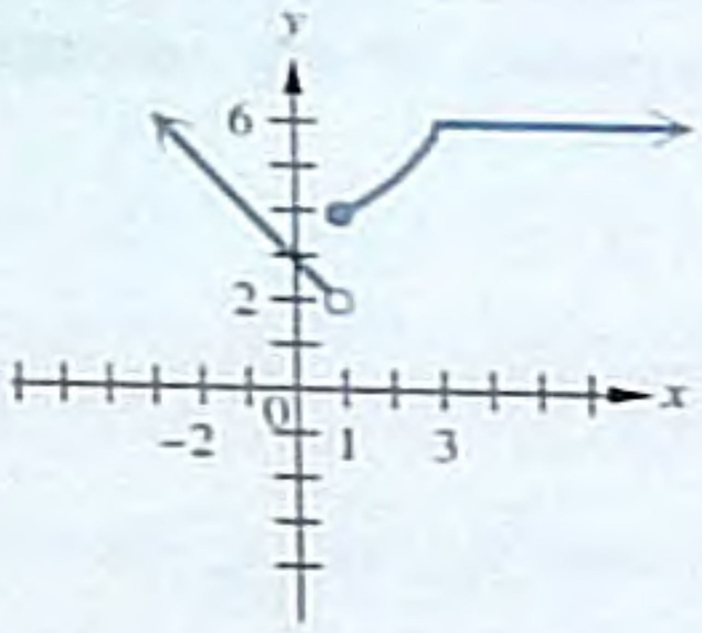


Figure 29

Figure 29 shows the graph of a function. Determine the intervals over which the function is increasing, decreasing, or constant.

SOLUTION We should ask, "What is happening to the y-values as the x-values are getting larger?" Moving from left to right on the graph, we see the following:

- On the interval $(-\infty, 1)$, the y-values are *decreasing*.
- On the interval $[1, 3]$, the y-values are *increasing*.
- On the interval $[3, \infty)$, the y-values are *constant* (and equal to 6).

Therefore, the function is decreasing on $(-\infty, 1)$, increasing on $[1, 3]$, and constant on $[3, \infty)$.

Now Try ***Exercise 77.***

EXAMPLE 10 Interpreting a Graph

Figure 30 shows the relationship between the number of gallons, $g(t)$, of water in a small swimming pool and time in hours, t. By looking at this graph of the function, we can answer questions about the water level in the pool at various times. For example, at time 0 the pool is empty. The water level then increases, stays constant for a while, decreases, and then becomes constant again. Use the graph to respond to the following.

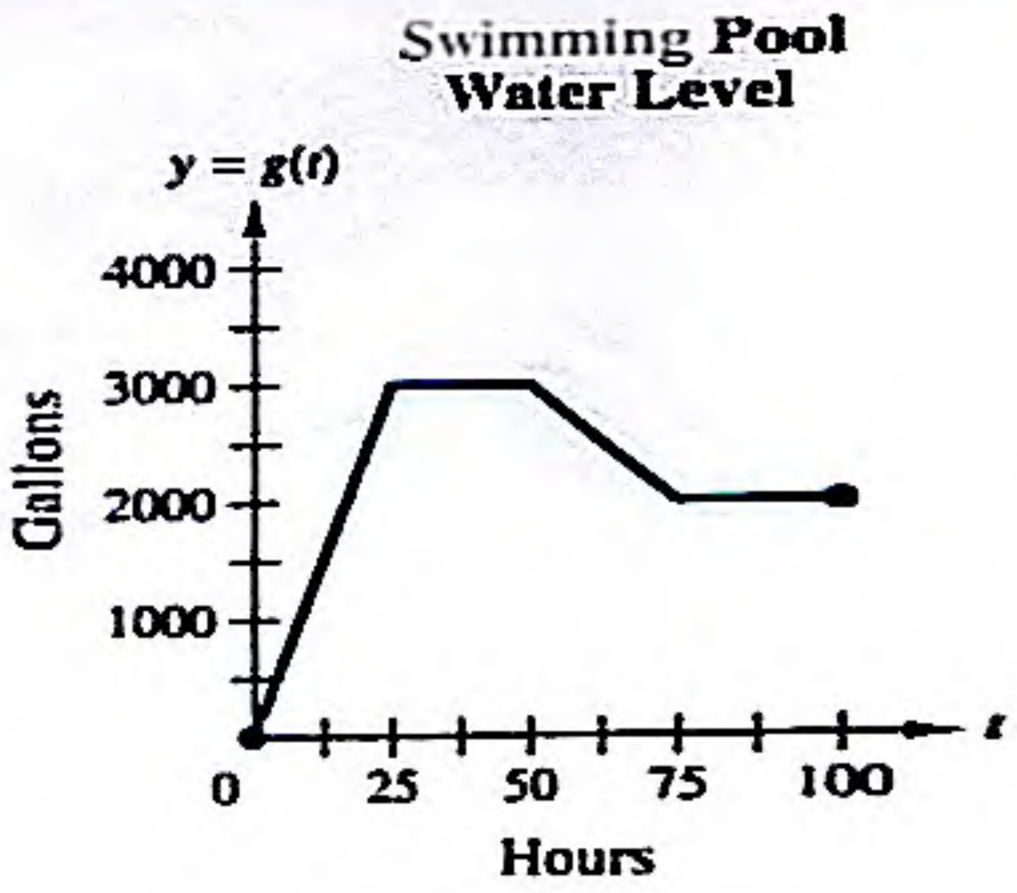


Figure 30

(a) What is the maximum number of gallons of water in the pool? When is the maximum water level first reached?

(b) For how long is the water level increasing? decreasing? constant?

(c) How many gallons of water are in the pool after 90 hr?

(d) Describe a series of events that could account for the water level changes shown in the graph.

SOLUTION

(a) The maximum range value is 3000, as indicated by the horizontal line segment for the hours 25 to 50. This maximum number of gallons, 3000, is first reached at $t = 25$ hr.

(b) The water level is increasing for $25 - 0 = 25$ hr and is decreasing for $75 - 50 = 25$ hr. It is constant for

$$(50 - 25) + (100 - 75) = 25 + 25, \quad \text{or} \quad 50 \text{ hr.}$$

(c) When $t = 90$, $y = g(90) = 2000$. There are 2000 gal after 90 hr.

(d) Looking at the graph in **Figure 30**, we might write the following description:

> The pool is empty at the beginning and then is filled to a level of 3000 gal during the first 25 hr. For the next 25 hr, the water level remains the same. At 50 hr, the pool starts to be drained, and this draining lasts for 25 hr, until only 2000 gal remain. For the next 25 hr, the water level is unchanged.

✓ *Now Try Exercise 83.*

2.3 Exercises

Decide whether each relation defines a function. See Example 1.

1. $\{(5, 1), (3, 2), (4, 9), (7, 8)\}$

2. $\{(8, 0), (5, 7), (9, 3), (3, 8)\}$

3. $\{(2, 4), (0, 2), (2, 6)\}$

4. $\{(9, -2), (-3, 5), (9, 1)\}$

5. $\{(-3, 1), (4, 1), (-2, 7)\}$

6. $\{(-12, 5), (-10, 3), (8, 3)\}$

7.

x	y
3	-4
7	-4
10	-4

8.

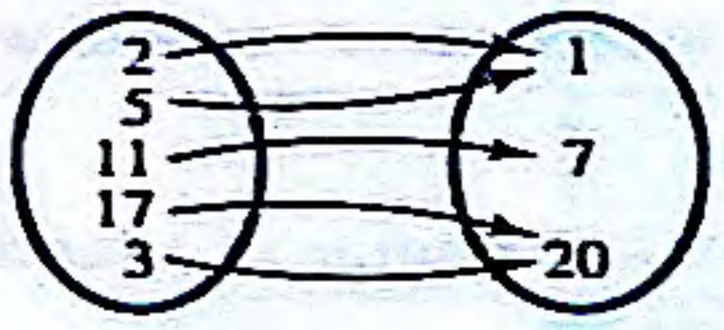
x	y
-4	$\sqrt{2}$
0	$\sqrt{2}$
4	$\sqrt{2}$

Decide whether each relation defines a function and give the domain and range. See Examples 1–4.

9. $\{(1, 1), (1, -1), (0, 0), (2, 4), (2, -4)\}$

10. $\{(2, 5), (3, 7), (3, 9), (5, 11)\}$

11.



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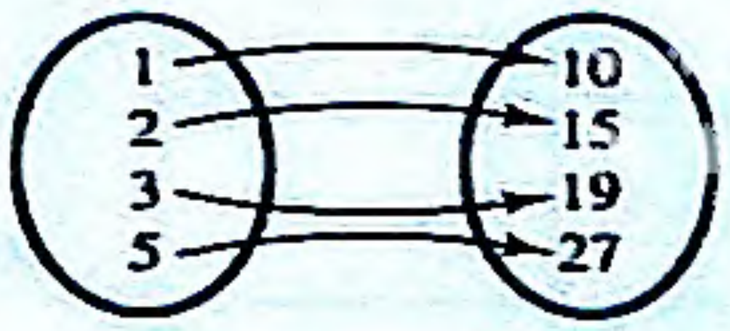

13.

x	y
0	0
-1	1
-2	2

14.

x	y
0	0
1	-1
2	-2

15. ***Number of Visits to U.S. National Parks***

Year (x)	Number of Visits (y) (millions)
2005	63.5
2006	60.4
2007	62.3
2008	61.2

Source: National Park Service.

16. ***Attendance at NCAA Women's College Basketball Games***

Season* (x)	Attendance (y)
2006	10,878,322
2007	11,120,822
2008	11,160,293
2009	11,134,738

Source: NCAA.

*Each season overlaps the starting year (given) with the following year.

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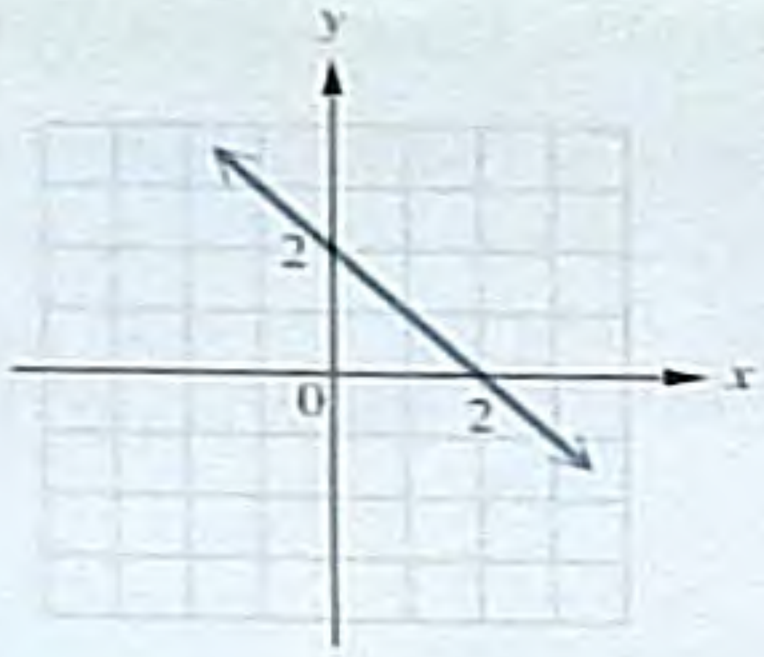

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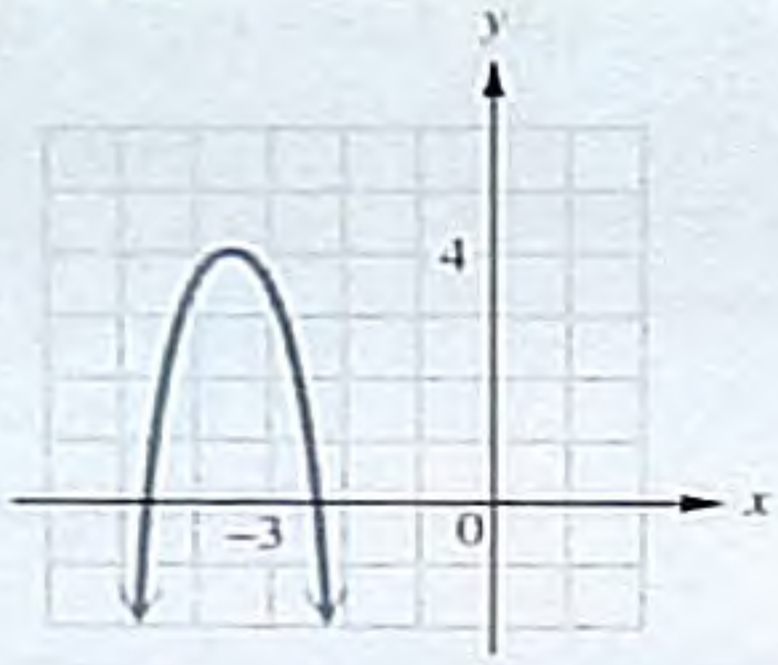

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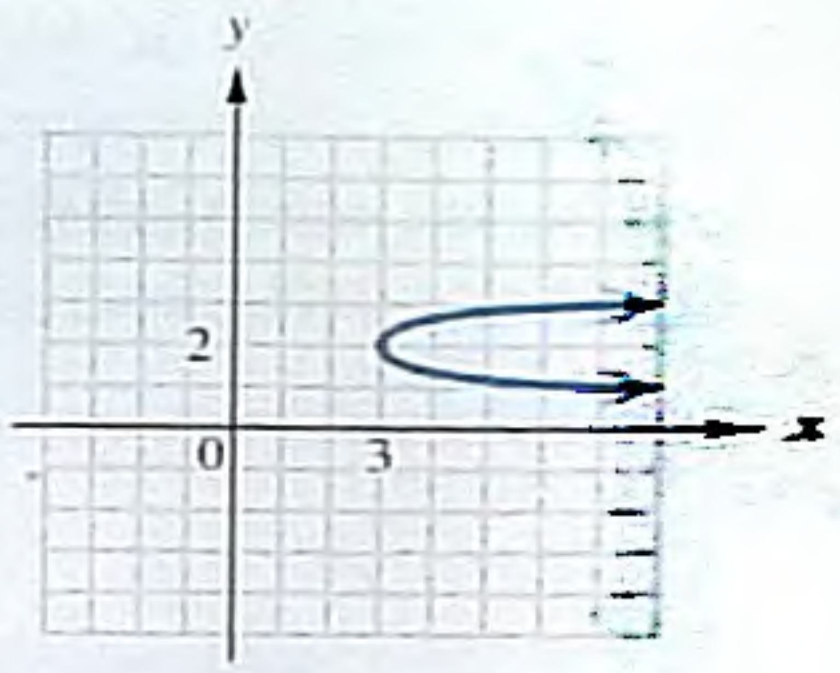

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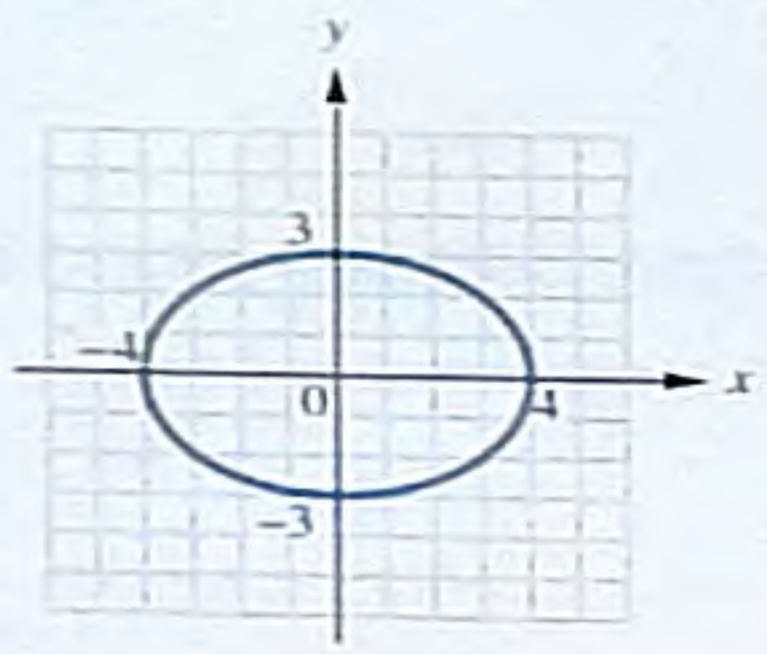

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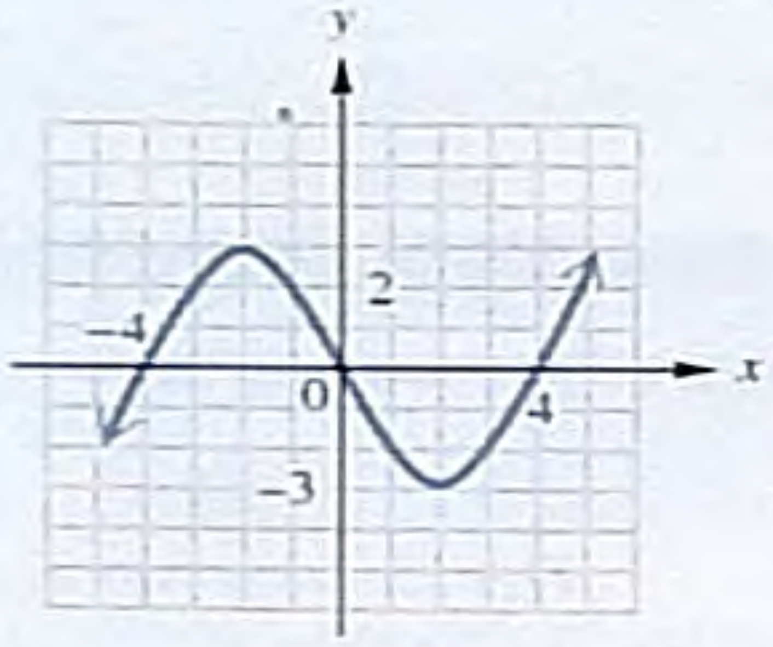

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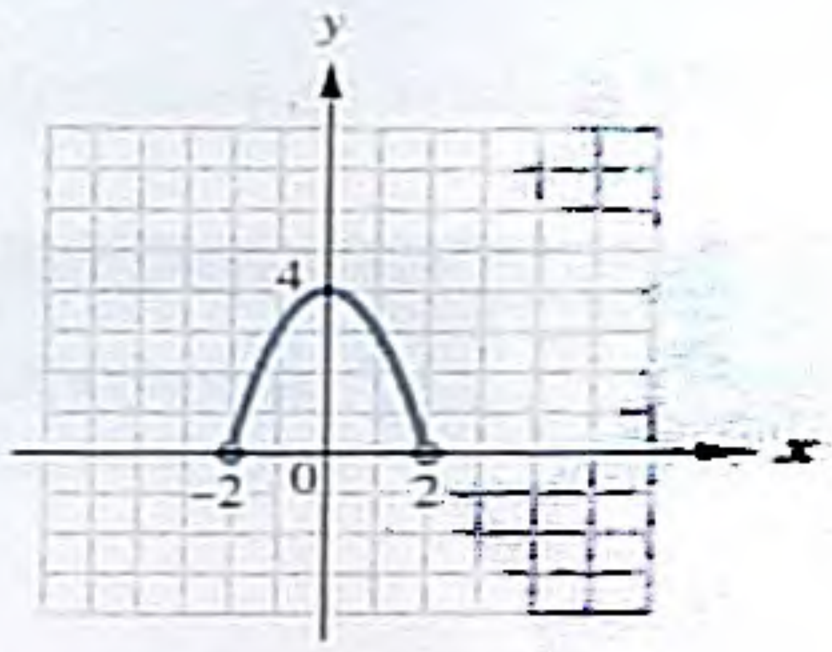


Decide whether each relation defines y as a function of x. Give the domain and range. See Example 5.

23. $y = x^2$ **24.** $y = x^3$ **25.** $x = y^6$ **26.** $x = y^4$

27. $y = 2x - 5$ **28.** $y = -6x + 4$ **29.** $x + y < 3$ **30.** $x - y < 4$

31. $y = \sqrt{x}$ **32.** $y = -\sqrt{x}$ **33.** $xy = 2$ **34.** $xy = -6$

35. $y = \sqrt{4x + 1}$ **36.** $y = \sqrt{7 - 2x}$ **37.** $y = \frac{2}{x - 3}$ **38.** $y = \frac{-7}{x - 5}$

39. *Concept Check* Choose the correct answer: For function f, the notation $f(3)$ means

- **A.** the variable f times 3, or $3f$.
- **B.** the value of the dependent variable when the independent variable is 3.
- **C.** the value of the independent variable when the dependent variable is 3.
- **D.** f equals 3.

40. *Concept Check* Give an example of a function from everyday life. (*Hint:* Fill in the blanks: _____ depends on _____, so _____ is a function of _____.)

Let $f(x) = -3x + 4$ and $g(x) = -x^2 + 4x + 1$. Find and simplify each of the following. See Example 6.

41. $f(0)$ **42.** $f(-3)$ **43.** $g(-2)$ **44.** $g(10)$

45. $f\left(\frac{1}{3}\right)$ **46.** $f\left(-\frac{7}{3}\right)$ **47.** $g\left(\frac{1}{2}\right)$ **48.** $g\left(-\frac{1}{4}\right)$

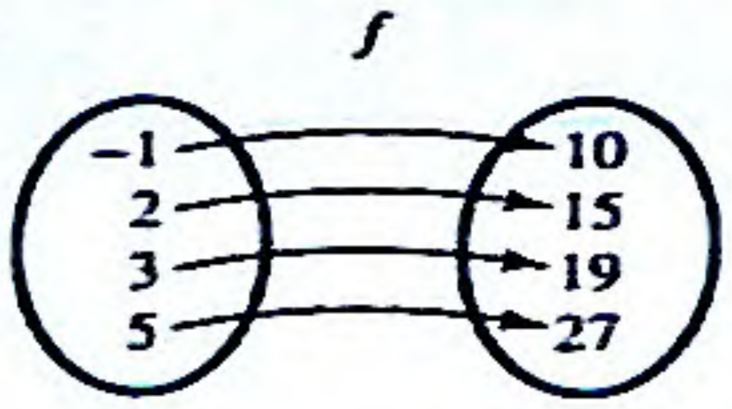
49. $f(p)$ **50.** $g(k)$ **51.** $f(-x)$ **52.** $g(-x)$

53. $f(x + 2)$ **54.** $f(a + 4)$ **55.** $f(2m - 3)$ **56.** $f(3t - 2)$

For each function, find (a) $f(2)$ and (b) $f(-1)$. See Example 7.

57. $f = \{(-1, 3), (4, 7), (0, 6), (2, 2)\}$ **58.** $f = \{(2, 5), (3, 9), (-1, 11), (5, 3)\}$

59.



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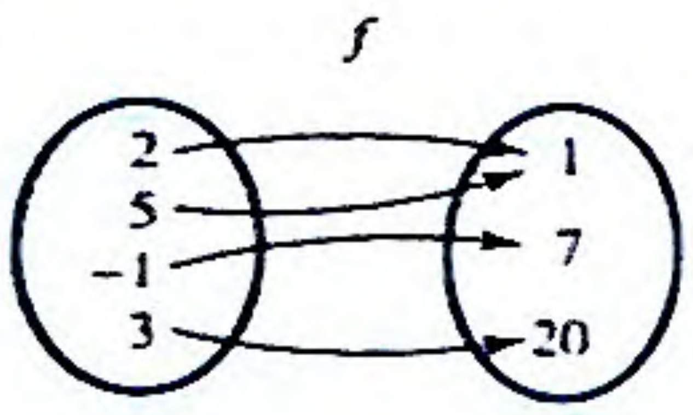

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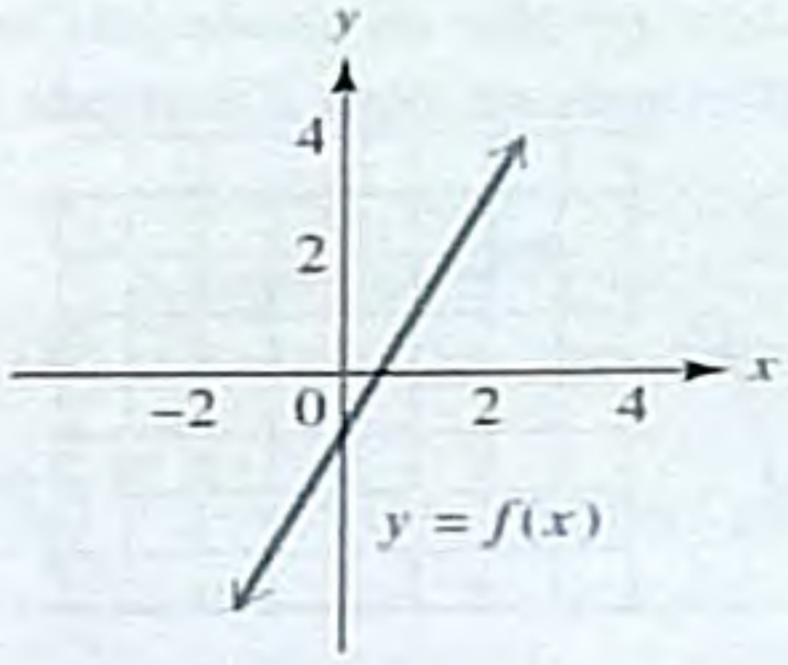

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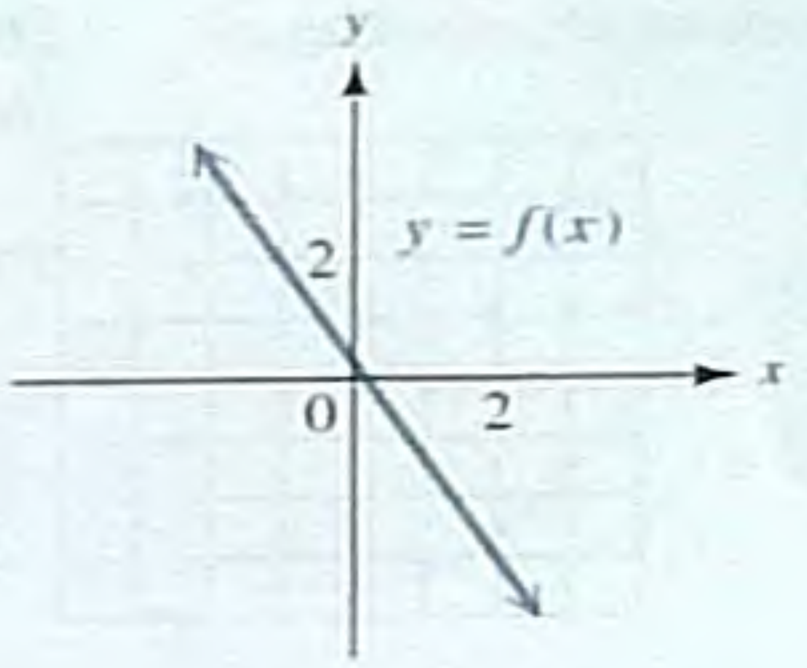


An equation that defines y as a function of x is given. (a) Solve for y in terms of x and replace y with the function notation $f(x)$. (b) Find $f(3)$. **See Example 8.**

63. $x + 3y = 12$

64. $x - 4y = 8$

65. $y + 2x^2 = 3 - x$

66. $y - 3x^2 = 2 + x$

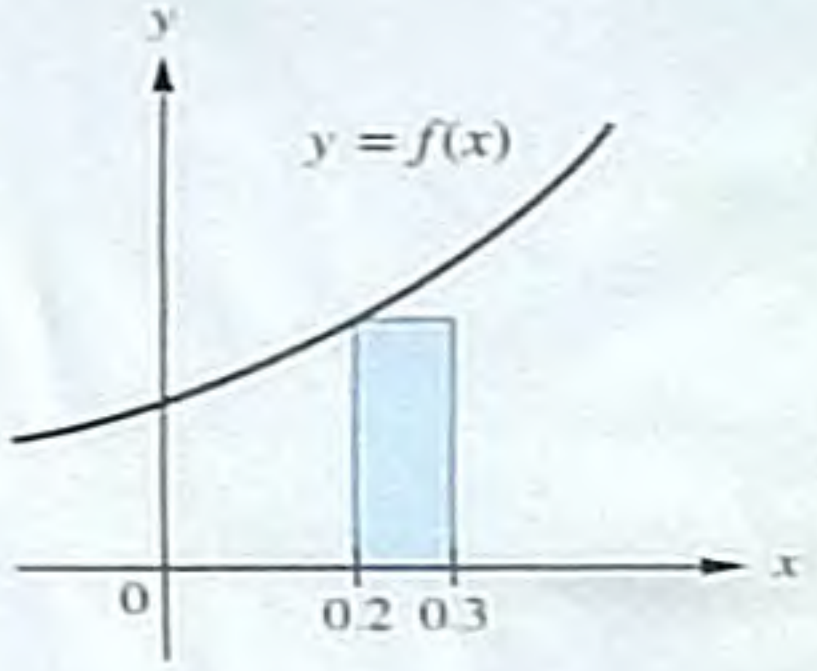
67. $4x - 3y = 8$

68. $-2x + 5y = 9$

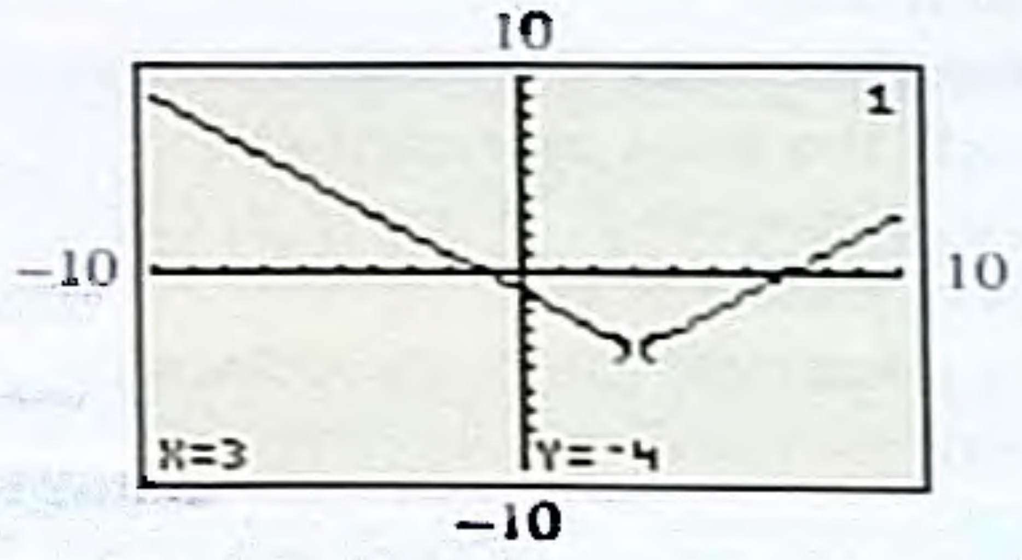
Concept Check Answer each question.

69. If $(3, 4)$ is on the graph of $y = f(x)$, which one of the following must be true: $f(3) = 4$ or $f(4) = 3$?

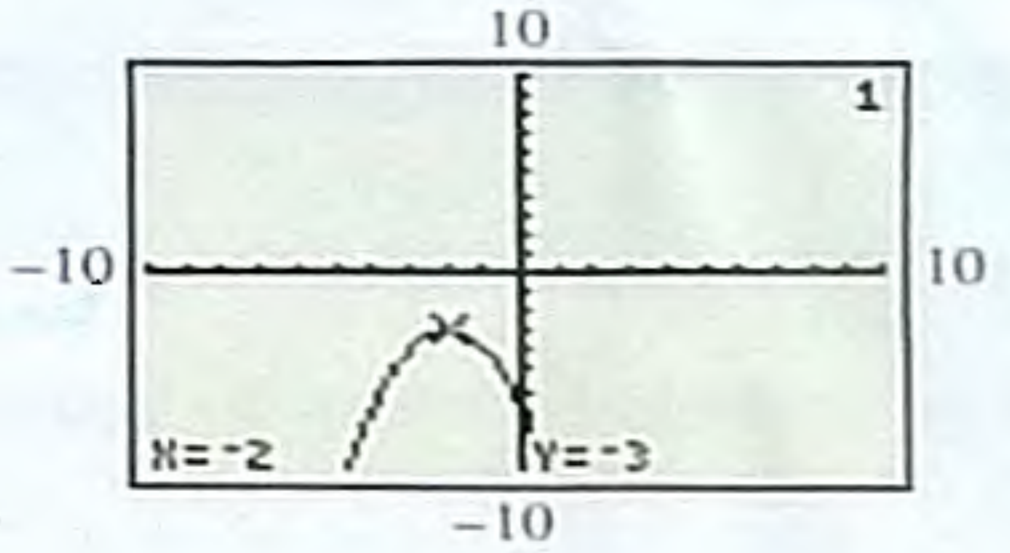
70. The figure shows a portion of the graph of $f(x) = x^2 + 3x + 1$ and a rectangle with its base on the x-axis and a vertex on the graph. What is the area of the rectangle? (*Hint:* $f(0.2)$ is the height.)



71. The graph of $Y_1 = f(X)$ is shown with a display at the bottom. What is $f(3)$?

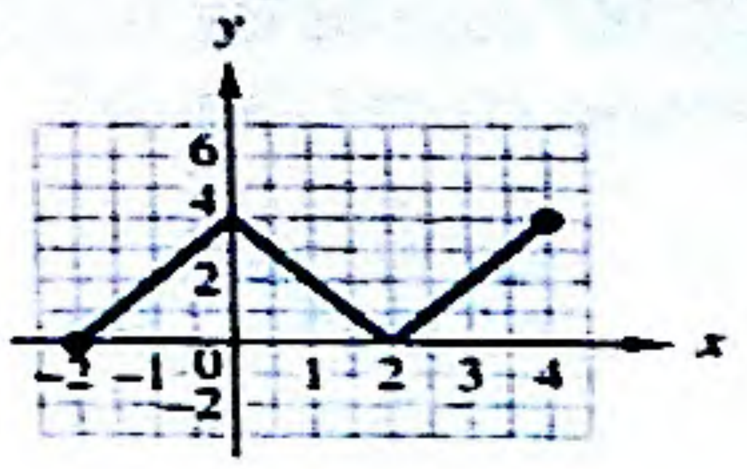


72. The graph of $Y_1 = f(X)$ is shown with a display at the bottom. What is $f(-2)$?



In Exercises 73–76, use the graph of $y = f(x)$ to find each function value: (a) $f(-2)$, (b) $f(0)$, (c) $f(1)$, and (d) $f(4)$. See Example 7(d).

73.



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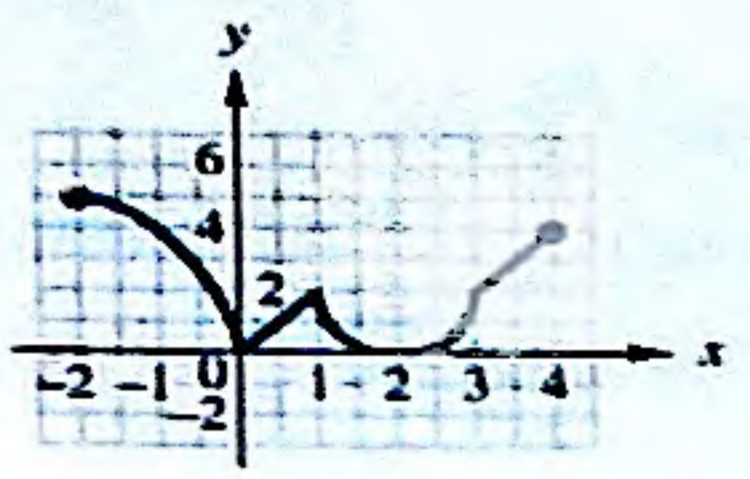

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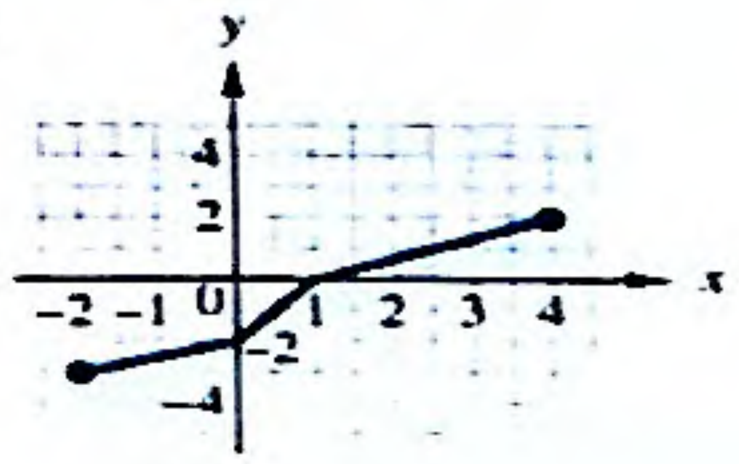

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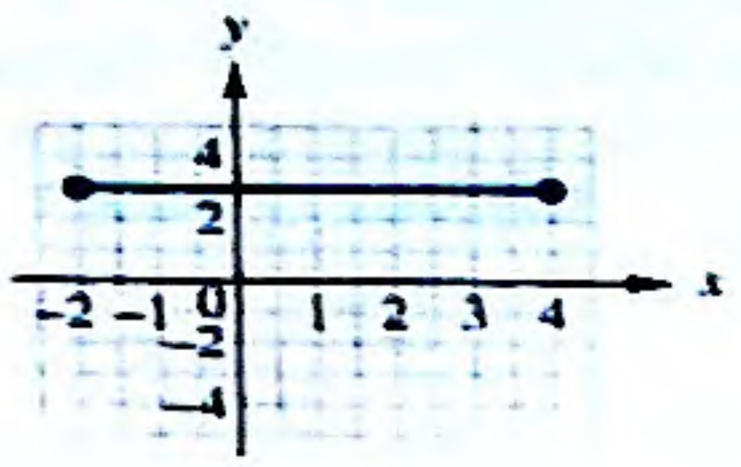

Determine the intervals of the domain for which each function is **(a) increasing**, *(b) decreasing, and (c) constant. See Example 9.*

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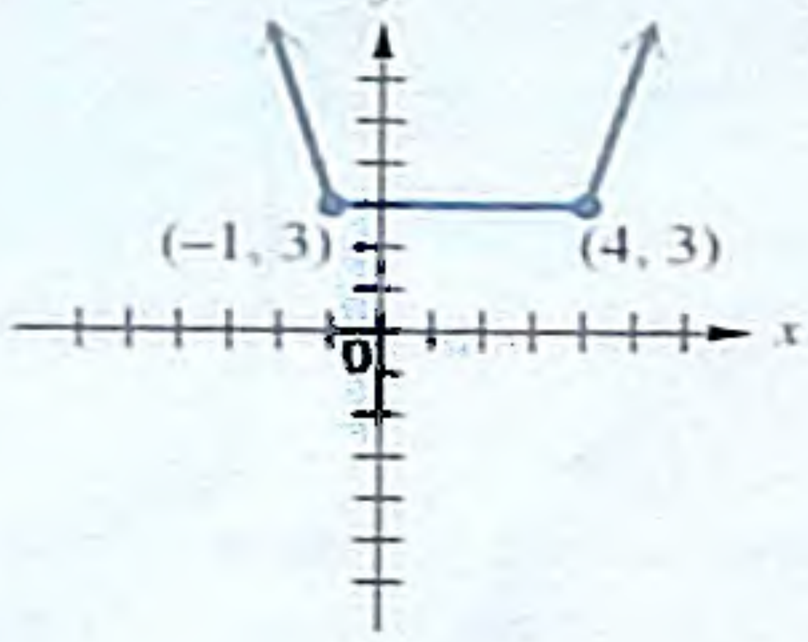

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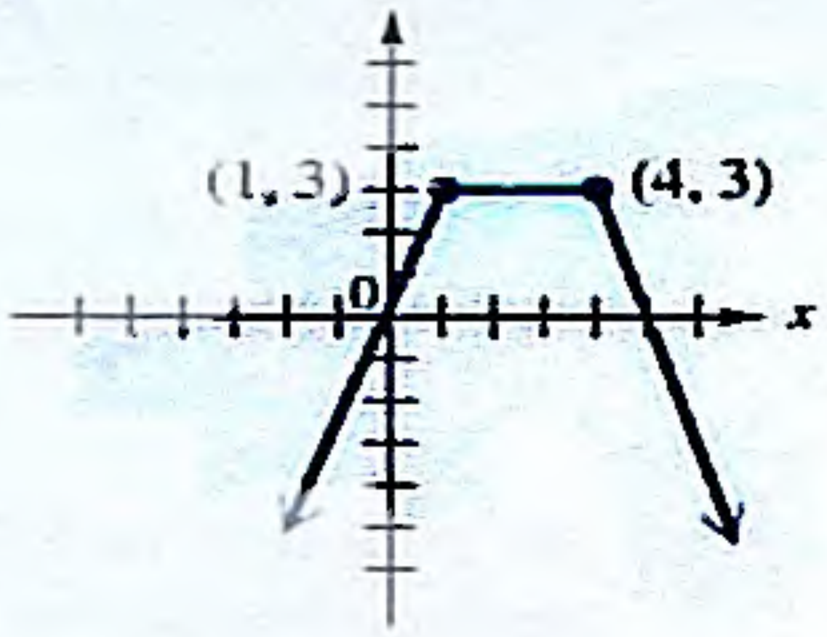

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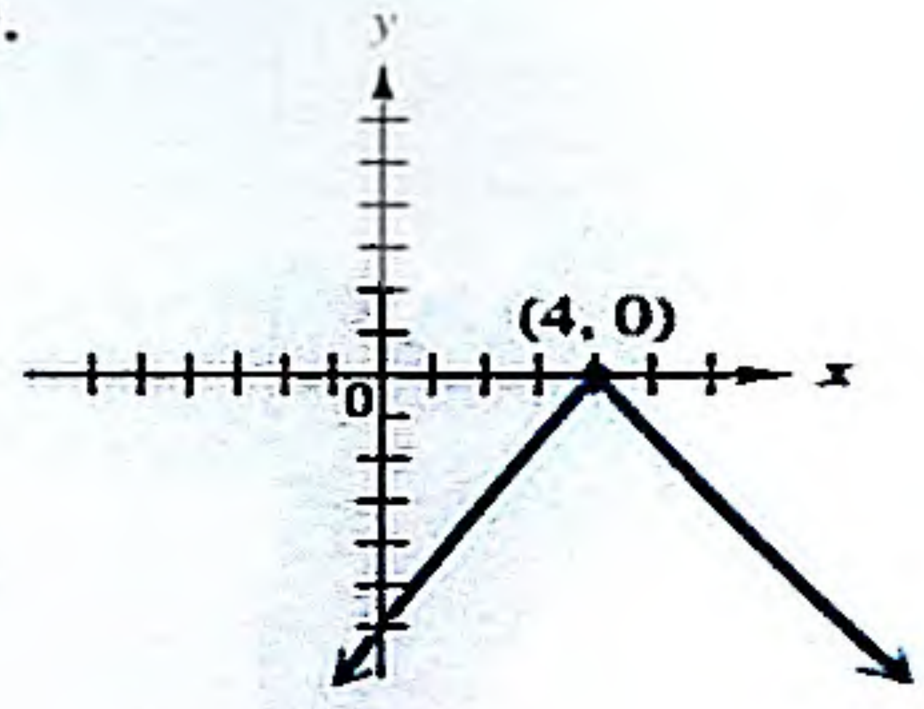

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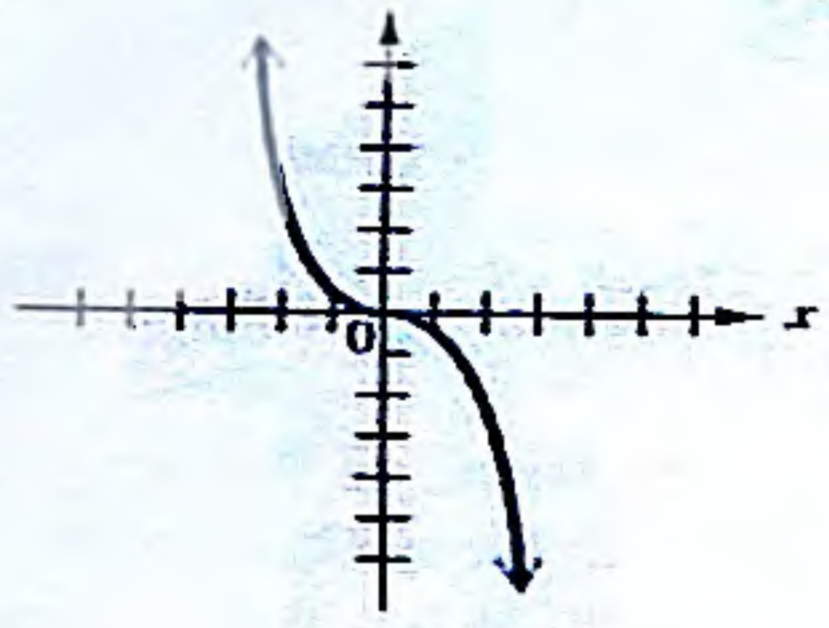

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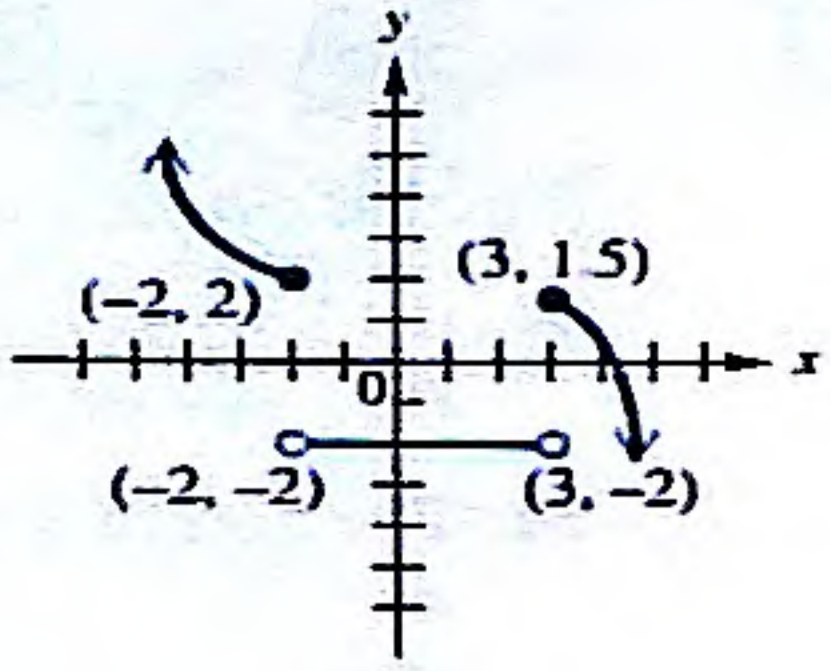

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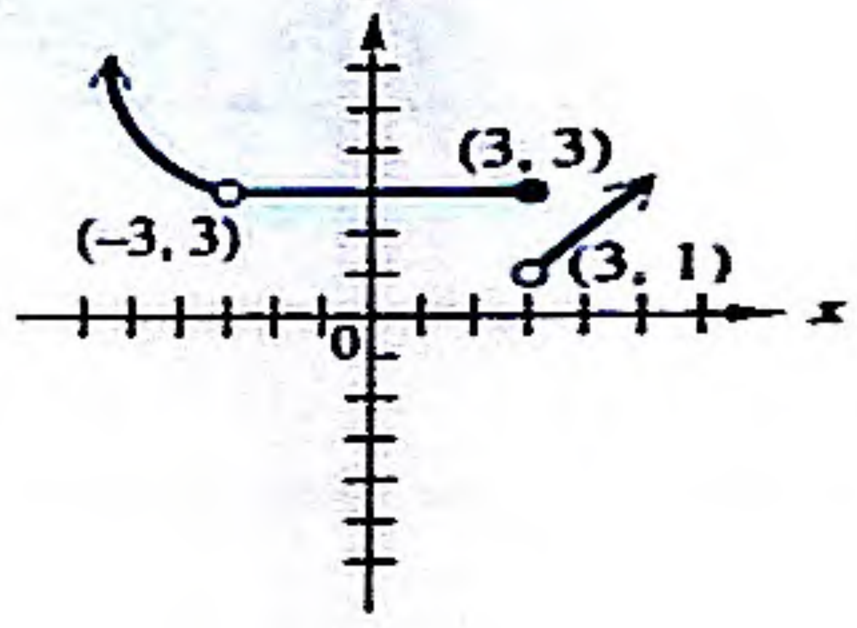
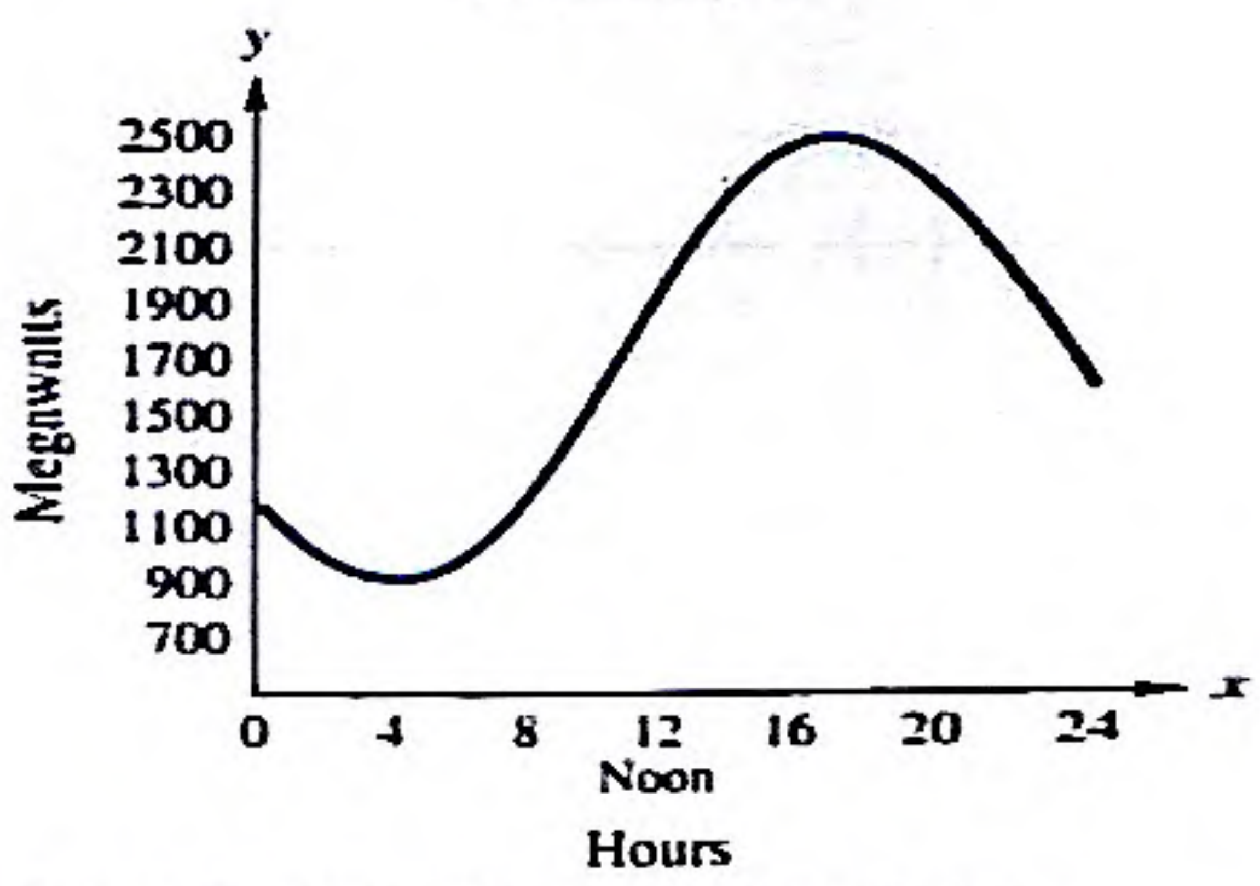


Solve each problem. See Example 10.

83. ***Electricity Usage*** The graph shows the daily megawatts of electricity used on a record-breaking summer day in Sacramento, California.

(a) Is this the graph of a function?
(b) What is the domain?
(c) Estimate the number of megawatts used at 8 A.M.
(d) At what time was the most electricity used? the least electricity?
(e) Call this function f. What is $f(12)$? Interpret your answer.
(f) During what time intervals is electricity usage increasing? decreasing?

Electricity Use

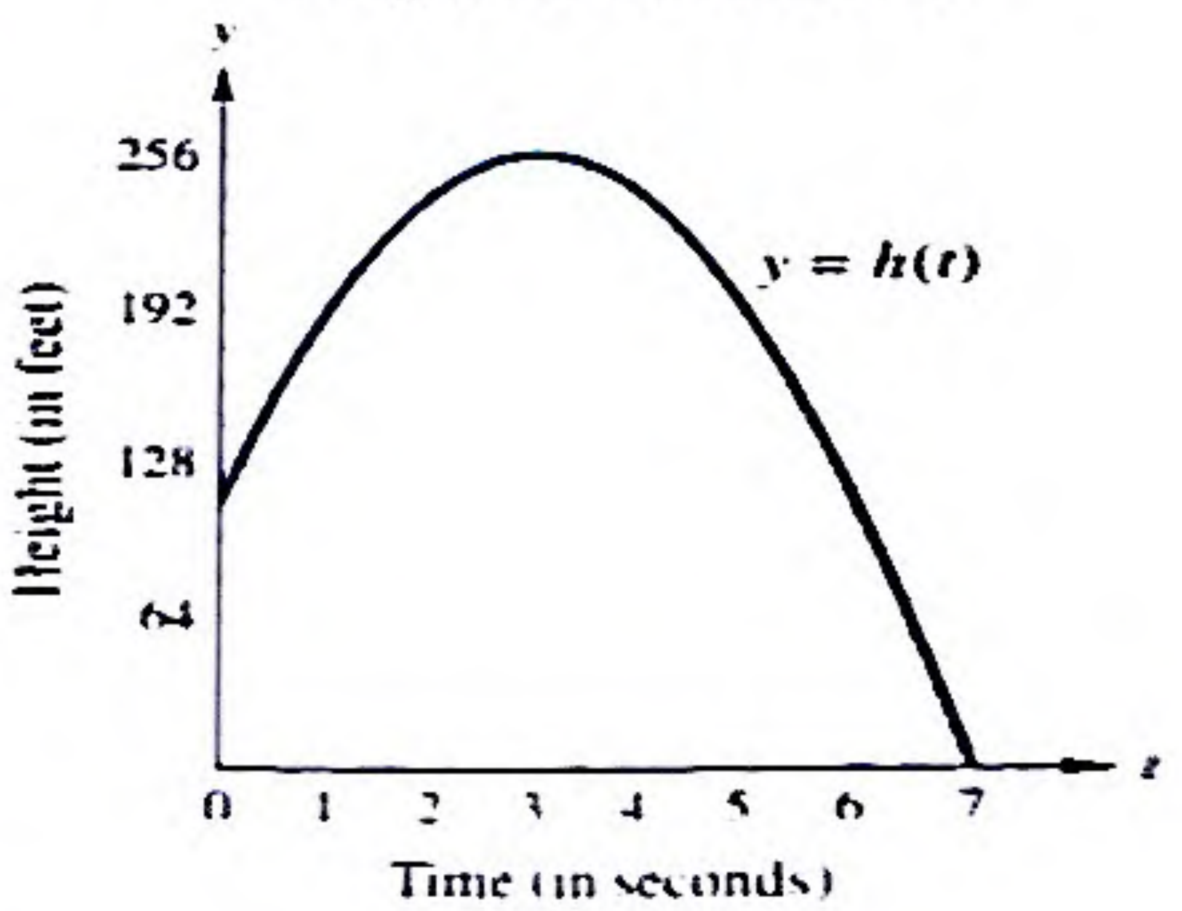


Source Sacramento Municipal Utility District

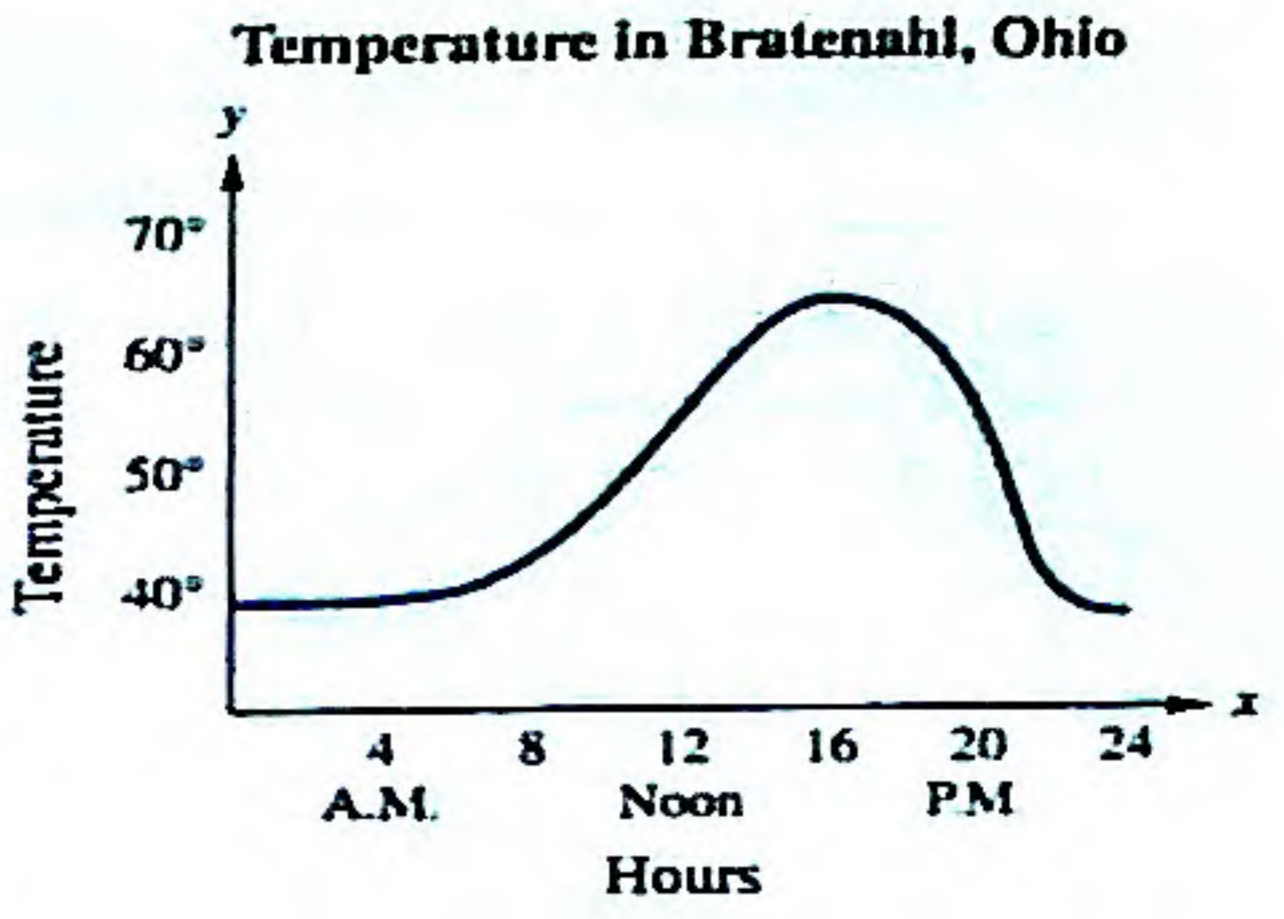
84. ***Height of a Ball*** A ball is thrown straight up into the air. The function defined by $y = h(t)$ in the graph gives the height of the ball (in feet) at t seconds. (*Note:* The graph does *not* show the path of the ball. The ball is rising straight up and then falling straight down.)

(a) What is the height of the ball at 2 sec?
(b) When will the height be 192 ft?
(c) During what time intervals is the ball going up? down?
(d) How high does the ball go, and when does the ball reach its maximum height?
(e) After how many seconds does the ball hit the ground?

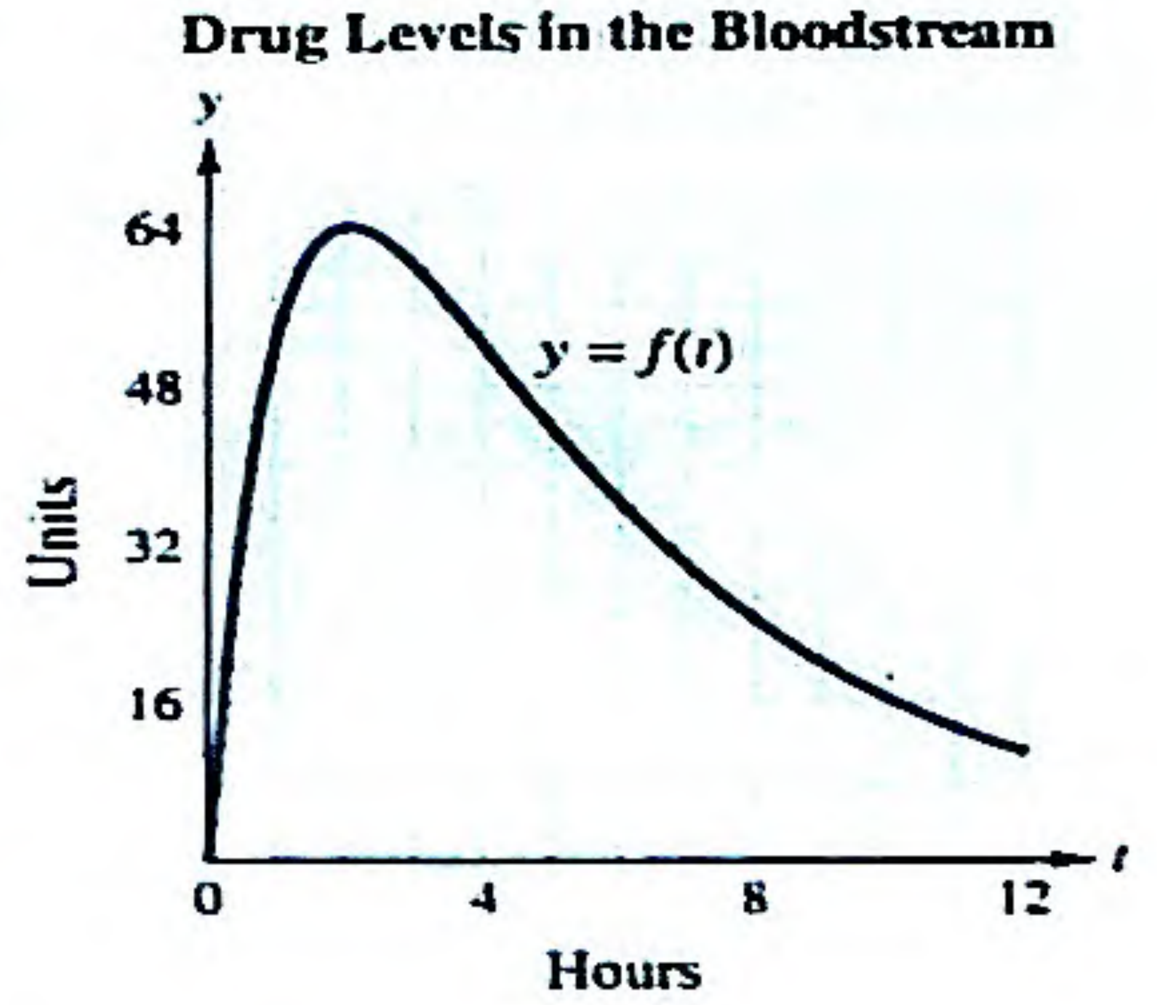
Height of a Thrown Ball



85. ***Temperature*** The graph shows temperatures on a given day in Bratenahl, Ohio.
    (a) At what times during the day was the temperature over 55°?
    (b) When was the temperature below 40°?
    (c) Greenville, South Carolina, is 500 mi south of Bratenahl, Ohio, and its temperature is 7° higher all day long. At what time was the temperature in Greenville the same as the temperature at noon in Bratenahl?
    (d) Use the graph to give a word description of the 24-hr period in Bratenahl.


Temperature in Bratenahl, Ohio

86. ***Drug Levels in the Bloodstream*** When a drug is taken orally, the amount of the drug in the bloodstream after t hours is given by the function defined by $y = f(t)$, as shown in the graph.
    (a) How many units of the drug are in the bloodstream at 8 hr?
    (b) During what time interval is the drug level in the bloodstream increasing? decreasing?
    (c) When does the level of the drug in the bloodstream reach its maximum value, and how many units are in the bloodstream at that time?
    (d) When the drug reaches its maximum level in the bloodstream, how many additional hours are required for the level to drop to 16 units?
    (e) Use the graph to give a word description of the 12-hr period.


Drug Levels in the Bloodstream

2.4 Linear Functions

- **Graphing Linear Functions**
- **Standard Form $Ax + By = C$**
- **Slope**
- **Average Rate of Change**
- **Linear Models**

Graphing Linear Functions We begin our study of specific functions by looking at *linear functions.*

Linear Function

A function f is a **linear function** if

$$f(x) = ax + b,$$

for real numbers a and b. If $a \neq 0$, the domain and the range of a linear function are both $(-\infty, \infty)$.

In **Section 2.1**, we graphed lines by finding ordered pairs and plotting them. Although only two points are necessary to graph a linear function, we usually plot a third point as a check. The intercepts are often good points to choose for graphing lines.

The **revenue function** for selling a product depends on the price per item p and the number of items sold x. It is given by the following function.

$$R(x) = px \qquad \text{Revenue function}$$

Profit is found by subtracting cost from revenue and is described by the **profit function.**

$$P(x) = R(x) - C(x) \qquad \text{Profit function}$$

EXAMPLE 9 Writing Linear Cost, Revenue, and Profit Functions

Assume that the cost to produce an item is a linear function and all items produced are sold. The fixed cost is \$1500, the variable cost per item is \$100, and the item sells for \$125. Write linear functions to model

(a) cost, **(b)** revenue, and **(c)** profit.

(d) How many items must be sold for the company to make a profit?

SOLUTION

(a) Since the cost function is linear, it will have the following form.

$$\begin{aligned} C(x) &= mx + b && \text{Cost function} \\ C(x) &= 100x + 1500 && \text{Let } m = 100 \text{ and } b = 1500. \end{aligned}$$

(b) The revenue function is defined by the product of 125 and x.

$$\begin{aligned} R(x) &= px && \text{Revenue function} \\ R(x) &= 125x && \text{Let } p = 125 \end{aligned}$$

(c) The profit function is found by subtracting the cost function from the revenue function.

$$\begin{aligned} P(x) &= R(x) - C(x) && \text{Use parentheses here.} \\ &= 125x - (100x + 1500) && \\ &= 125x - 100x - 1500 && \text{Distributive property} \\ P(x) &= 25x - 1500 && \text{Combine like terms} \end{aligned}$$

ALGEBRAIC SOLUTION

(d) To make a profit, $P(x)$ must be positive.

$$P(x) = 25x - 1500 \qquad \text{Profit function from part (c)}$$

Set $P(x) > 0$ and solve.

$$\begin{aligned} P(x) &> 0 && \\ 25x - 1500 &> 0 && P(x) = 25x - 1500 \\ 25x &> 1500 && \text{Add 1500 to each side (Section 1.7)} \\ x &> 60 && \text{Divide by 25} \end{aligned}$$

Since the number of items must be a whole number, at least 61 items must be sold for the company to make a profit.

GRAPHING CALCULATOR SOLUTION

(d) Define Y_1 as $25X - 1500$ and graph the line. Use the capability of your calculator to locate the x-intercept. See **Figure 42.** As the graph shows, y-values for x less than 60 are negative, and y-values for x greater than 60 are positive, so at least 61 items must be sold for the company to make a profit.

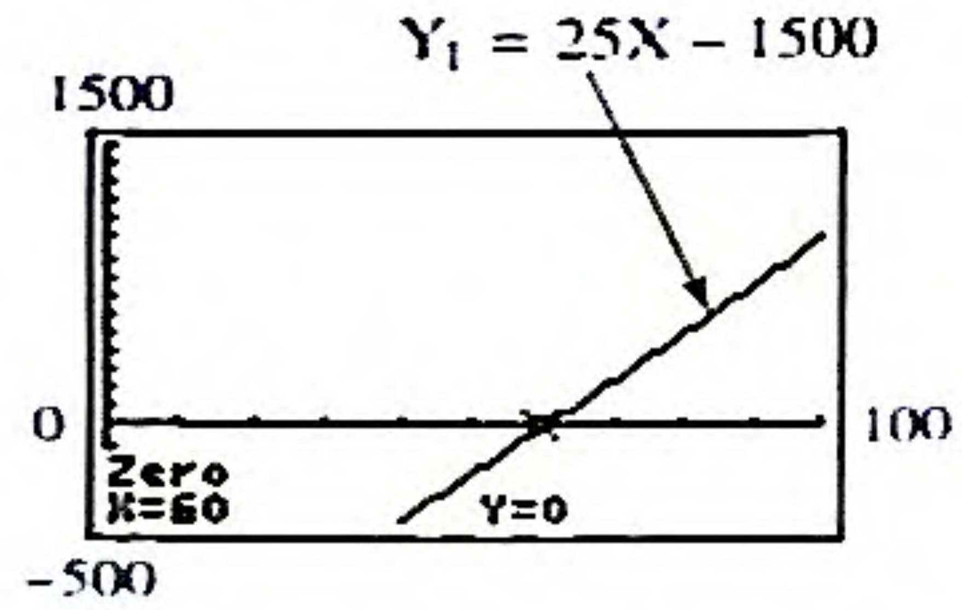


Figure 42

✔ *Now Try Exercise 87.*

2.4 Exercises

Concept Check *Match the description in Column I with the correct response in Column II. Some choices may not be used.*

I	II
B **1.** a linear function whose graph has y-intercept 6	~~**A.**~~ $f(x) = 5x$
X **2.** a vertical line	~~**B.**~~ $f(x) = 3x + 6$
C **3.** a constant function	~~**C.**~~ $f(x) = -8$
G **4.** a linear function whose graph has x-intercept -2 and y-intercept 4	~~**D.**~~ $f(x) = x^2$
A **5.** a linear function whose graph passes through the origin (0, 0)	**E.** $x + y = -6$
D **6.** a function that is not linear	~~**F.**~~ $f(x) = 3x + 4$
	~~**G.**~~ $2x - y = -4$
	~~**H.**~~ $x = 9$

Graph each linear function. Identify any constant functions. Give the domain and range. ***See Examples 1 and 2.***

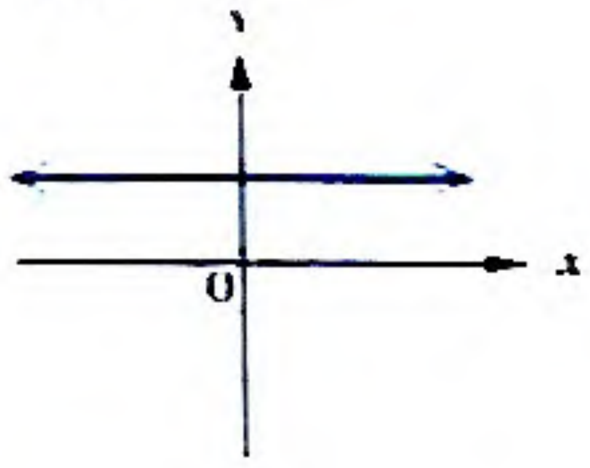
7. $f(x) = x - 4$
8. $f(x) = -x + 4$
9. $f(x) = \frac{1}{2}x - 6$
10. $f(x) = \frac{2}{3}x + 2$
11. $f(x) = 3x$
12. $f(x) = -2x$
13. $f(x) = -4$
14. $f(x) = 3$

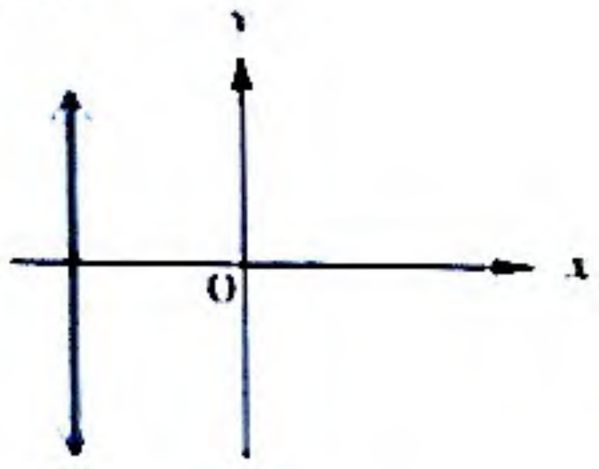
Graph each line. Give the domain and range. ***See Examples 3 and 4.***

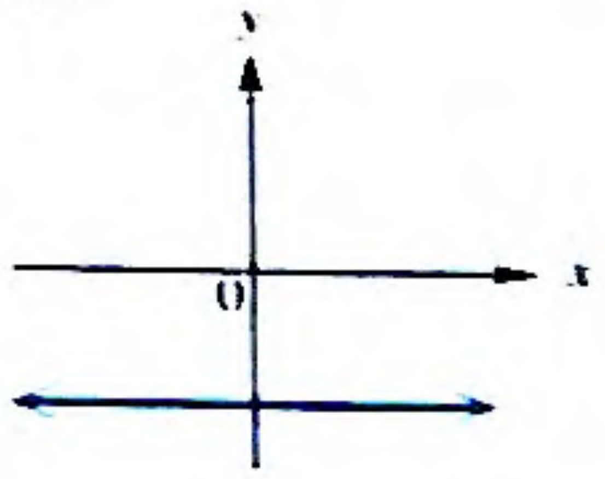
15. $-4x + 3y = 12$
16. $2x + 5y = 10$
17. $3y - 4x = 0$
18. $3x + 2y = 0$
19. $x = 3$
20. $x = -4$
21. $2x + 4 = 0$ $2x = -4$ $x = -2$
22. $-3x + 6 = 0$
23. $-x + 5 = 0$
24. $3 + x = 0$

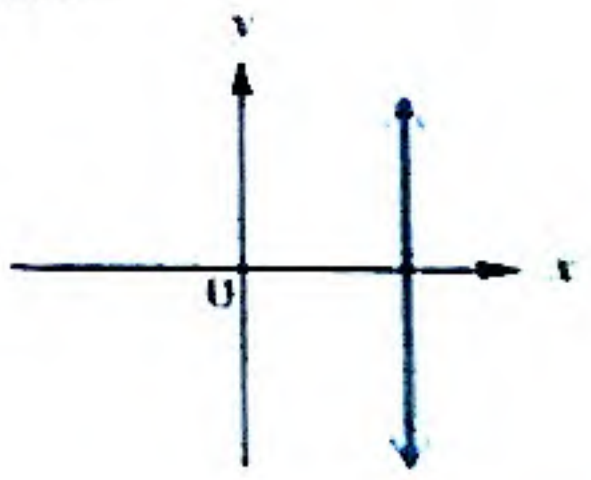
Match each equation with the sketch that most closely resembles its graph. ***See Examples 2 and 3.***

25. $y = 5$
26. $y = -5$
27. $x = 5$
28. $x = -5$

A.


B.


C.


D.


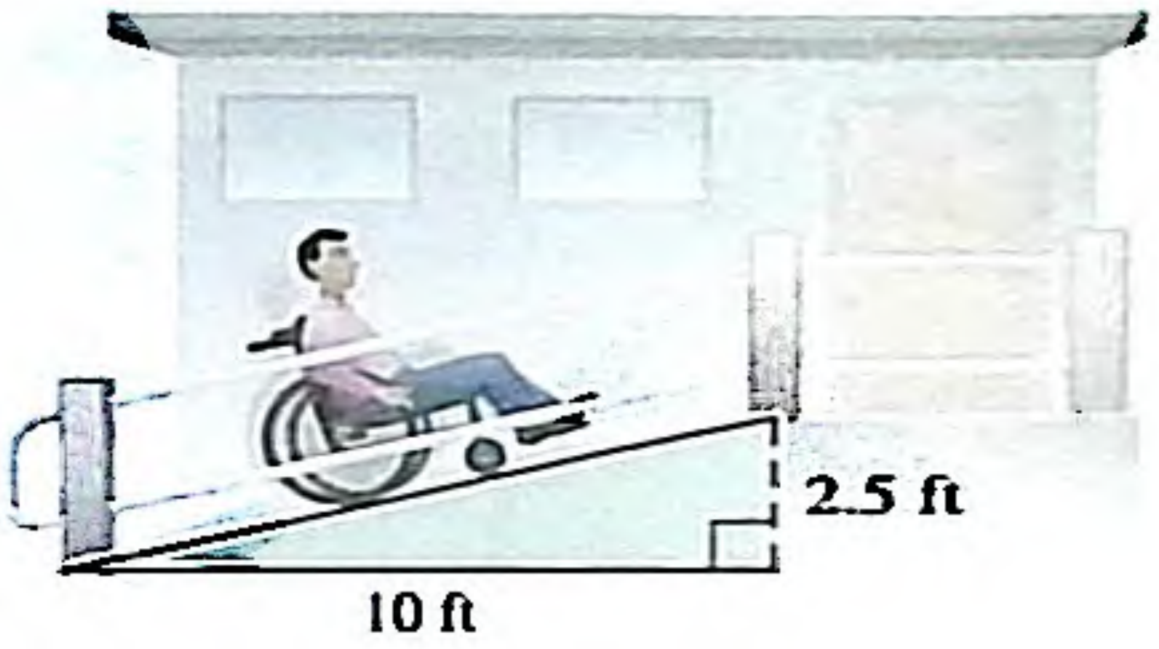
Use a graphing calculator to graph each equation in the standard viewing window. ***See Examples 1, 2, and 4.***

29. $y = 3x + 4$
30. $y = -2x + 3$
31. $3x + 4y = 6$
32. $-2x + 5y = 10$

33. ***Concept Check*** If a walkway rises 2.5 ft for every 10 ft on the horizontal, which of the following express its slope (or grade)? (There are several correct choices.)

A. 0.25 **B.** 4 **C.** $\frac{2.5}{10}$ **D.** 25%

E. $\frac{1}{4}$ **F.** $\frac{10}{2.5}$ **G.** 400% **H.** 2.5%



34. ***Concept Check*** If the pitch of a roof is $\frac{1}{4}$, how many feet in the horizontal direction correspond to a rise of 4 ft?

Find the slope of the line satisfying the given conditions. See Example 5.

35. through $(2, -1)$ and $(-3, -3)$

36. through $(-3, 4)$ and $(2, -8)$

37. through $(5, 8)$ and $(3, 12)$

38. through $(5, -3)$ and $(1, -7)$

39. through $(5, 9)$ and $(-2, 9)$

40. through $(-2, 4)$ and $(6, 4)$

41. horizontal, through $(5, 1)$

42. horizontal, through $(3, 5)$

43. vertical, through $(4, -7)$

44. vertical, through $(-8, 5)$

For each line, (a) find the slope and (b) sketch the graph. See Examples 6 and 7.

45. $y = 3x + 5$

46. $y = 2x - 4$

47. $2y = -3x$

48. $-4y = 5x$

49. $5x - 2y = 10$

50. $4x + 3y = 12$

Graph the line passing through the given point and having the indicated slope. Plot two points on the line. See Example 7.

51. through $(-1, 3)$, $m = \frac{3}{2}$

52. through $(-2, 8)$, $m = \frac{2}{5}$

53. through $(3, -4)$, $m = -\frac{1}{3}$

54. through $(-2, -3)$, $m = -\frac{3}{4}$

55. through $\left(-\frac{1}{2}, 4\right)$, $m = 0$

56. through $\left(\frac{3}{2}, 2\right)$, $m = 0$

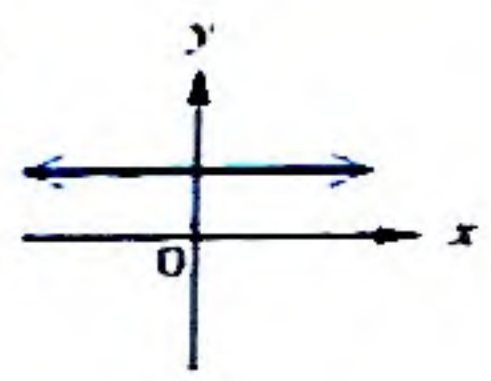
57. through $\left(-\frac{5}{2}, 3\right)$, undefined slope

58. through $\left(\frac{9}{4}, 2\right)$, undefined slope

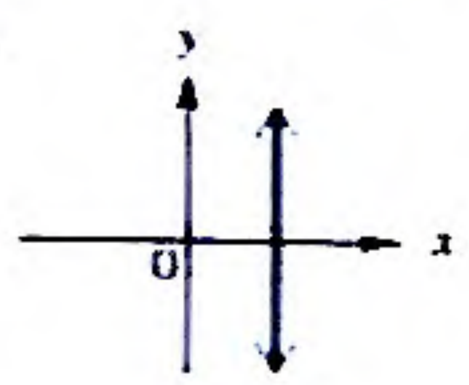
Concept Check *For each given slope in Exercises 59–64, identify the line in A–F that could have this slope.*

59. $\frac{1}{3}$ **60.** -3 **61.** 0 **62.** $-\frac{1}{3}$ **63.** 3 **64.** undefined

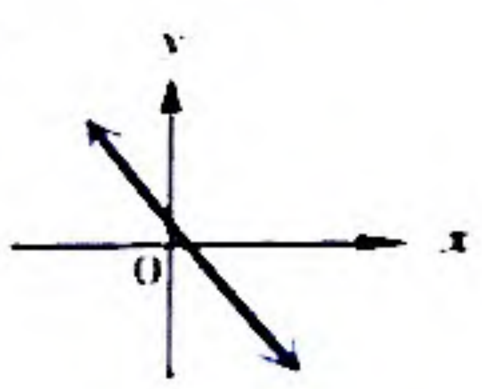
A.



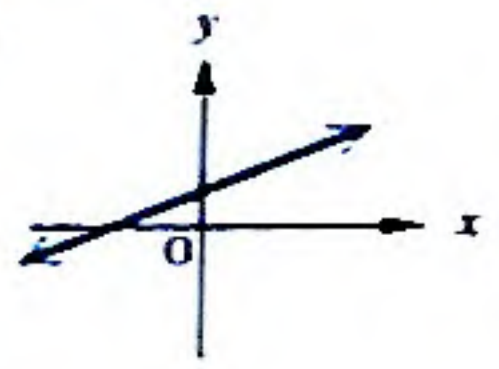
B.



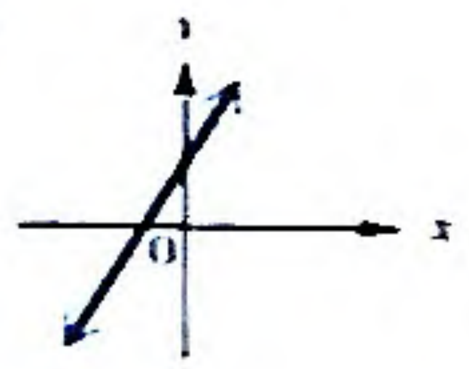
C.



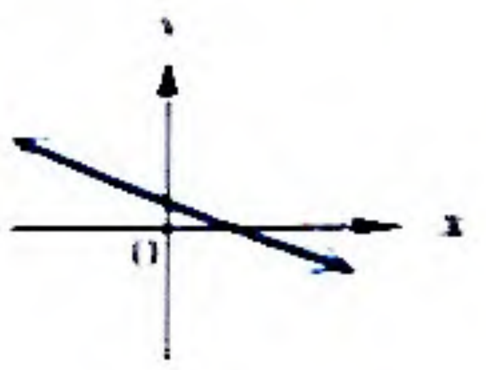
D.



E.

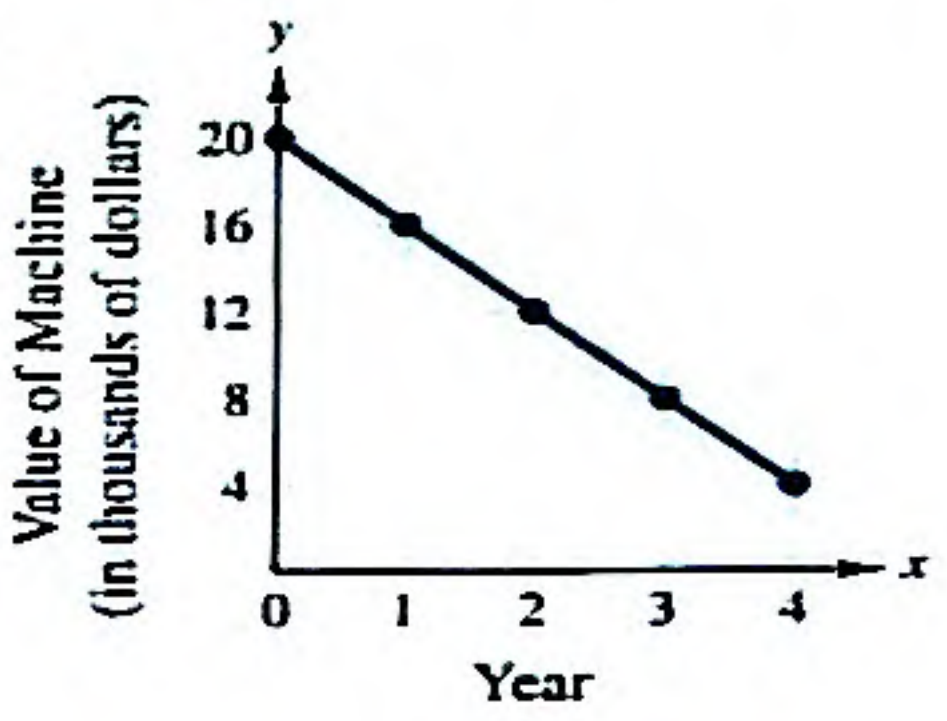


F.

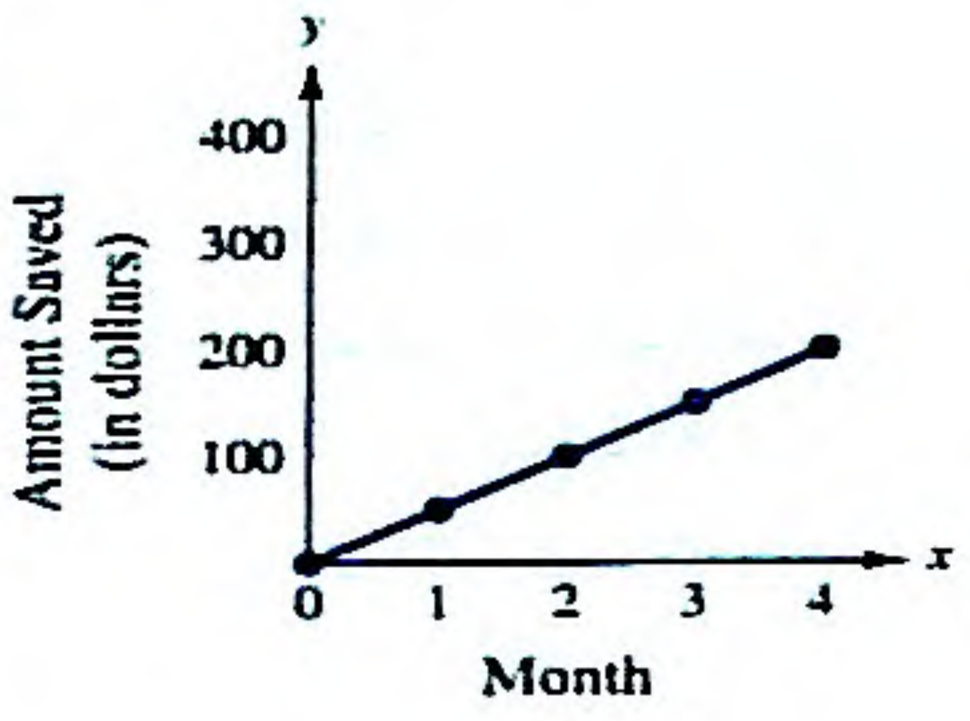


Concept Check Find and interpret the average rate of change illustrated in each graph.

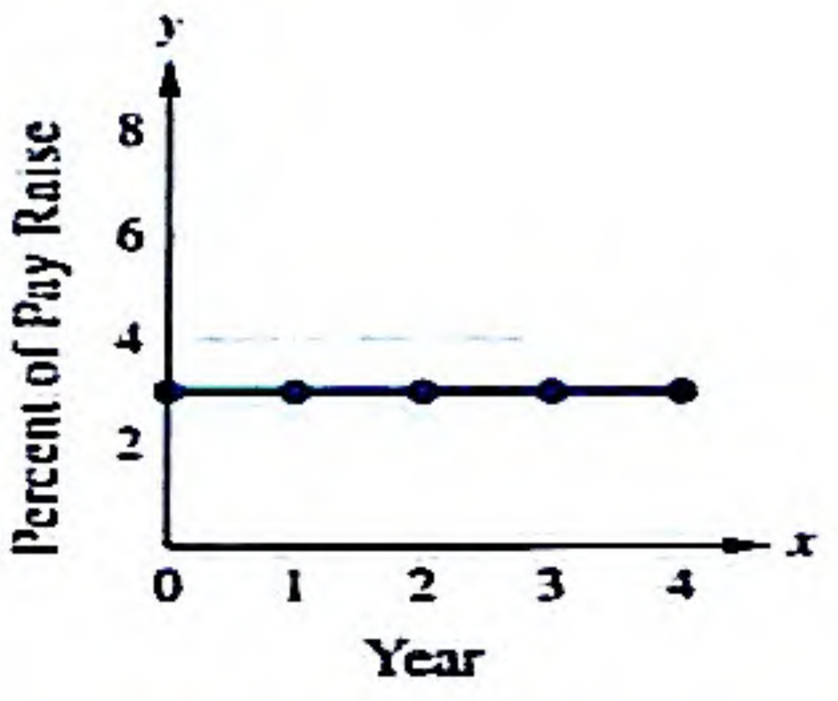
65.



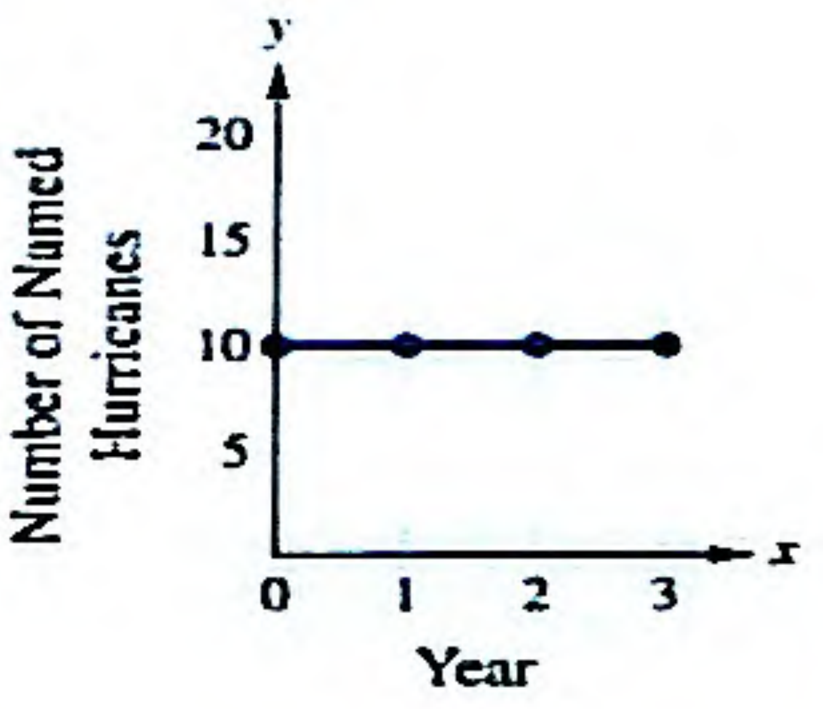
66.



67.



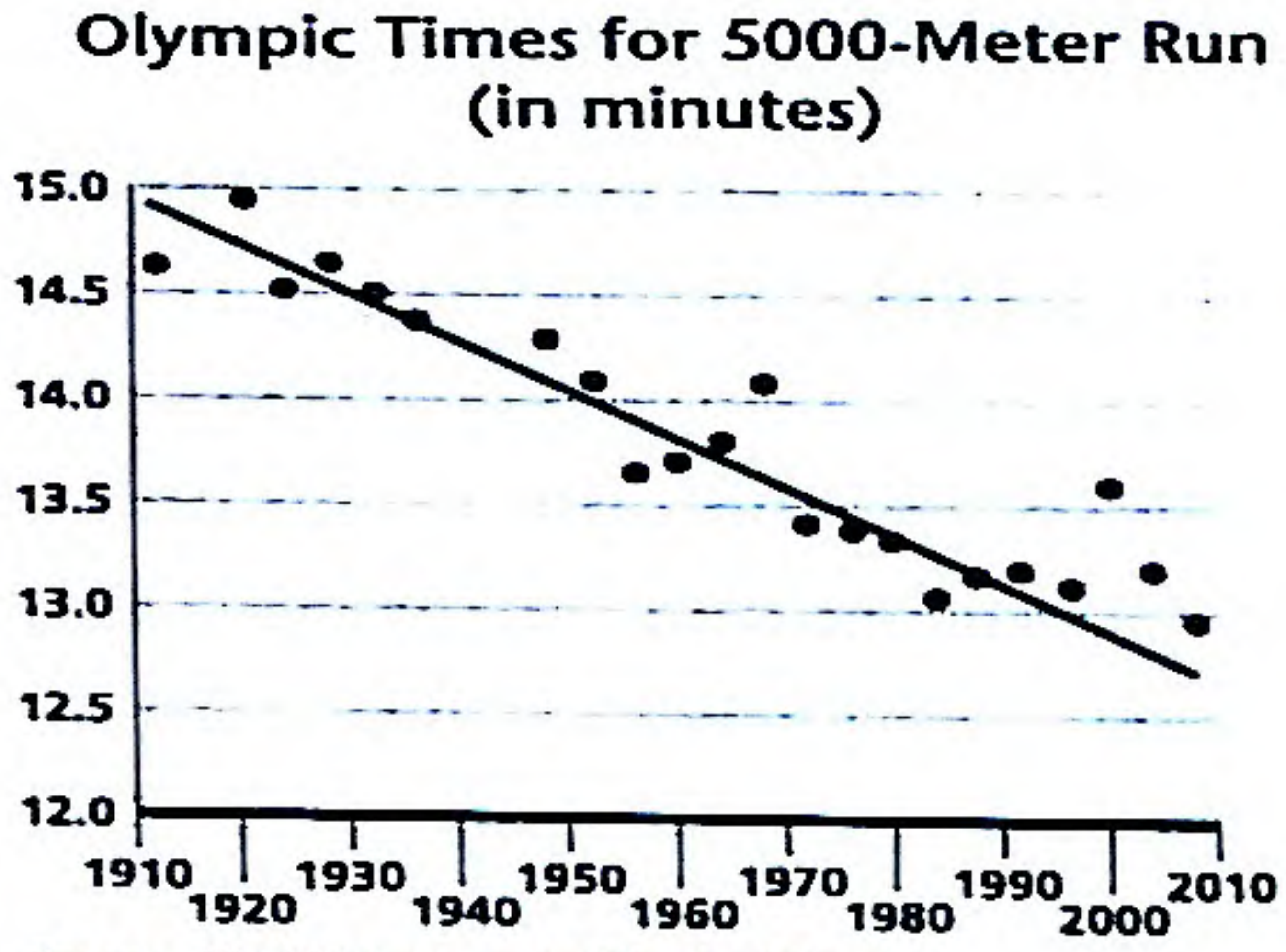
68.



69. *Concept Check* For a constant function, is the average rate of change positive, negative, or zero? (Choose one.)

Solve each problem. See Example 8.

70. *(Modeling) Olympic Times for 5000-Meter Run* The graph shows the winning times (in minutes) at the Olympic Games for the men's 5000-m run, together with a linear approximation of the data.



Source: World Almanac and Book of Facts.

(a) An equation for the linear model, based on data from 1912–2008 (where x represents the year), is

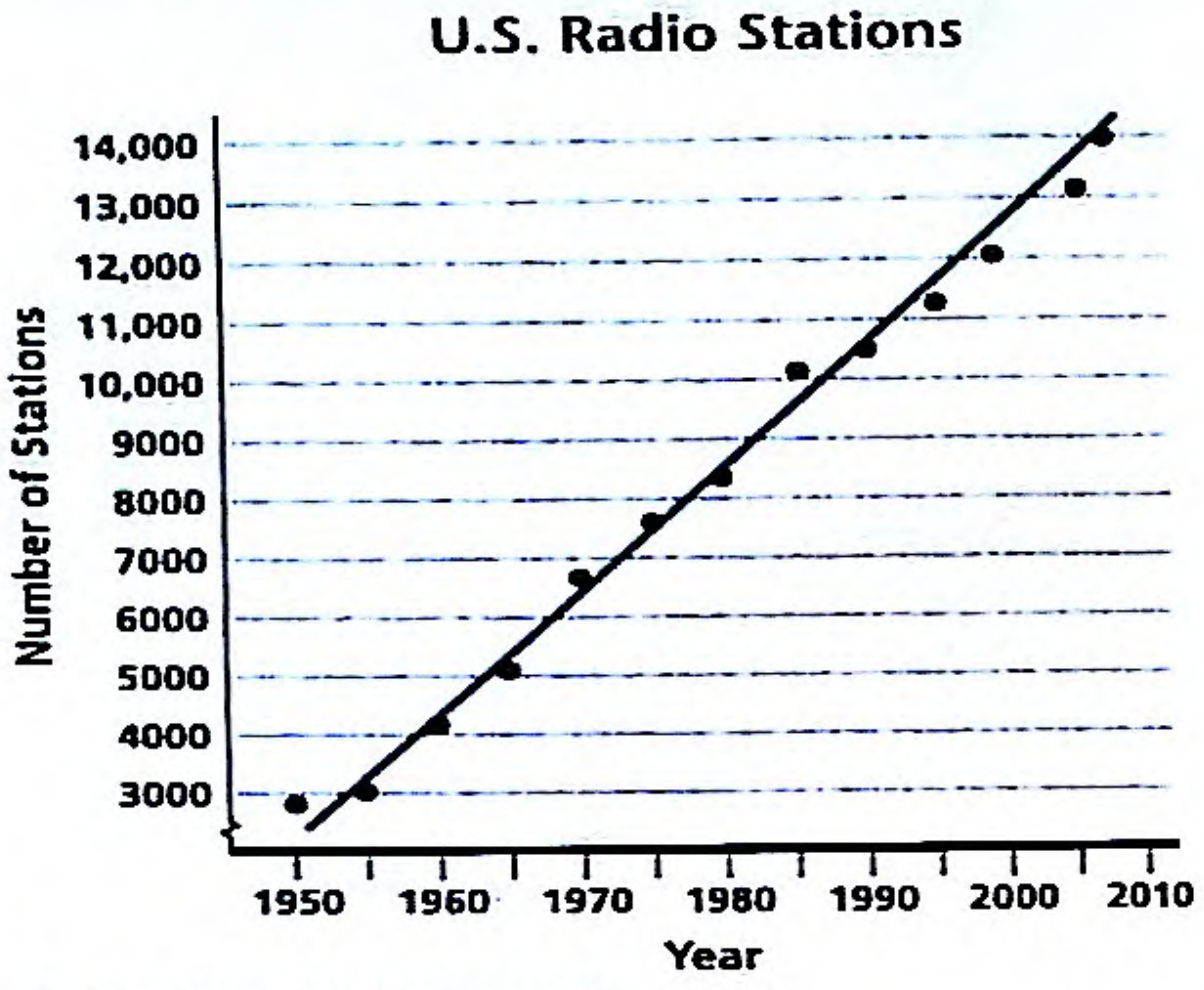
$$y = -0.0193x + 51.73.$$

Determine the slope. (**See Example 6.**) What does the slope of this line represent? Why is the slope negative?

(b) Can you think of any reason why there are no data points for the years 1916, 1940, and 1944?

(c) The winning time for the 1996 Olympic Games was 13.13 min. What does the model predict? How far is the prediction from the actual value?

71. ***(Modeling) U.S. Radio Stations*** The graph shows the number of U.S. radio stations on the air, along with the graph of a linear function that models the data.



Source National Association of Broadcasters.

(a) Discuss the predictive accuracy of the linear function.

(b) Use the two data points (1950, 2773) and (2007, 13,977) to find the approximate slope of the line shown. Interpret this number.

72. ***Cellular Telephone Subscribers*** The table gives the number of cellular telephone subscribers in the U.S. (in thousands) from 2004 through 2009.

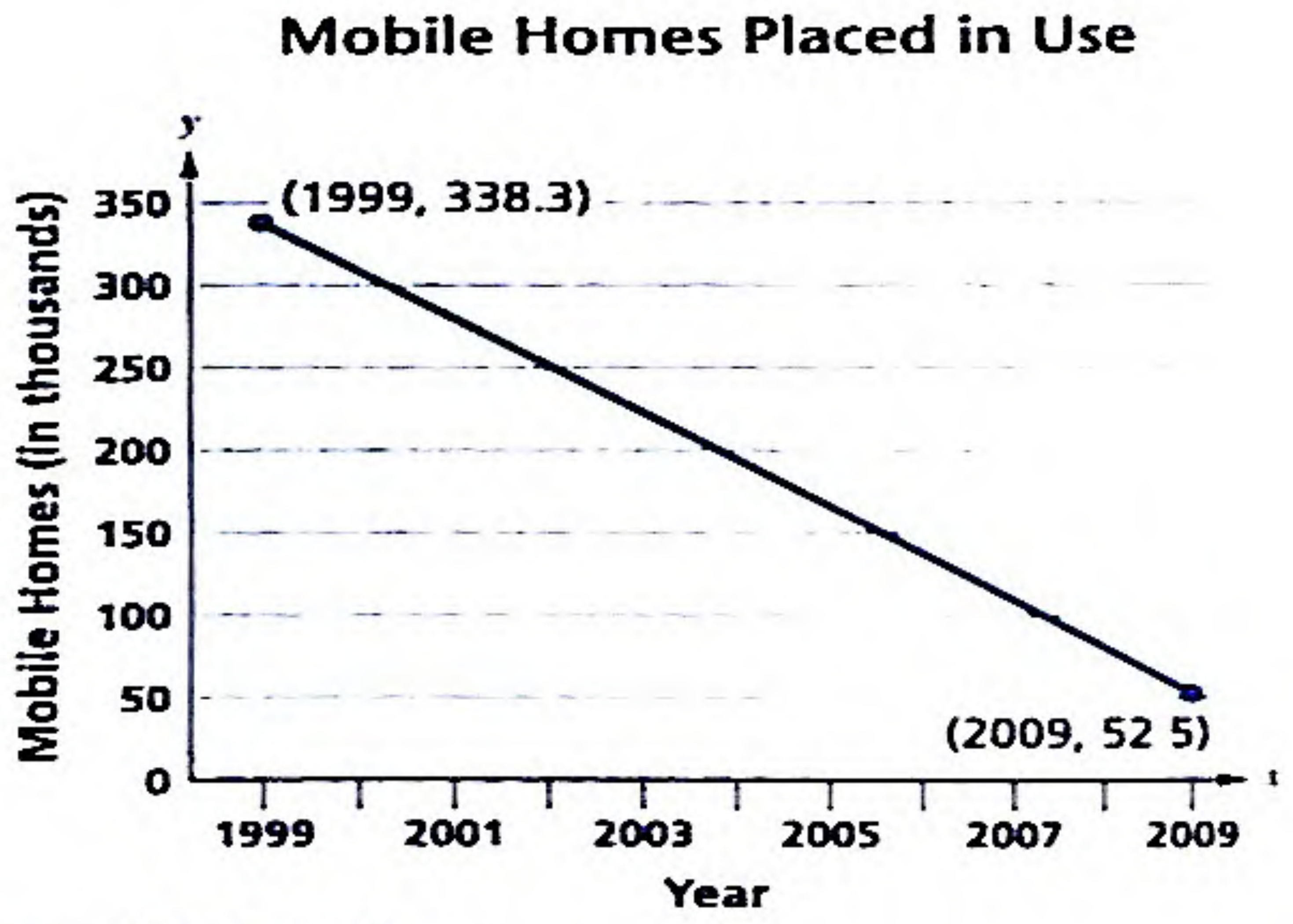
(a) Find the change in subscribers for 2004–2005, 2005–2006, and so on.

(b) Are the changes in successive years approximately the same? If the ordered pairs in the table were plotted, could an approximately straight line be drawn through them?

Year	Subscribers (in thousands)
2004	182,140
2005	207,896
2006	233,041
2007	255,396
2008	270,334
2009	285,646

Source. CTIA-The Wireless Association.

73. ***Mobile Homes*** The graph provides a good approximation of the number of mobile homes (in thousands) placed in use in the United States from 1999 through 2009.



Source U.S. Census Bureau

(a) Use the given ordered pairs to find the average rate of change in the number of mobile homes per year during this period.
(b) Interpret what a negative slope means in this situation.

74. ***Olympic Times for 5000-Meter Run*** Exercise 70 showed the winning times for the Olympic men's 5000-m run. Find and interpret the average rate of change for the following periods.
(a) The winning time in 1912 was 14.6 min. In 1996 it was 13.1 min.
(b) The winning time in 1912 was 14.6 min. In 2008 it was 13.0 min.

75. ***High School Dropouts*** In 1980, the number of high school dropouts in the United States was 5212 thousand. By 2008, this number had decreased to 3118 thousand. Find and interpret the average rate of change per year in the number of high school dropouts. (*Source:* U.S. Census Bureau.)

76. ***Plasma Flat-Panel TV Sales*** The total amount spent on plasma flat-panel TVs in the United States changed from $5302 million in 2006 to $2907 million in 2009. Find and interpret the average rate of change in sales, in millions of dollars per year. Round your answer to the nearest hundredth. (*Source:* Consumer Electronics Association.)

Relating Concepts

For individual or collaborative investigation *(Exercises 77–86)*

The table shows several points on the graph of a linear function. **Work Exercises 77–86 in order,** *to see connections between the slope formula, the distance formula, the midpoint formula, and linear functions.*

x	y
0	−6
1	−3
2	0
3	3
4	6
5	9
6	12

77. Use the first two points in the table to find the slope of the line.

78. Use the second and third points in the table to find the slope of the line.

79. Make a conjecture by filling in the blank: If we use any two points on a line to find its slope, we find that the slope is ______ in all cases.

80. Find the distance between the first two points in the table. (*Hint:* Use the distance formula.)

81. Find the distance between the second and fourth points in the table.

82. Find the distance between the first and fourth points in the table.

83. Add the results in **Exercises 80 and 81,** and compare the sum to the answer you found in **Exercise 82.** What do you notice?

84. Fill in the following blanks, basing your answers on your observations in **Exercises 80–83:** If points A, B, and C lie on a line in that order, then the distance between A and B added to the distance between ______ and ______ is equal to the distance between ______ and ______.

85. Find the midpoint of the segment joining $(0, -6)$ and $(6, 12)$. Compare your answer to the middle entry in the table. What do you notice?

86. If the table were set up to show an x-value of 4.5, what would be the corresponding y-value?

(Modeling) Cost, Revenue, and Profit Analysis *A firm will break even (no profit and no loss) as long as revenue just equals cost. The value of x (the number of items produced and sold) where $C(x) = R(x)$ is called the* ***break-even point****. Assume that each of the following can be expressed as a linear function. Find*

(a) *the cost function,* ***(b)*** *the revenue function, and* ***(c)*** *the profit function.*

(d) *Find the break-even point and decide whether the product should be produced, given the restrictions on sales.*

See Example 9.

	Fixed Cost	Variable Cost	Price of Item	
87.	\$ 500	\$ 10	\$ 35	No more than 18 units can be sold.
88.	\$2700	\$150	\$280	No more than 25 units can be sold.
89.	\$1650	\$400	\$305	All units produced can be sold.
90.	\$ 180	\$ 11	\$ 20	No more than 30 units can be sold.

(Modeling) Break-Even Point *The manager of a small company that produces roof tile has determined that the total cost in dollars, $C(x)$, of producing x units of tile is given by*

$$C(x) = 200x + 1000,$$

while the revenue in dollars, $R(x)$, from the sale of x units of tile is given by

$$R(x) = 240x.$$

91. Find the break-even point and the cost and revenue at the break-even point.

92. Suppose the variable cost is actually \$220 per unit, instead of \$200. How does this affect the break-even point? Is the manager better off or not?

Chapter 2 Quiz (Sections 2.1–2.4)

1. For $A(-4, 2)$ and $B(-8, -3)$, find $d(A, B)$, the distance between A and B.

2. *Two-Year College Enrollment* Enrollments in two-year colleges for selected years are shown in the table. Use the midpoint formula to estimate the enrollments for 2002 and 2006.

Year	Enrollment (in millions)
2000	5.95
2004	6.55
2008	6.97

Source: U.S. Center for Education Statistics.

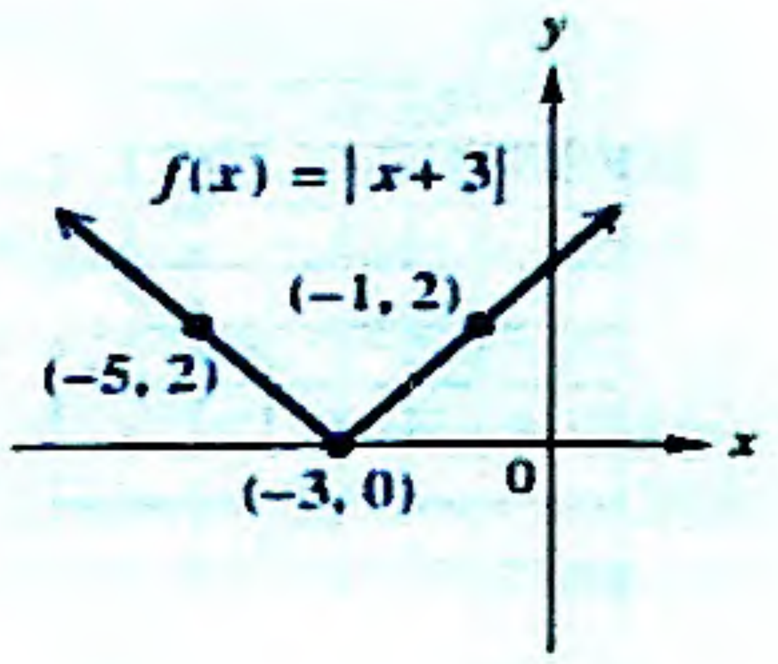
3. Sketch the graph of $y = -x^2 + 4$ by plotting points.

4. Sketch the graph of $x^2 + y^2 = 16$.

5. Determine the radius and the coordinates of the center of the circle with equation

$$x^2 + y^2 - 4x + 8y + 3 = 0.$$

For Exercises 6–8, refer to the graph of $f(x) = |x + 3|$.



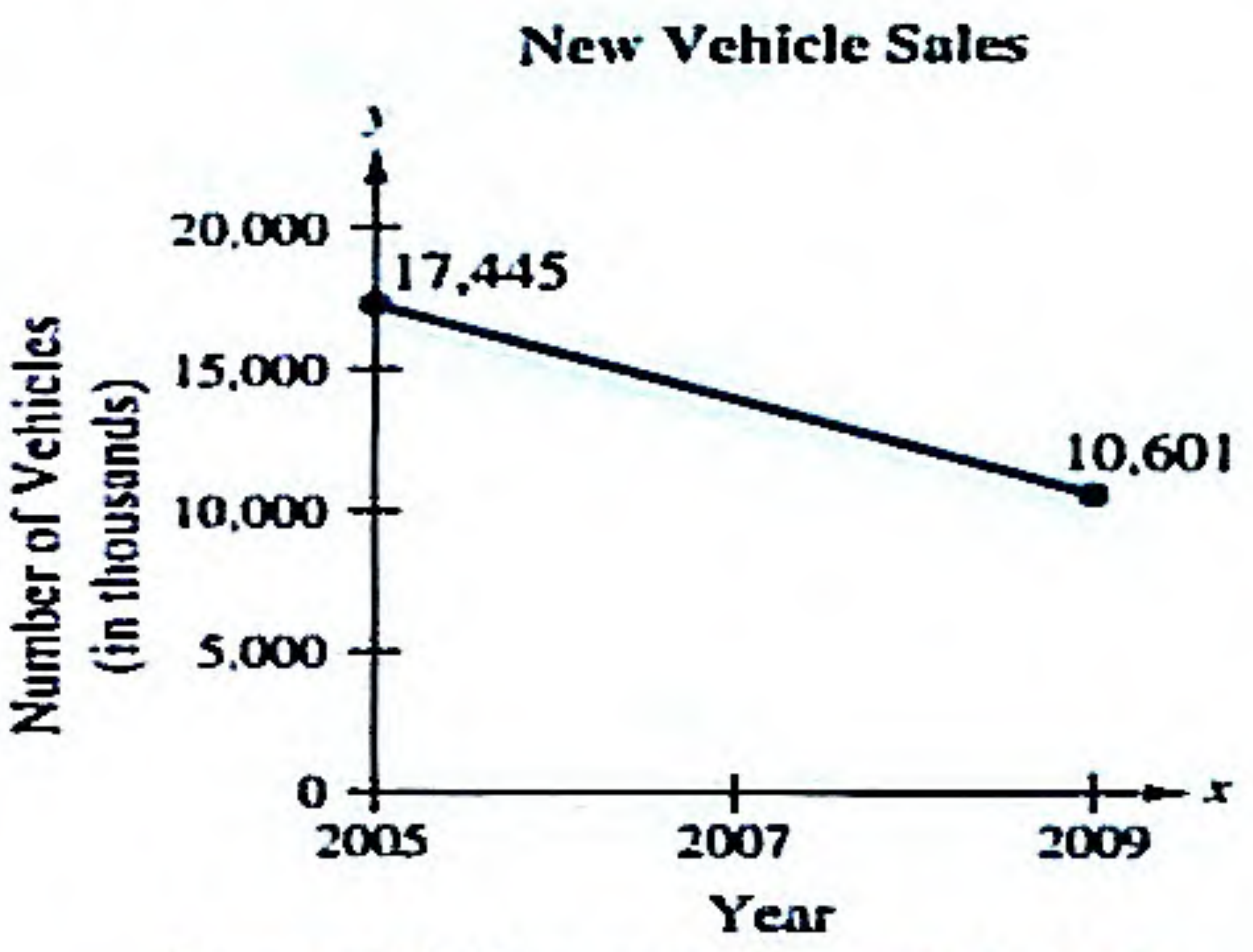
6. Find $f(-1)$.

7. Give the domain and the range of f.

8. Give the largest interval over which the function f is (a) decreasing, (b) increasing, (c) constant.

9. Find the slope of the line through the given points.
(a) $(1, 5)$ and $(5, 11)$ (b) $(-7, 4)$ and $(-1, 4)$ (c) $(6, 12)$ and $(6, -4)$

10. *Motor Vehicle Sales* The graph shows a straight line segment that approximates new motor vehicle sales in the United States from 2005 to 2009. Determine the average rate of change from 2005 to 2009, and interpret the results.



Source: U.S. Bureau of Economic Analysis

2.5 Equations of Lines and Linear Models

- **Point-Slope Form**
- **Slope-Intercept Form**
- **Vertical and Horizontal Lines**
- **Parallel and Perpendicular Lines**
- **Modeling Data**
- **Solving Linear Equations in One Variable by Graphing**

Point-Slope Form The graph of a linear function is a straight line. We now develop various forms for the equation of a line.

Figure 43 shows the line passing through the fixed point (x_1, y_1) having slope m. (Assuming that the line has a slope guarantees that it is not vertical.) Let (x, y) be any other point on the line. Since the line is not vertical, $x - x_1 \neq 0$. Now use the definition of slope.

$$m = \frac{y - y_1}{x - x_1} \qquad \text{Slope formula (Section 2.4)}$$

$$m(x - x_1) = y - y_1 \qquad \text{Multiply each side by } x - x_1.$$

or

$$y - y_1 = m(x - x_1) \qquad \text{Interchange sides}$$

This result is the *point-slope form* of the equation of a line.

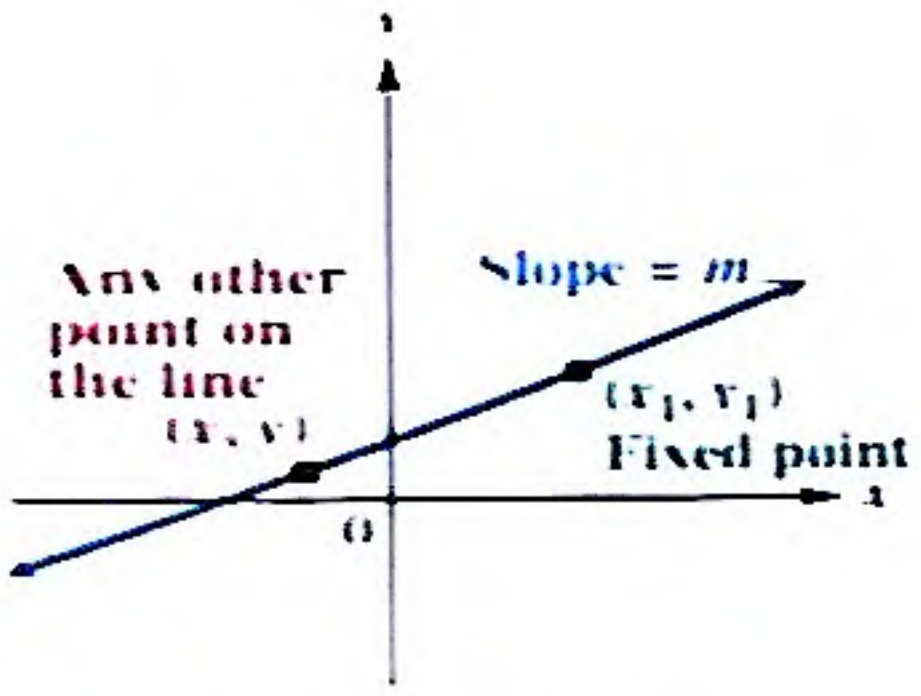


Figure 43

Point-Slope Form

The **point-slope form** of the equation of the line with slope m passing through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1).$$

2.5 Exercises

Concept Check *Match each equation in Exercises 1–4 to the correct graph in A–D.*

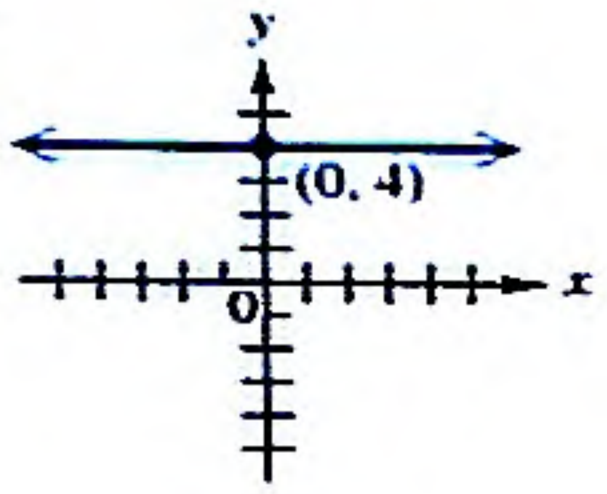
1. $y = \frac{1}{4}x + 2$

2. $4x + 3y = 12$

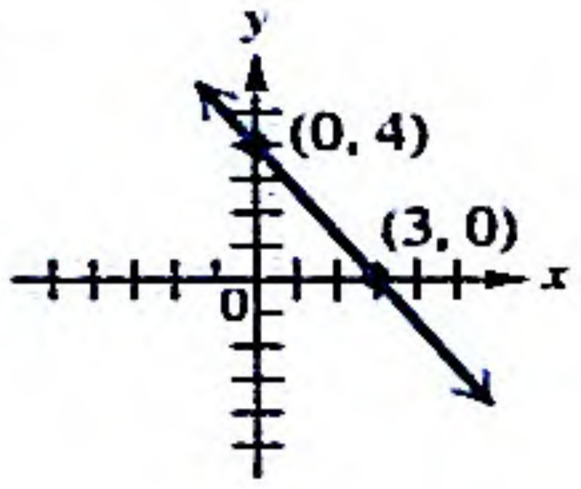
3. $y - (-1) = \frac{3}{2}(x - 1)$

4. $y = 4$

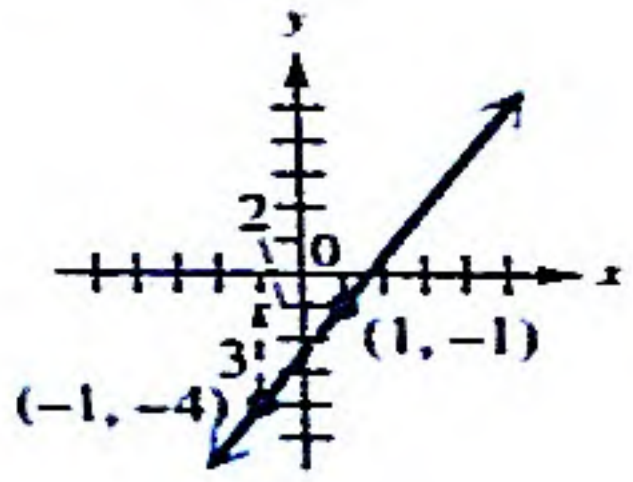
A.



B.



C.



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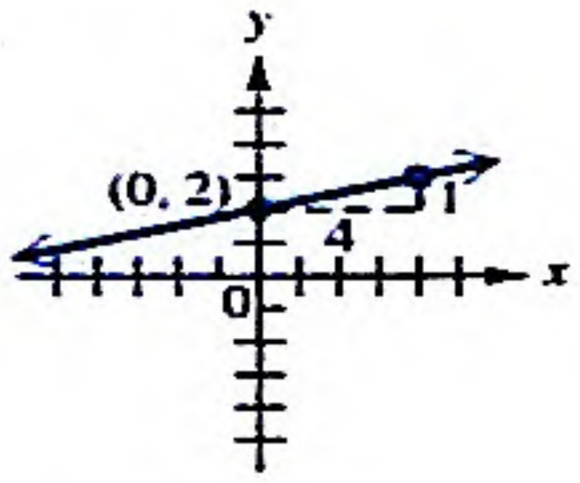


In Exercises 5–26, write an equation for the line described. Give answers in standard form for Exercises 5–14 and in slope-intercept form (if possible) for Exercises 15–26. See Examples 1–4.

5. through $(1, 3)$, $m = -2$

6. through $(2, 4)$, $m = -1$

7. through $(-5, 4)$, $m = -\frac{3}{2}$

8. through $(-4, 3)$, $m = \frac{3}{4}$

9. through $(-8, 4)$, undefined slope

10. through $(5, 1)$, undefined slope

11. through $(5, -8)$, $m = 0$

12. through $(-3, 12)$, $m = 0$

13. through $(-1, 3)$ and $(3, 4)$

14. through $(2, 3)$ and $(-1, 2)$

15. x-intercept 3, y-intercept -2

16. x-intercept -4, y-intercept 3

17. vertical, through $(-6, 4)$

18. vertical, through $(2, 7)$

19. horizontal, through $(-7, 4)$

20. horizontal, through $(-8, -2)$

21. $m = 5$, $b = 15$

22. $m = -2$, $b = 12$

23. through $(-2, 5)$ having slope -4

24. through $(4, -7)$ having slope -2

25. slope 0, y-intercept $\frac{3}{2}$

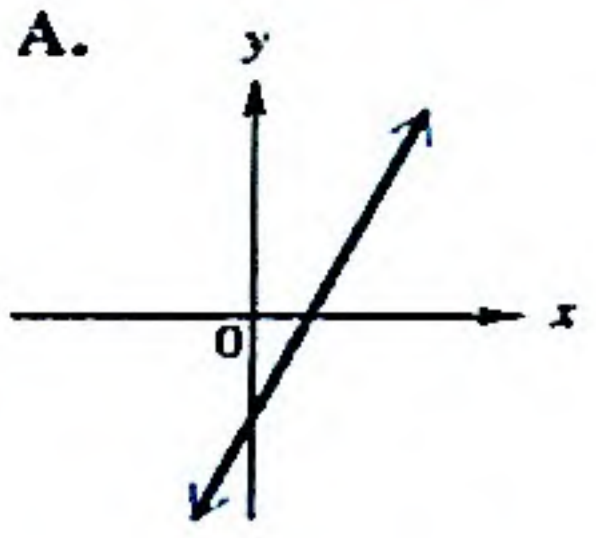
26. slope 0, y-intercept $-\frac{5}{4}$

27. ***Concept Check*** Fill in each blank with the appropriate response. The line $x + 2 = 0$ has x-intercept ______. It ______ (does/does not) have a y-intercept. The slope of this line is ______ (0/undefined). The line $4y = 2$ has y-intercept ______. It ______ (does/does not) have an x-intercept. The slope of this line is ______ (0/undefined).

28. *Concept Check* Match each equation with the line that would most closely resemble its graph. (*Hint:* Consider the signs of m and b in the slope-intercept form.)

(a) $y = 3x + 2$ **(b)** $y = -3x + 2$ **(c)** $y = 3x - 2$ **(d)** $y = -3x - 2$

A.



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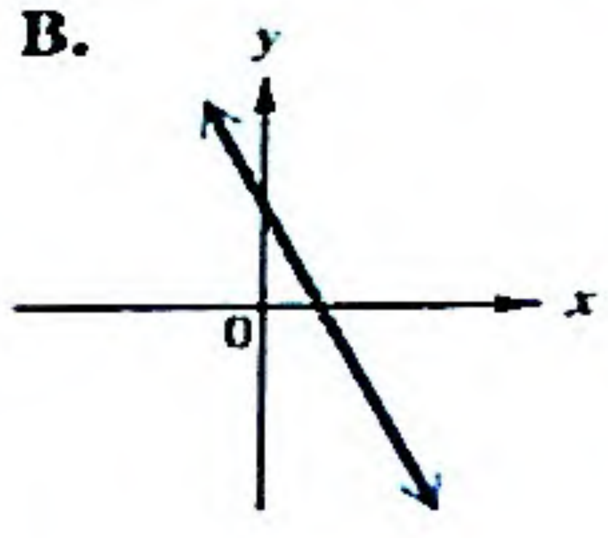

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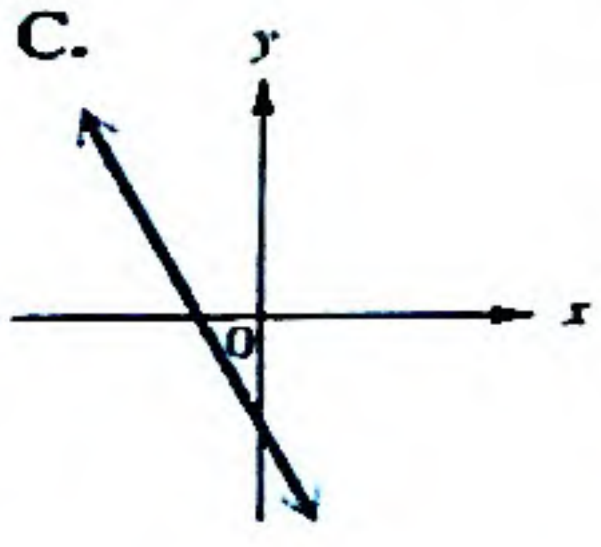

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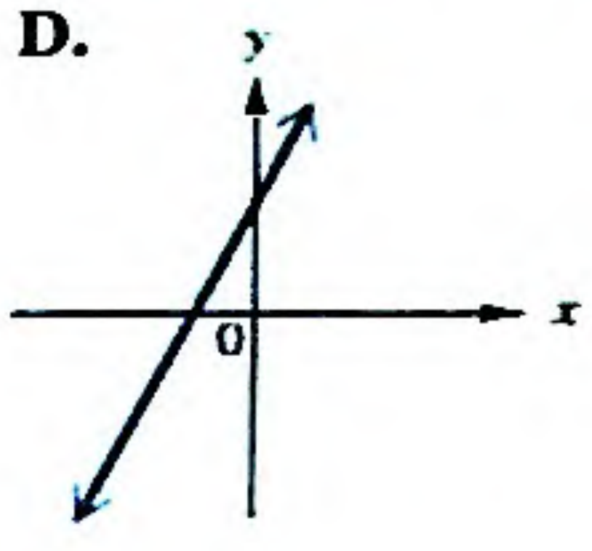


Give the slope and y-intercept of each line, and graph it. See Example 3.

29. $y = 3x - 1$

30. $y = -2x + 7$

31. $4x - y = 7$

32. $2x + 3y = 16$

33. $4y = -3x$

34. $2y = x$

35. $x + 2y = -4$

36. $x + 3y = -9$

37. $y - \frac{3}{2}x - 1 = 0$

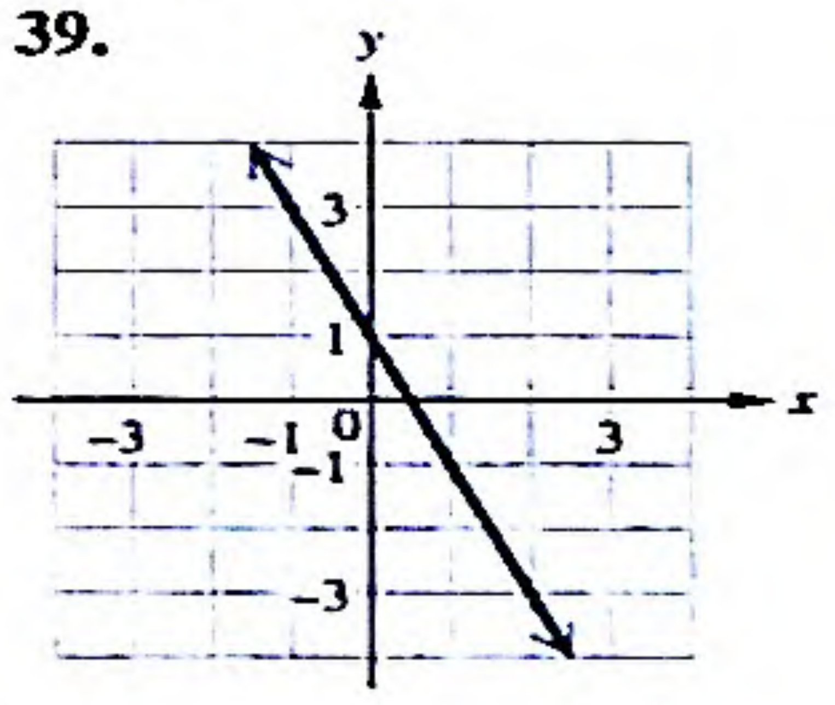
38. *Concept Check* The table represents a linear function f.

(a) Find the slope of the line defined by $y = f(x)$.
(b) Find the y-intercept of the line.
(c) Find the equation for this line in slope-intercept form.

x	y
−2	−11
−1	−8
0	−5
1	−2
2	1
3	4

Connecting Graphs with Equations The graph of a linear function f is shown. (a) Identify the slope, y-intercept, and x-intercept. (b) Write the equation that defines f. See Example 5.

39.



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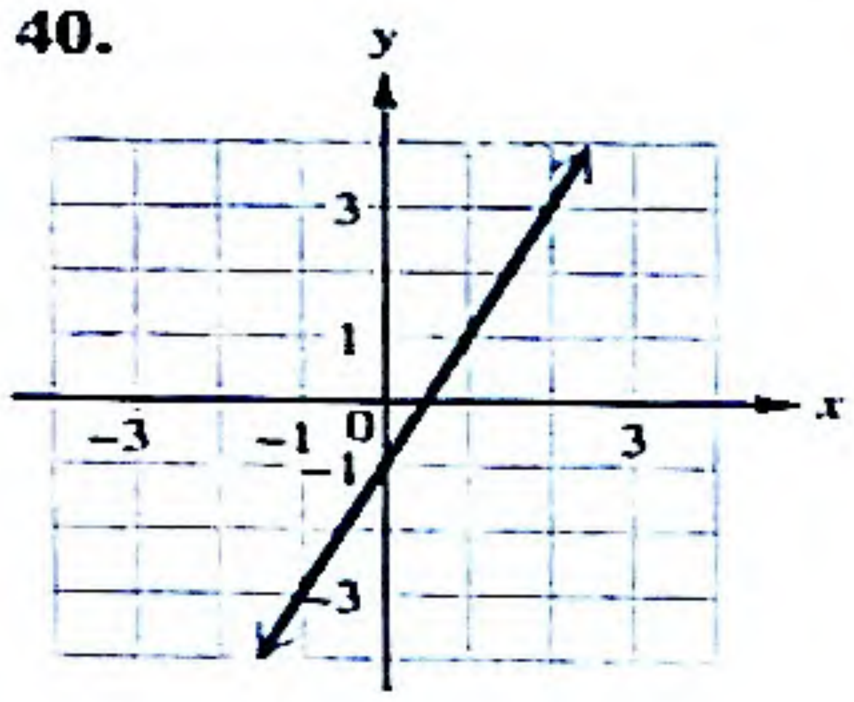

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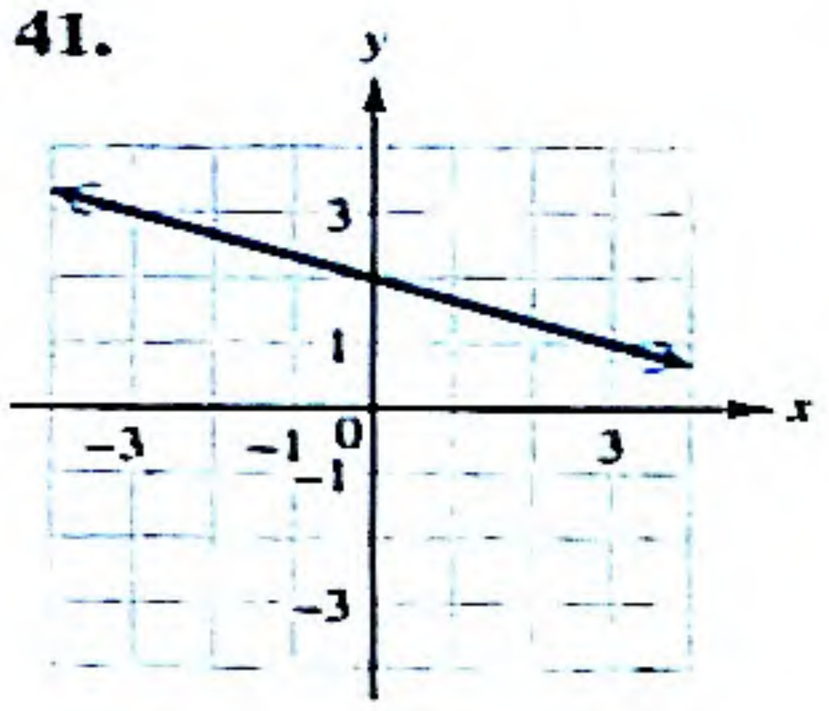

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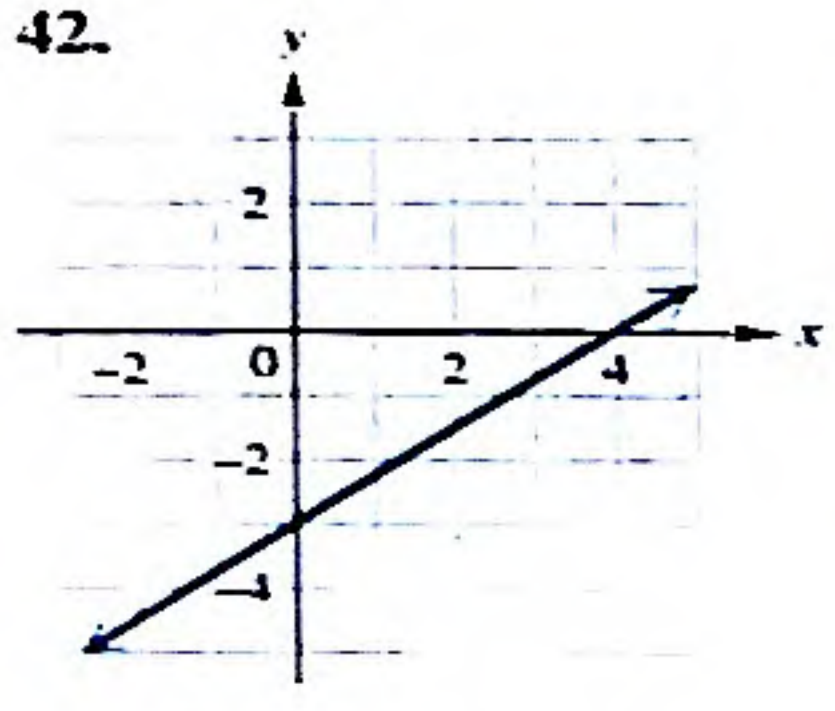

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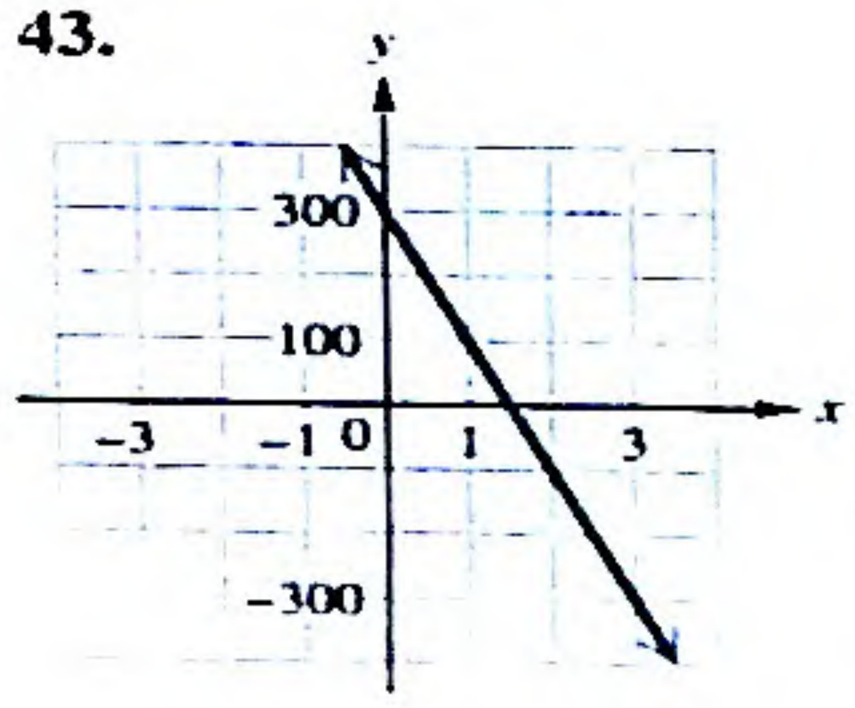

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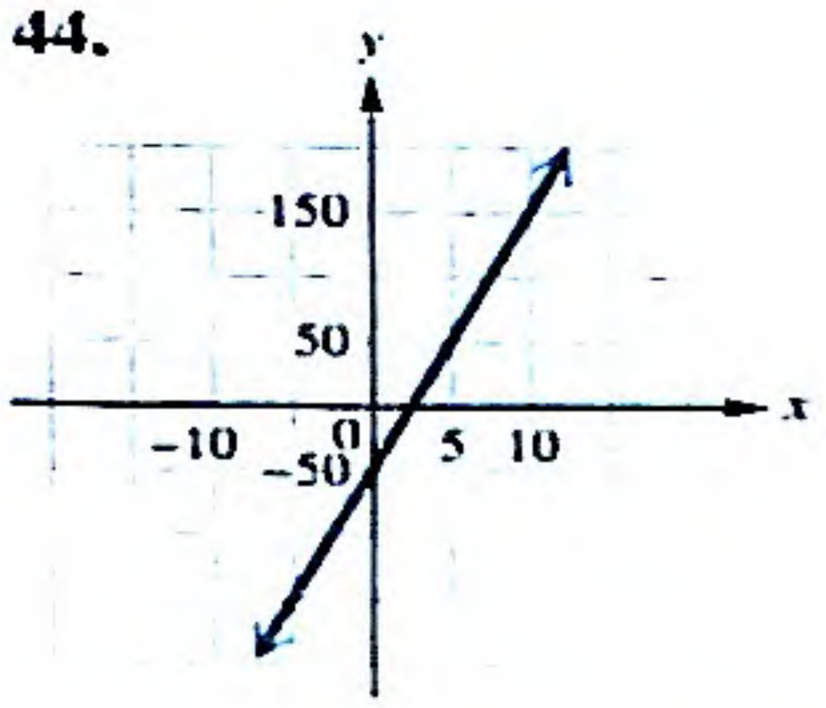


In Exercises 45–52, write an equation (a) in standard form and (b) in slope-intercept form for the line described. See Example 6.

45. through $(-1, 4)$, parallel to $x + 3y = 5$

46. through $(3, -2)$, parallel to $2x - y = 5$

47. through $(1, 6)$, perpendicular to $3x + 5y = 1$

48. through $(-2, 0)$, perpendicular to $8x - 3y = 7$

49. through $(4, 1)$, parallel to $y = -5$

50. through $(-2, -2)$, parallel to $y = 3$

51. through $(-5, 6)$, perpendicular to $x = -2$

52. through $(4, -4)$, perpendicular to $x = 4$

53. Find k so that the line through $(4, -1)$ and $(k, 2)$ is
(a) parallel to $3y + 2x = 6$; (b) perpendicular to $2y - 5x = 1$.

54. Find r so that the line through $(2, 6)$ and $(-4, r)$ is
(a) parallel to $2x - 3y = 4$; (b) perpendicular to $x + 2y = 1$.

(Modeling) Solve each problem. See Example 7.

55. ***Annual Tuition and Fees*** Use the data points $(0, 5492)$ and $(4, 7050)$ to find a linear equation that models the data shown in the table accompanying **Figure 49** in **Example 7.** Then use it to predict the average annual tuition and fees for in-state students at public four-year colleges in 2010. How does the result compare to the actual figure given in the table, \$7605?

56. ***Annual Tuition and Fees*** Repeat Exercise 55 using the data points for the years 2006 and 2008 to predict the average annual tuition and fees for 2010. How does the result compare to the actual figure given in the table, \$7605?

57. ***Cost of Private College Education*** The table lists the average annual cost (in dollars) of tuition and fees at private four-year colleges for selected years.

(a) Determine a linear function $f(x) = mx + b$ that models the data, where $x = 0$ represents 1996, $x = 1$ represents 1997, and so on. Use the points $(0, 12{,}881)$ and $(12, 22{,}449)$ to graph f and a scatter diagram of the data on the same coordinate axes. (You may wish to use a graphing calculator.) What does the slope of the graph of f indicate?

(b) Use this function to approximate tuition and fees in 2007. Compare your approximation to the actual value of \$21,979.

(c) Use the linear regression feature of a graphing calculator to find the equation of the line of best fit.

Year	Tuition and Fees (in dollars)
1996	12,881
1998	13,973
2000	15,470
2002	16,826
2004	18,604
2006	20,517
2008	22,449

Source: The College Board, *Annual Survey of Colleges*

58. ***Distances and Velocities of Galaxies*** The table lists the distances (in megaparsecs, 1 megaparsec $= 3.085 \times 10^{24}$ cm, and 1 megaparsec $= 3.26$ million light-years) and velocities (in kilometers per second) of four galaxies moving rapidly away from Earth.

Galaxy	Distance	Velocity
Virgo	15	1600
Ursa Minor	200	15,000
Corona Borealis	290	24,000
Bootes	520	40,000

Source: Acker, A., and C. Jaschek, *Astronomical Methods and Calculations*, John Wiley and Sons. Karttunen, H. (editor), *Fundamental Astronomy*, Springer-Verlag.

(a) Plot the data using distances for the x-values and velocities for the y-values. What type of relationship seems to hold between the data?

(b) Find a linear equation in the form $y = mx$ that models these data using the points $(520, 40{,}000)$ and $(0, 0)$. Graph your equation with the data on the same coordinate axes.

(c) The galaxy Hydra has a velocity of 60,000 km per sec. How far away is it according to the model in part (b)?

(d) The value of m is called the **Hubble constant.** The Hubble constant can be used to estimate the age of the universe A (in years) using the formula

$$A = \frac{9.5 \times 10^{11}}{m}.$$

Approximate A using your value of m.

(e) Astronomers currently place the value of the Hubble constant between 50 and 100. What is the range for the age of the universe A?

59. ***Celsius and Fahrenheit Temperatures*** When the Celsius temperature is 0°, the corresponding Fahrenheit temperature is 32°. When the Celsius temperature is 100°, the corresponding Fahrenheit temperature is 212°. Let C represent the Celsius temperature and F the Fahrenheit temperature.

(a) Express F as an exact linear function of C.
(b) Solve the equation in part (a) for C, thus expressing C as a function of F.
(c) For what temperature is $F = C$ a true statement?

60. ***Water Pressure on a Diver*** The pressure p of water on a diver's body is a linear function of the diver's depth, x. At the water's surface, the pressure is 1 atmosphere. At a depth of 100 ft, the pressure is about 3.92 atmospheres.

(a) Find the linear function that relates p to x.
(b) Compute the pressure at a depth of 10 fathoms (60 ft).

61. ***Consumption Expenditures*** In Keynesian macroeconomic theory, total consumption expenditure on goods and services, C, is assumed to be a linear function of national personal income, I. The table gives the values of C and I for 2004 and 2009 in the United States (in billions of dollars).

Year	2004	2009
Total consumption (C)	\$8285	\$10,089
National income (I)	\$9937	\$12,026

Source: U.S. Bureau of Economic Analysis.

(a) Find the formula for C as a function of I.
(b) The slope of the linear function is called the **marginal propensity to consume.** What is the marginal propensity to consume for the United States from 2004–2009?

Use a graphing calculator to solve each linear equation. See Example 8.

62. $2x + 7 - x = 4x - 2$

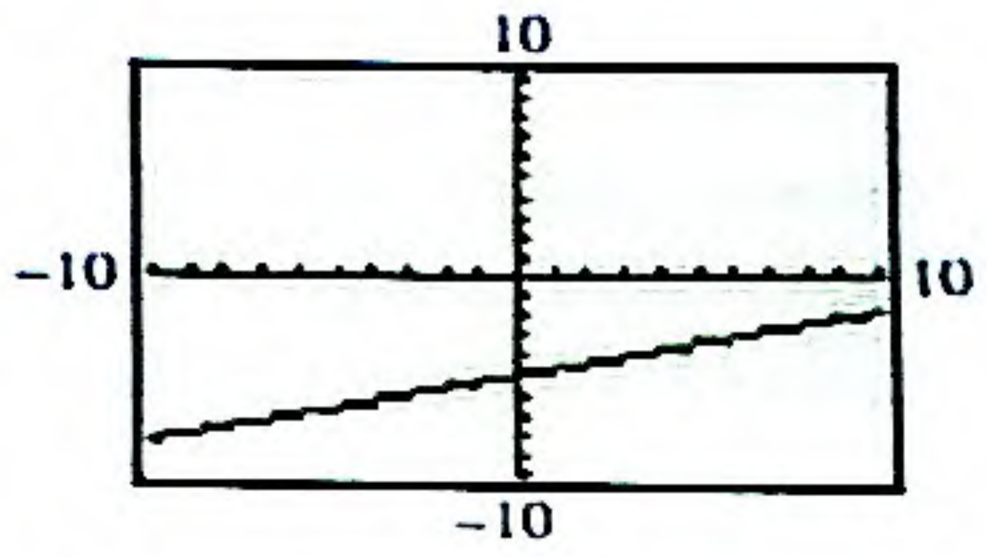
63. $7x - 2x + 4 - 5 = 3x + 1$

64. $3(2x + 1) - 2(x - 2) = 5$

65. $4x - 3(4 - 2x) = 2(x - 3) + 6x + 2$

66. The graph of $y = f(x)$ is shown in the standard viewing window. Which is the only value of x that could possibly be the solution of the equation $f(x) = 0$?

A. -15 B. 0 C. 5 D. 15



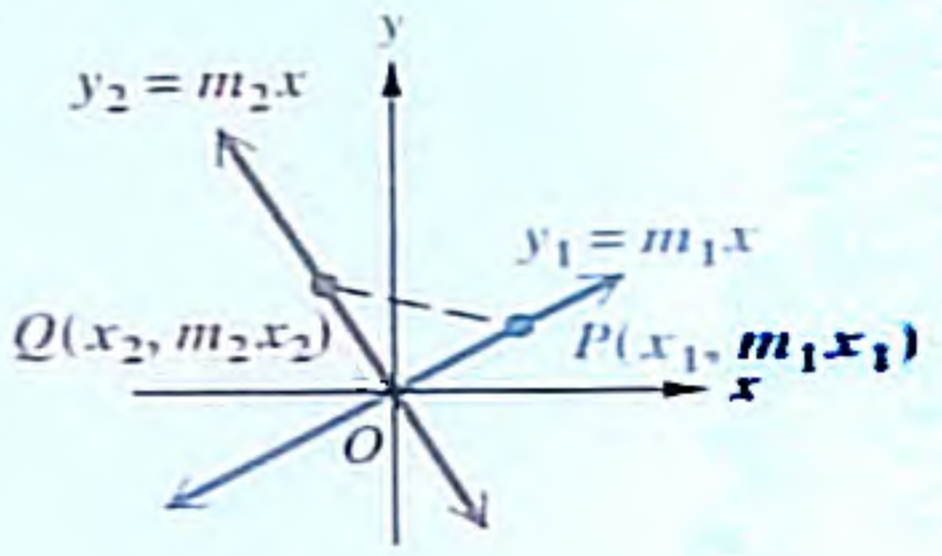
67. (a) Solve $-2(x - 5) = -x - 2$ using the methods of **Chapter 1.**
(b) Explain why the standard viewing window of a graphing calculator cannot graphically support the solution found in part (a). What minimum and maximum x-values would make it possible for the solution to be seen?

68. Using a graphing calculator, try to solve $-3(2x + 6) = -4x + 8 - 2x$. Explain what happens. What is the solution set?

Relating Concepts

For individual or collaborative investigation *(Exercises 69–75)*

In this section we state that two lines, neither of which is vertical, are perpendicular if and only if their slopes have a product of −1. In Exercises 69–75, we outline a partial proof of this for the case where the two lines intersect at the origin. ***Work these exercises in order,*** *and refer to the figure as needed.*



By the converse of the Pythagorean theorem, if

$$[d(O, P)]^2 + [d(O, Q)]^2 = [d(P, Q)]^2,$$

then triangle POQ is a right triangle with right angle at O.

69. Find an expression for the distance $d(O, P)$.

70. Find an expression for the distance $d(O, Q)$.

71. Find an expression for the distance $d(P, Q)$.

72. Use your results from **Exercises 69–71,** and substitute into the equation from the Pythagorean theorem. Simplify to show that this leads to the equation

$$-2m_1m_2x_1x_2 - 2x_1x_2 = 0.$$

73. Factor $-2x_1x_2$ from the final form of the equation in **Exercise 72.**

74. Use the property that if $ab = 0$ then $a = 0$ or $b = 0$ to solve the equation in **Exercise 73,** showing that $m_1m_2 = -1$.

75. State your conclusion based on **Exercises 69–74.**

76. Refer to **Example 4** in **Section 2.1,** and prove that the three points are collinear by taking them two at a time and showing that in all three cases, the slope is the same.

Determine whether the three points are collinear by using slopes as in Exercise 76. (Note: These problems were first seen in Exercises 25–28 in Section 2.1.)

77. $(-1, 4)$, $(-2, -1)$, $(1, 14)$

78. $(0, -7)$, $(-3, 5)$, $(2, -15)$

79. $(-1, -3)$, $(-5, 12)$, $(1, -11)$

80. $(0, 9)$, $(-3, -7)$, $(2, 19)$

Summary Exercises on Graphs, Circles, Functions, and Equations

These summary exercises provide practice with some of the concepts from Sections 2.1–2.5.

For the points P and Q, find (a) the distance d(P, Q), (b) the coordinates of the midpoint of the segment PQ, and (c) an equation for the line through the two points. Write the equation in slope-intercept form if possible.

1. $P(3, 5)$, $Q(2, -3)$

2. $P(-1, 0)$, $Q(4, -2)$

3. $P(-2, 2)$, $Q(3, 2)$

4. $P(2\sqrt{2}, \sqrt{2})$, $Q(\sqrt{2}, 3\sqrt{2})$

5. $P(5, -1)$, $Q(5, 1)$

6. $P(1, 1)$, $Q(-3, -3)$

7. $P(2\sqrt{3}, 3\sqrt{5})$, $Q(6\sqrt{3}, 3\sqrt{5})$

8. $P(0, -4)$, $Q(3, 1)$

Write an equation for each of the following, and sketch the graph.

9. the line through $(-2, 1)$ and $(4, -1)$

10. the horizontal line through $(2, 3)$

11. the circle with center $(2, -1)$ and radius 3

12. the circle with center $(0, 2)$ and tangent to the x-axis

13. the line through $(3, -5)$ with slope $-\frac{5}{6}$

14. the line through the origin and perpendicular to the line $3x - 4y = 2$

15. the line through $(-3, 2)$ and parallel to the line $2x + 3y = 6$

16. the vertical line through $(-4, 3)$

Decide whether or not each equation has a circle as its graph. If it does, give the center and the radius.

17. $x^2 + y^2 - 4x + 2y = 4$

18. $x^2 + y^2 + 6x + 10y + 36 = 0$

19. $x^2 + y^2 - 12x + 20 = 0$

20. $x^2 + y^2 + 2x + 16y = -61$

21. $x^2 + y^2 - 2x + 10 = 0$

22. $x^2 + y^2 - 8y - 9 = 0$

23. Find the coordinates of the points of intersection of the line $y = 2$ and the circle with center at $(4, 5)$ and radius 4.

24. Find the shortest distance from the origin to the graph of the circle with equation

$$x^2 + y^2 - 10x - 24y + 144 = 0.$$

For each relation, (a) find the domain and range, and (b) if the relation defines y as a function f of x, rewrite the relation using function notation and find $f(-2)$.

25. $x - 4y = -6$

26. $y^2 - x = 5$

27. $(x + 2)^2 + y^2 = 25$

28. $x^2 - 2y = 3$

2.6 Graphs of Basic Functions

- **Continuity**
- **The Identity, Squaring, and Cubing Functions**
- **The Square Root and Cube Root Functions**
- **The Absolute Value Function**
- **Piecewise-Defined Functions**
- **The Relation $x = y^2$**

Continuity Earlier in this chapter we graphed linear functions. The graph of a linear function, a straight line, may be drawn by hand over any interval of its domain without picking the pencil up from the paper. In mathematics we say that a function with this property is *continuous* over any interval. The formal definition of continuity requires concepts from calculus, but we can give an informal definition at the college algebra level.

Continuity (Informal Definition)

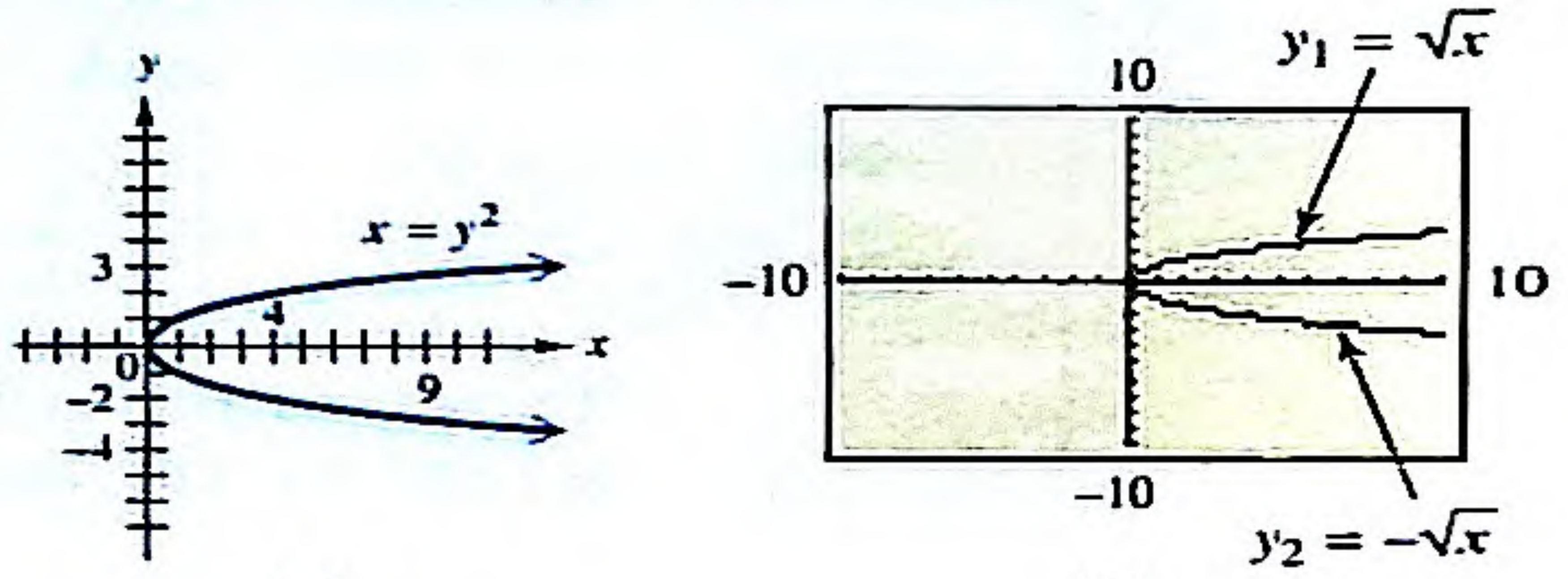
A function is **continuous** over an interval of its domain if its hand-drawn graph over that interval can be sketched without lifting the pencil from the paper.

If we plot the points from the table and join them with a smooth curve, we find that the graph of $x = y^2$ is a parabola opening to the right with vertex $(0, 0)$. See **Figure 67(a)**. The domain is $[0, \infty)$ and the range is $(-\infty, \infty)$.

Selected Ordered Pairs for $x = y^2$

x	y
0	0
1	± 1
4	± 2
9	± 3

There are two different y-values for the same x-value.



(a) **(b)**

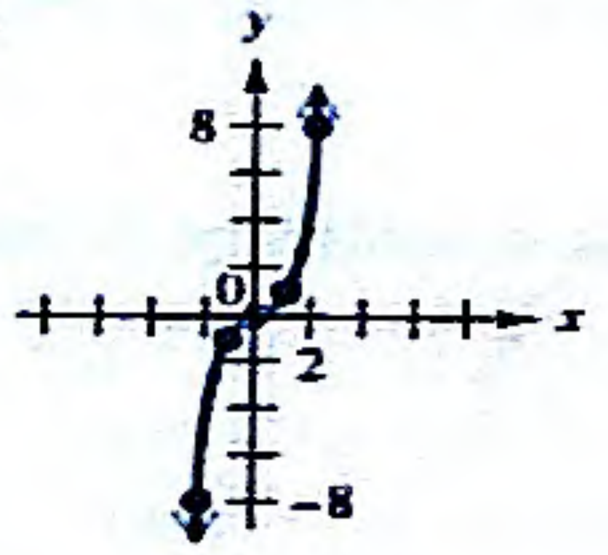
Figure 67

To use a calculator in function mode to graph the relation $x = y^2$, we graph the two functions $y_1 = \sqrt{x}$ (to generate the top half of the parabola) and $y_2 = -\sqrt{x}$ (to generate the bottom half). See **Figure 67(b)**. ■

2.6 Exercises

Concept Check For Exercises 1–10, refer to the following basic graphs.

A.



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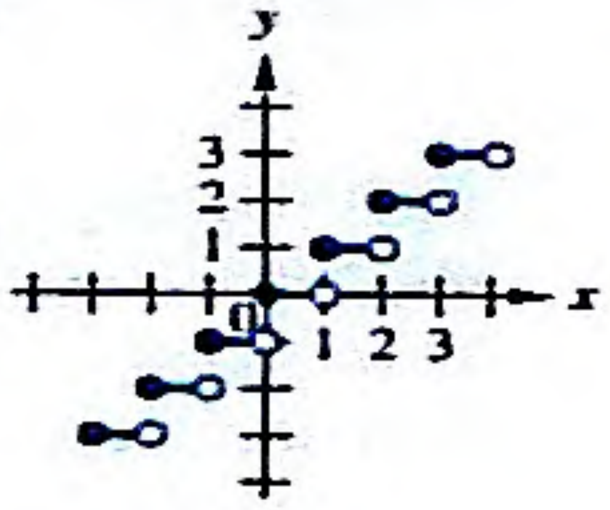

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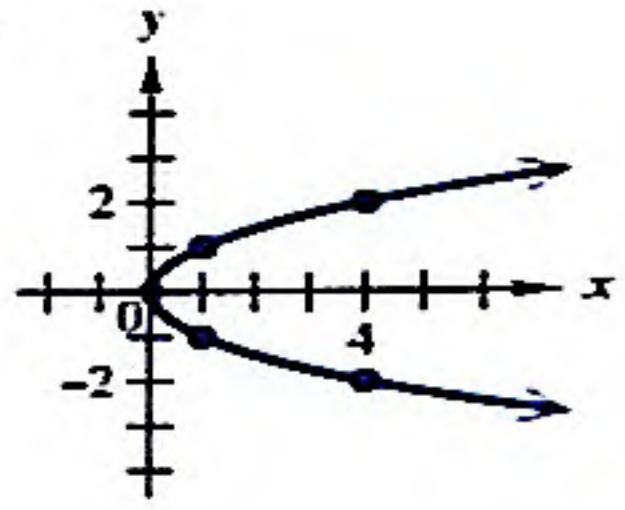

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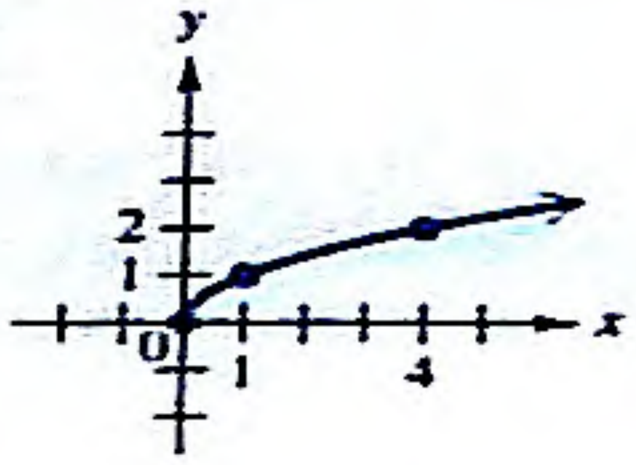

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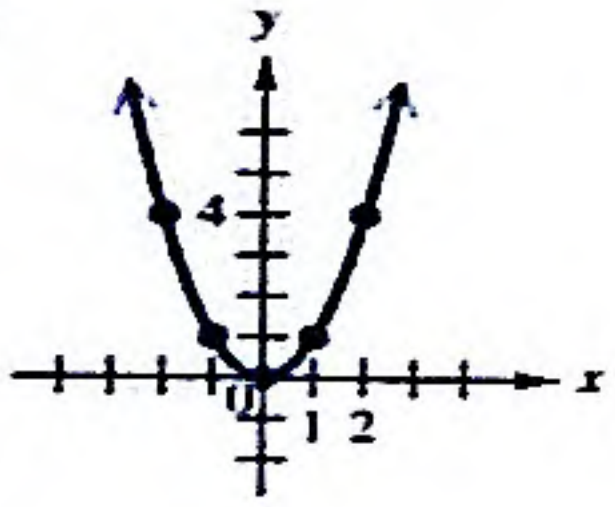

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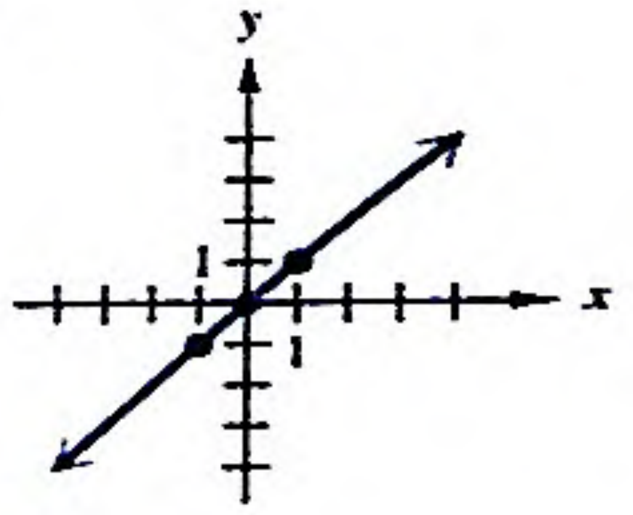

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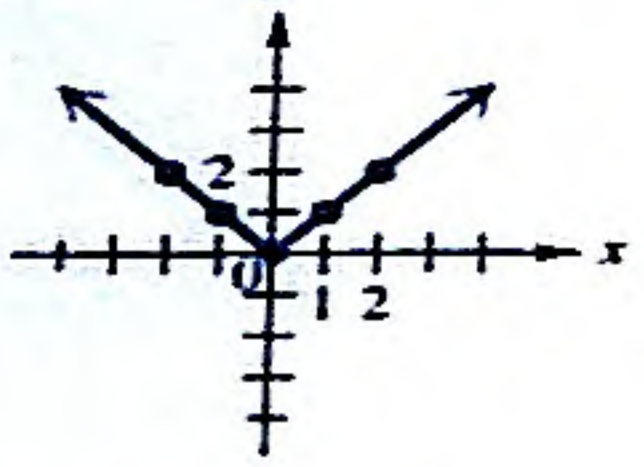

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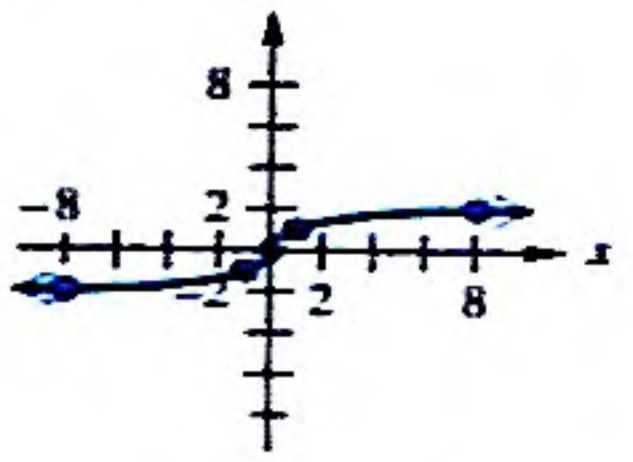
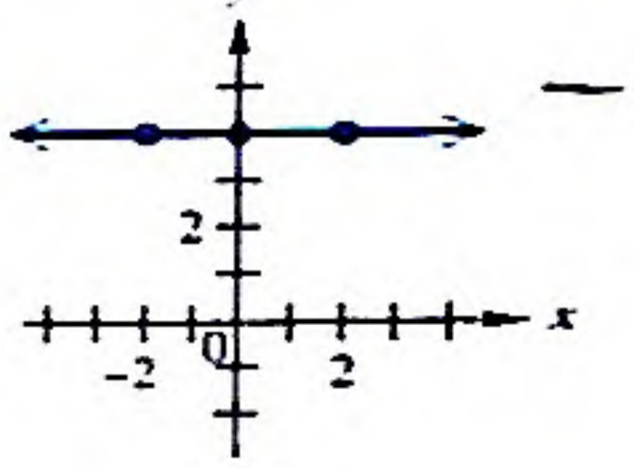


I.



1. Which one is the graph of $y = x^2$? What is its domain?
2. Which one is the graph of $y = |x|$? On what interval is it increasing?
3. Which one is the graph of $y = x^3$? What is its range?
4. Which one is not the graph of a function? What is its equation?
5. Which one is the identity function? What is its equation?

6. Which one is the graph of $y = [\![x]\!]$? What is the value of y when $x = 1.5$?

7. Which one is the graph of $y = \sqrt[3]{x}$? Is there any interval over which the function is decreasing?

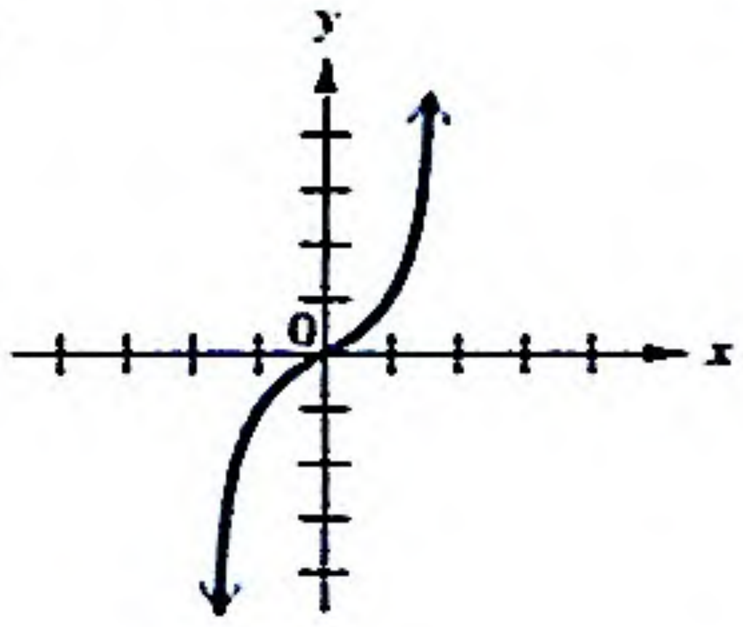
8. Which one is the graph of $y = \sqrt{x}$? What is its domain?

9. Which one is discontinuous at many points? What is its range?

10. Which graphs of functions decrease over part of the domain and increase over the rest of the domain? On what intervals do they increase? decrease?

Determine the intervals of the domain over which each function is continuous. See Example 1.

11.



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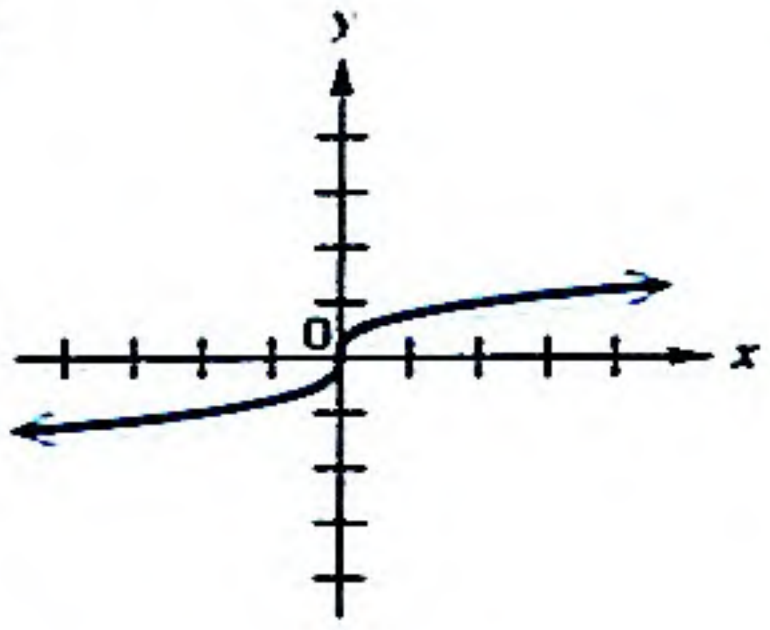

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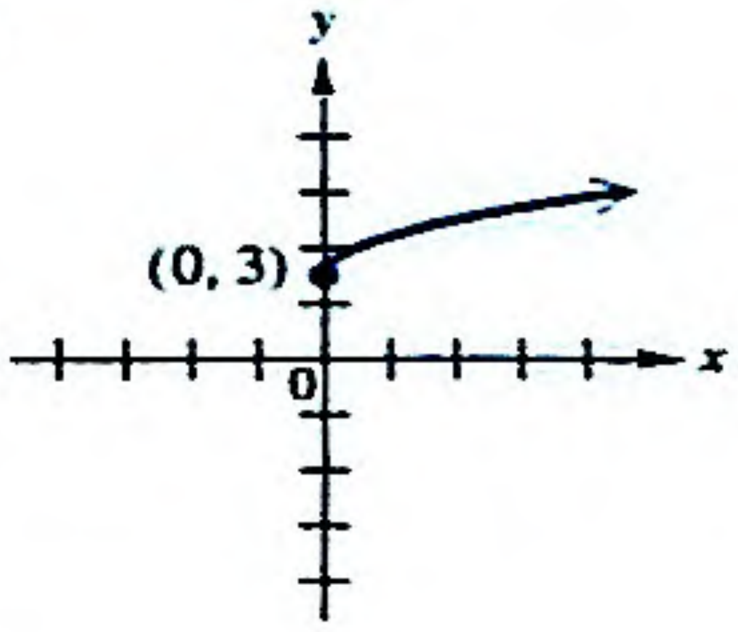

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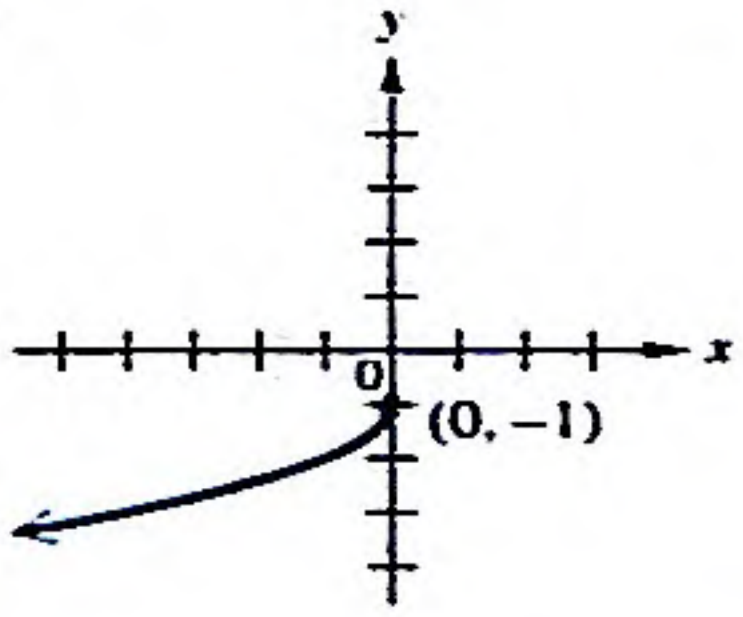

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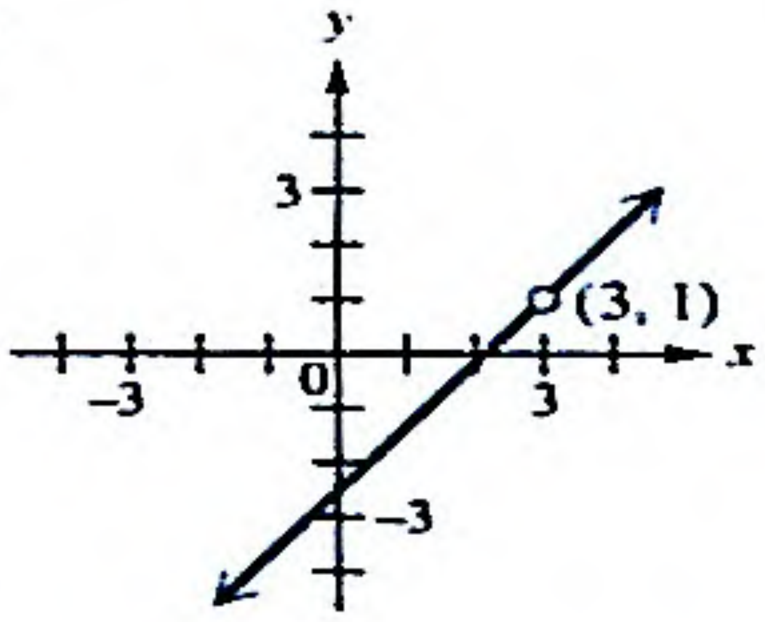

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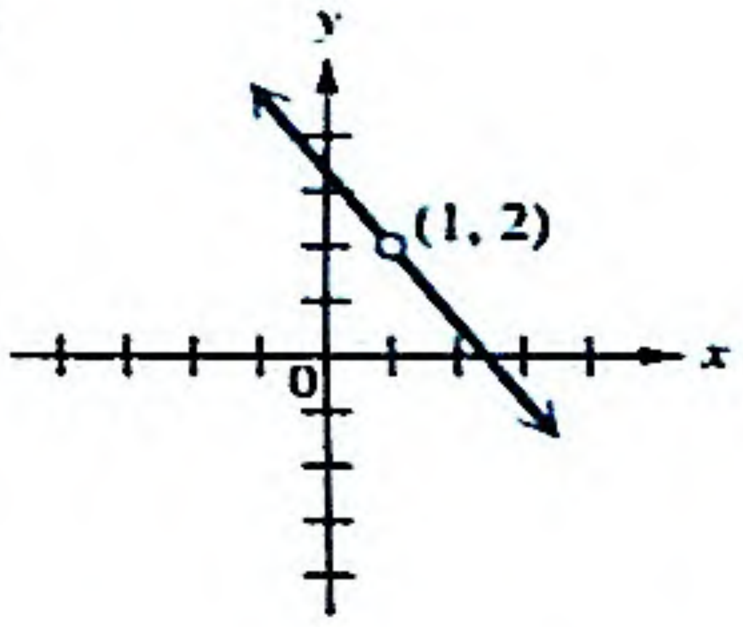


For each piecewise-defined function, find (a) $f(-5)$, (b) $f(-1)$, (c) $f(0)$, and (d) $f(3)$. See Example 2.

17. $f(x) = \begin{cases} 2x & \text{if } x \le -1 \\ x - 1 & \text{if } x > -1 \end{cases}$

18. $f(x) = \begin{cases} x - 2 & \text{if } x < 3 \\ 5 - x & \text{if } x \ge 3 \end{cases}$

19. $f(x) = \begin{cases} 2 + x & \text{if } x < -4 \\ -x & \text{if } -4 \le x \le 2 \\ 3x & \text{if } x > 2 \end{cases}$

20. $f(x) = \begin{cases} -2x & \text{if } x < -3 \\ 3x - 1 & \text{if } -3 \le x \le 2 \\ -4x & \text{if } x > 2 \end{cases}$

Graph each piecewise-defined function. See Example 2.

21. $f(x) = \begin{cases} x - 1 & \text{if } x \le 3 \\ 2 & \text{if } x > 3 \end{cases}$

22. $f(x) = \begin{cases} 6 - x & \text{if } x \le 3 \\ 3 & \text{if } x > 3 \end{cases}$

23. $f(x) = \begin{cases} 4 - x & \text{if } x < 2 \\ 1 + 2x & \text{if } x \ge 2 \end{cases}$

24. $f(x) = \begin{cases} 2x + 1 & \text{if } x \ge 0 \\ x & \text{if } x < 0 \end{cases}$

25. $f(x) = \begin{cases} -3 & \text{if } x \le 1 \\ -1 & \text{if } x > 1 \end{cases}$

26. $f(x) = \begin{cases} -2 & \text{if } x \le 1 \\ 2 & \text{if } x > 1 \end{cases}$

27. $f(x) = \begin{cases} 2 + x & \text{if } x < -4 \\ -x & \text{if } -4 \le x \le 5 \\ 3x & \text{if } x > 5 \end{cases}$

28. $f(x) = \begin{cases} -2x & \text{if } x < -3 \\ 3x - 1 & \text{if } -3 \le x \le 2 \\ -4x & \text{if } x > 2 \end{cases}$

29. $f(x) = \begin{cases} -\frac{1}{2}x^2 + 2 & \text{if } x \le 2 \\ \frac{1}{2}x & \text{if } x > 2 \end{cases}$

30. $f(x) = \begin{cases} x^3 + 5 & \text{if } x \le 0 \\ -x^2 & \text{if } x > 0 \end{cases}$

31. $f(x) = \begin{cases} 2x & \text{if } -5 \le x < -1 \\ -2 & \text{if } -1 \le x < 0 \\ x^2 - 2 & \text{if } 0 \le x \le 2 \end{cases}$

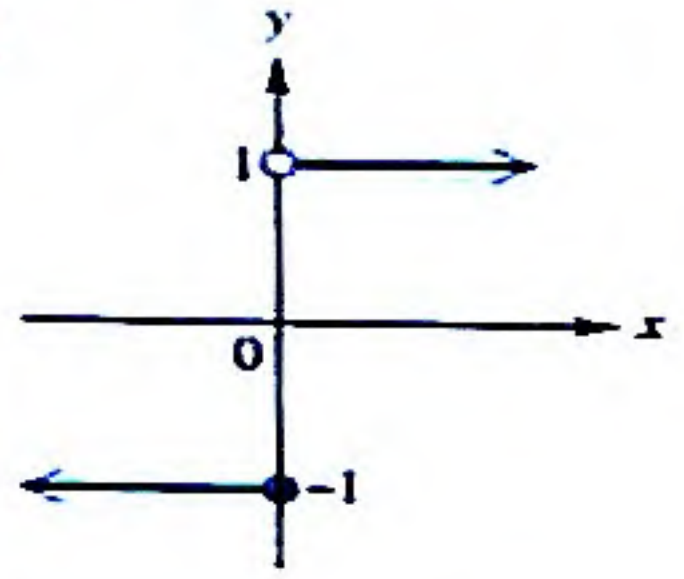
32. $f(x) = \begin{cases} 0.5x^2 & \text{if } -4 \le x \le -2 \\ x & \text{if } -2 < x < 2 \\ x^2 - 4 & \text{if } 2 \le x \le 4 \end{cases}$

33. $f(x) = \begin{cases} x^3 + 3 & \text{if } -2 \le x \le 0 \\ x + 3 & \text{if } 0 < x < 1 \\ 4 + x - x^2 & \text{if } 1 \le x \le 3 \end{cases}$

34. $f(x) = \begin{cases} -2x & \text{if } -3 \le x < -1 \\ x^2 + 1 & \text{if } -1 \le x \le 2 \\ \frac{1}{2}x^3 + 1 & \text{if } 2 < x \le 3 \end{cases}$

Connecting Graphs with Equations *Give a rule for each piecewise-defined function. Also give the domain and range.*

35.



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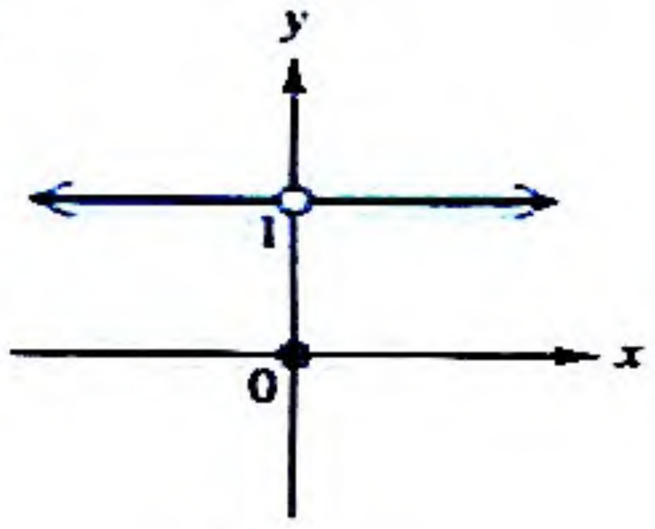

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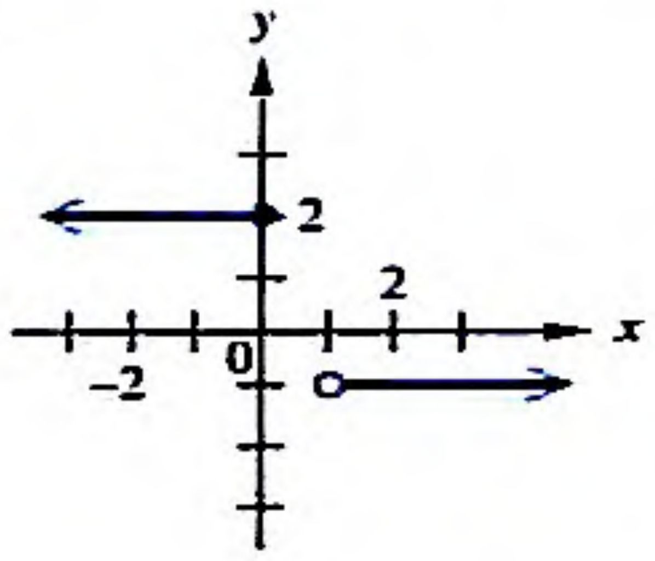

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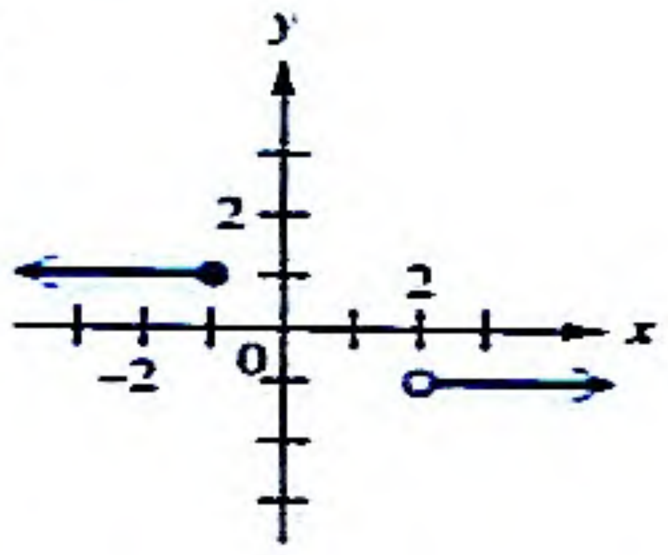

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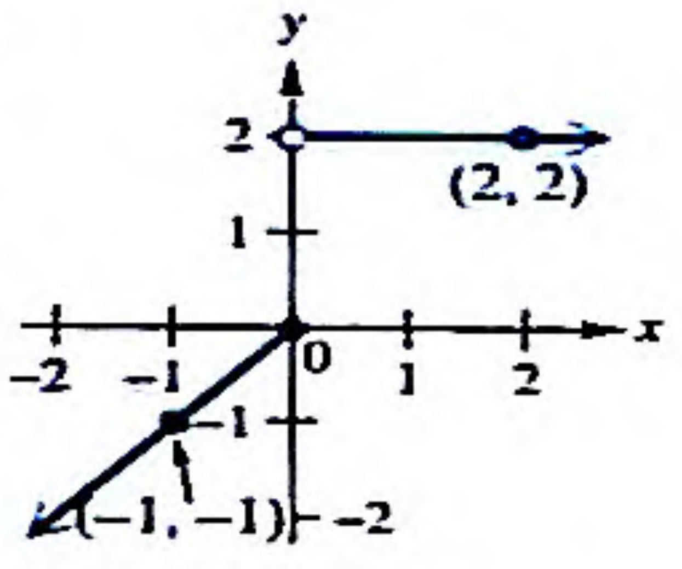

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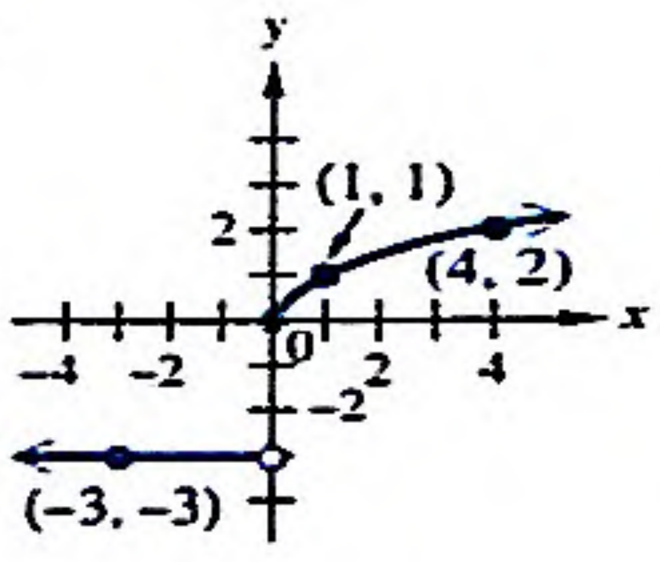

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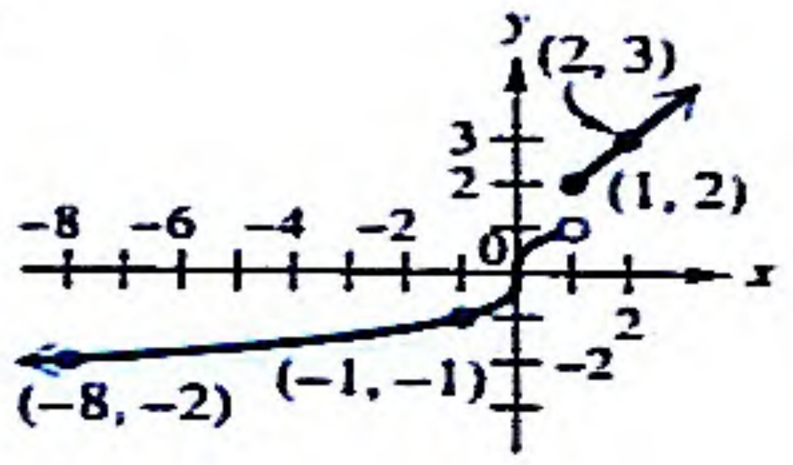

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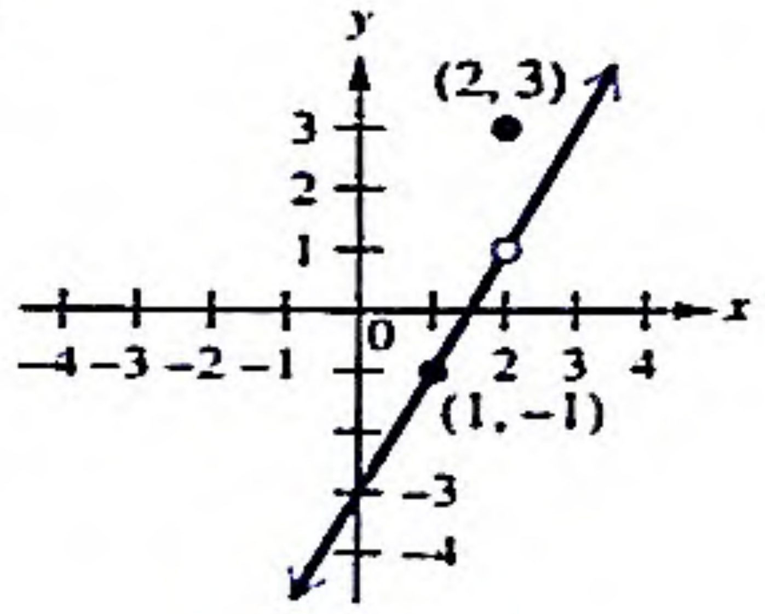


Graph each function. Give the domain and range. See Example 3.

43. $f(x) = [\![-x]\!]$

44. $f(x) = -[\![x]\!]$

45. $f(x) = [\![2x]\!]$

46. $g(x) = [\![2x - 1]\!]$

(Modeling) ***Solve each problem. See Example 4.***

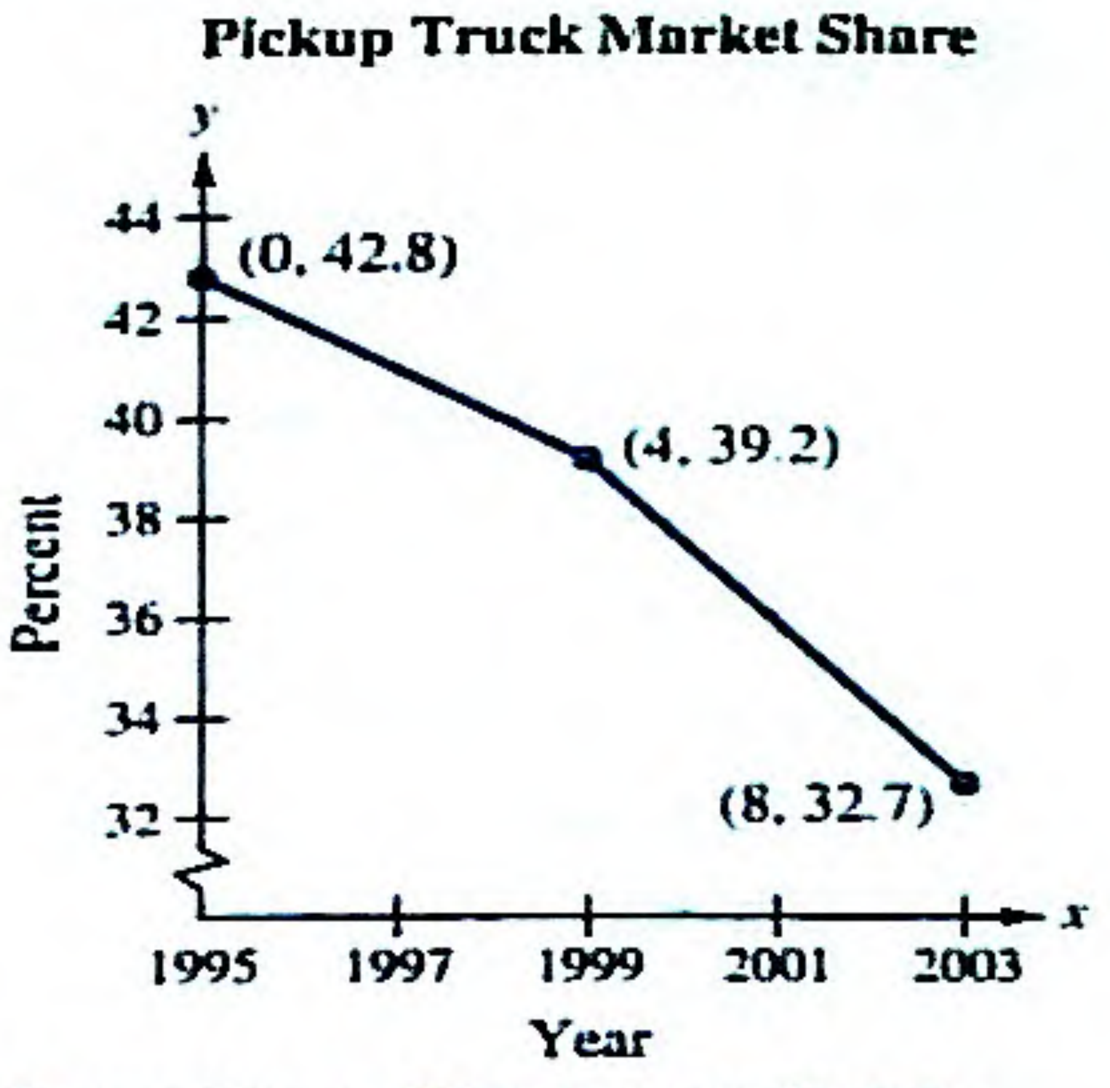
47. ***Postage Charges*** Assume that postage rates are \$0.44 for the first ounce, plus \$0.20 for each additional ounce, and that each letter carries one \$0.44 stamp and as many \$0.20 stamps as necessary. Graph the function f that models the number of stamps on a letter weighing x ounces over the interval $(0, 5]$.

48. ***Airport Parking Charges*** The cost of parking a car at an airport hourly parking lot is \$3 for the first half-hour and \$2 for each additional half-hour or fraction of a half-hour. Graph the function f that models the cost of parking a car for x hours over the interval $(0, 2]$.

49. ***Water in a Tank*** Sketch a graph that depicts the amount of water in a 100-gal tank. The tank is initially empty and then filled at a rate of 5 gal per minute. Immediately after it is full, a pump is used to empty the tank at 2 gal per minute.

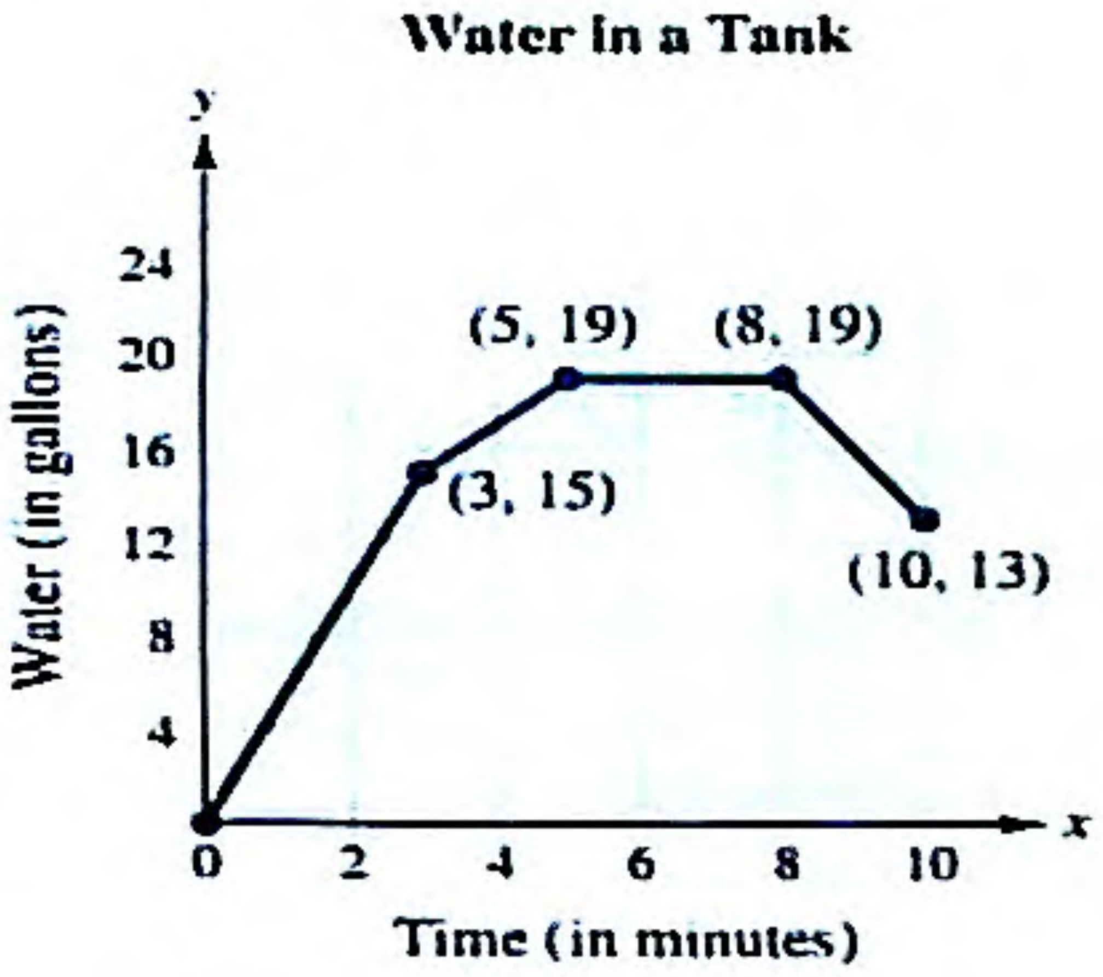
50. *Distance from Home* Sketch a graph showing the distance a person is from home after x hours if he or she drives on a straight road at 40 mph to a park 20 mi away, remains at the park for 2 hr, and then returns home at a speed of 20 mph.

51. *Pickup Truck Market Share* The light vehicle market share (in percent) in the United States for pickup trucks is shown in the graph. Let $x = 0$ represent 1995, $x = 4$ represent 1999, and so on.
   (a) Use the points on the graph to write equations for the line segments in the intervals $[0, 4]$ and $(4, 8]$.
   (b) Define this graph as a piecewise-defined function f.

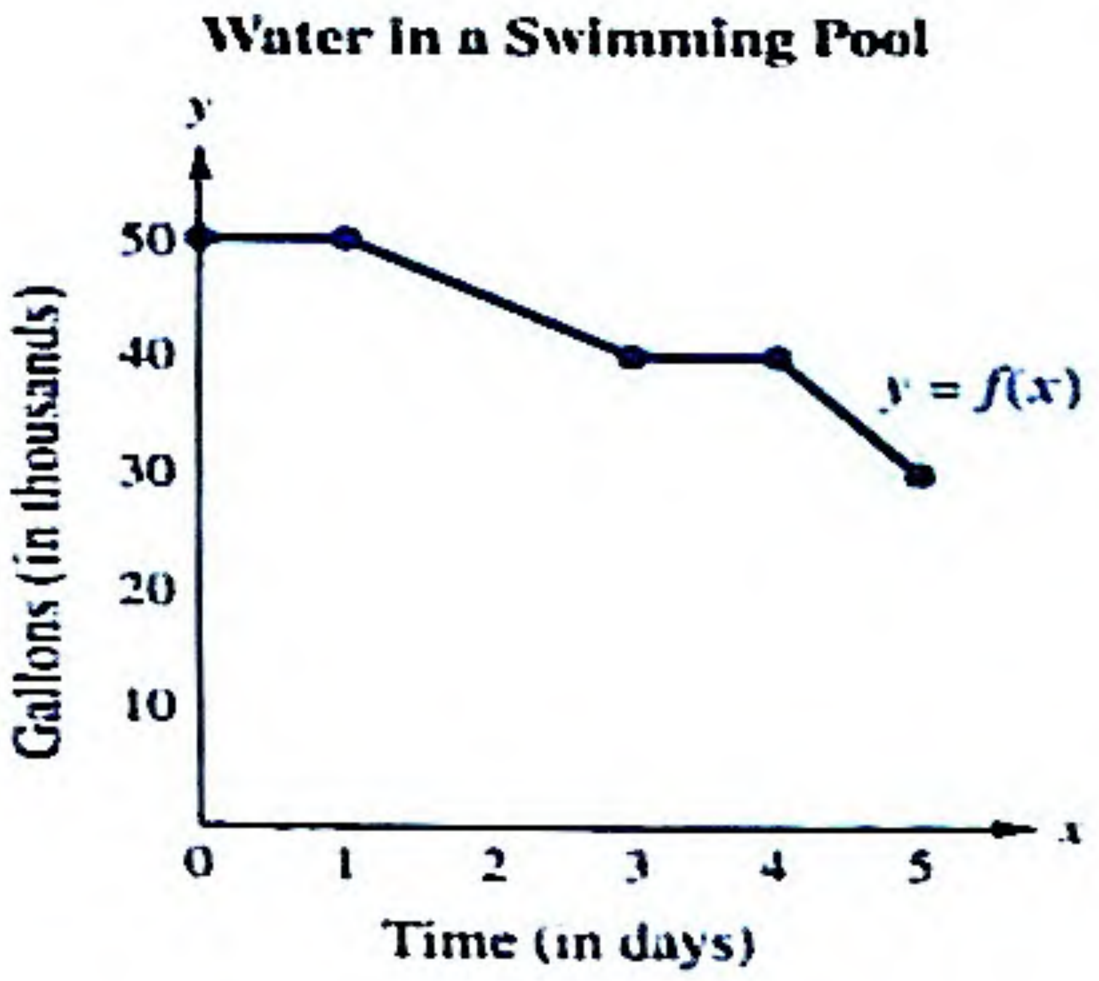


Source: Bureau of Transportation Statistics.

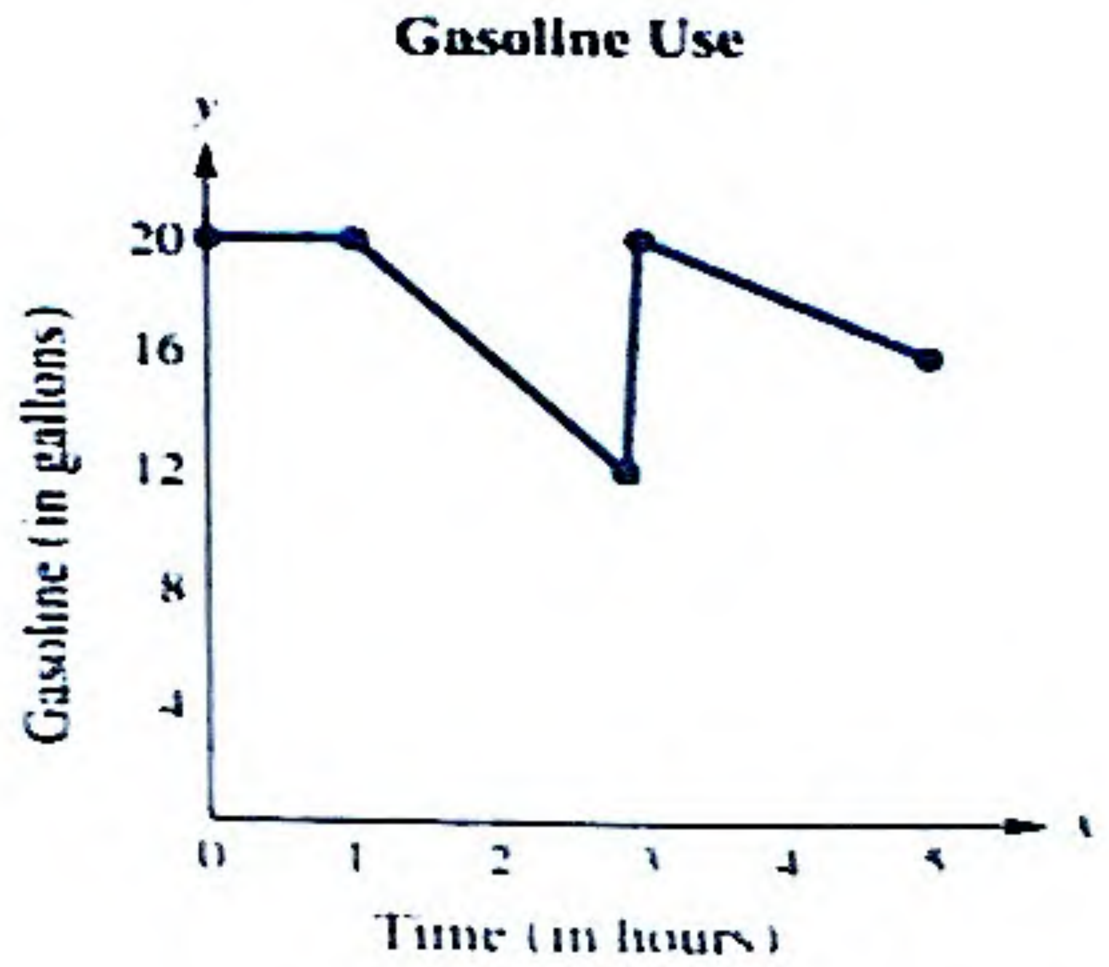
52. *Flow Rates* A water tank has an inlet pipe with a flow rate of 5 gal per minute and an outlet pipe with a flow rate of 3 gal per minute. A pipe can be either closed or completely open. The graph shows the number of gallons of water in the tank after x minutes. Use the concept of slope to interpret each piece of this graph.



53. *Swimming Pool Levels* The graph of $y = f(x)$ represents the amount of water in thousands of gallons remaining in a swimming pool after x days.
   (a) Estimate the initial and final amounts of water contained in the pool.
   (b) When did the amount of water in the pool remain constant?
   (c) Approximate $f(2)$ and $f(4)$.
   (d) At what rate was water being drained from the pool when $1 \leq x \leq 3$?



54. *Gasoline Usage* The graph shows the gallons of gasoline y in the gas tank of a car after x hours.
   (a) Estimate how much gasoline was in the gas tank when $x = 3$.
   (b) When did the car burn gasoline at the greatest rate?



55. ***Lumber Costs*** Lumber that is used to frame walls of houses is frequently sold in multiples of 2 ft. If the length of a board is not exactly a multiple of 2 ft, there is often no charge for the additional length. For example, if a board measures at least 8 ft, but less than 10 ft, then the consumer is charged for only 8 ft.

(a) Suppose that the cost of lumber is \$0.80 every 2 ft. Find a formula for a function f that computes the cost of a board x feet long for $6 \le x \le 18$.
(b) Determine the costs of boards with lengths of 8.5 ft and 15.2 ft.

56. ***Snow Depth*** The snow depth in Michigan's Isle Royale National Park varies throughout the winter. In a typical winter, the snow depth in inches is approximated by the following function.

$$f(x) = \begin{cases} 6.5x & \text{if } 0 \le x \le 4 \\ -5.5x + 48 & \text{if } 4 < x \le 6 \\ -30x + 195 & \text{if } 6 < x \le 6.5 \end{cases}$$

Here, x represents the time in months with $x = 0$ representing the beginning of October, $x = 1$ representing the beginning of November, and so on.

(a) Graph $y = f(x)$.
(b) In what month is the snow deepest? What is the deepest snow depth?
(c) In what months does the snow begin and end?

2.7 Graphing Techniques

- Stretching and Shrinking
- Reflecting
- Symmetry
- Even and Odd Functions
- Translations

Graphing techniques presented in this section show how to graph functions that are defined by altering the equation of a basic function.

Stretching and Shrinking We begin by considering how the graphs of $y = af(x)$ and $y = f(ax)$ compare to the graph of $y = f(x)$, where $a > 0$.

EXAMPLE 1 Stretching or Shrinking a Graph

Graph each function.

(a) $g(x) = 2|x|$ (b) $h(x) = \frac{1}{2}|x|$ (c) $k(x) = |2x|$

SOLUTION

(a) Comparing the tables of values for $f(x) = |x|$ and $g(x) = 2|x|$ in **Figure 68,** we see that for corresponding x-values, the y-values of g are each twice those of f. The graph of $f(x) = |x|$ is *vertically stretched.* The graph of $g(x)$, shown in blue in **Figure 68,** is narrower than that of $f(x)$, shown in red for comparison.

x	$f(x) = \lvert x \rvert$	$g(x) = 2\lvert x \rvert$
−2	2	4
−1	1	2
0	0	0
1	1	2
2	2	4

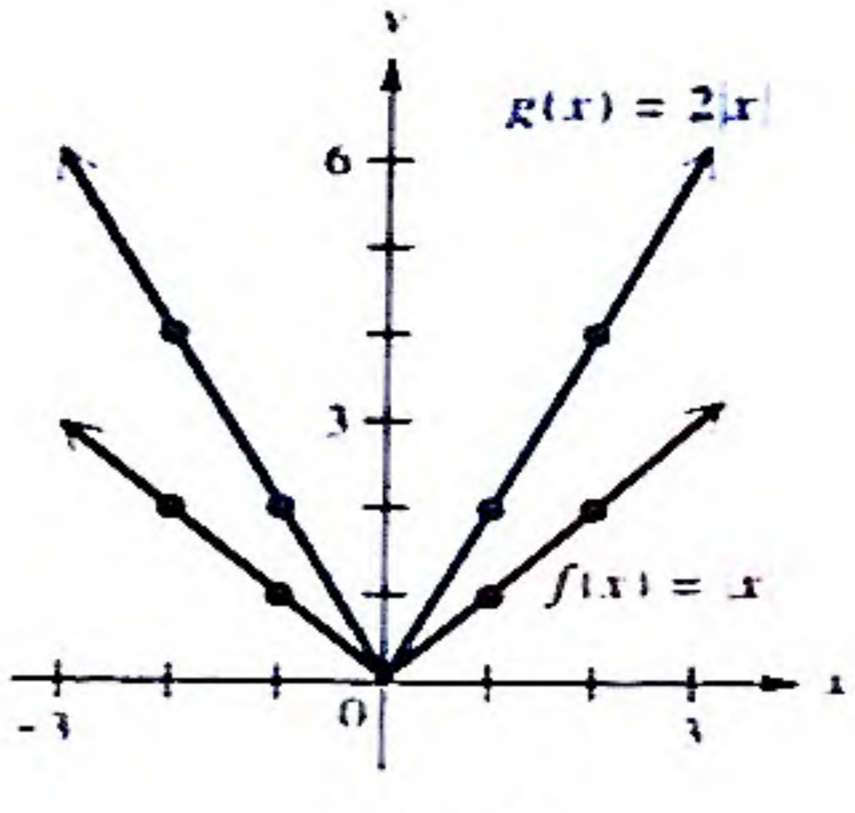


Figure 68

(b) The graph of $h(x) = \frac{1}{2}|x|$ is also the same general shape as that of $f(x)$, but here the coefficient $\frac{1}{2}$ is between 0 and 1 and causes a *vertical shrink*. The graph of $h(x)$ is wider than the graph of $f(x)$, as we see by comparing the tables of values. See **Figure 69.**

x	$f(x) = \|x\|$	$h(x) = \frac{1}{2}\|x\|$
-2	2	1
-1	1	$\frac{1}{2}$
0	0	0
1	1	$\frac{1}{2}$
2	2	1

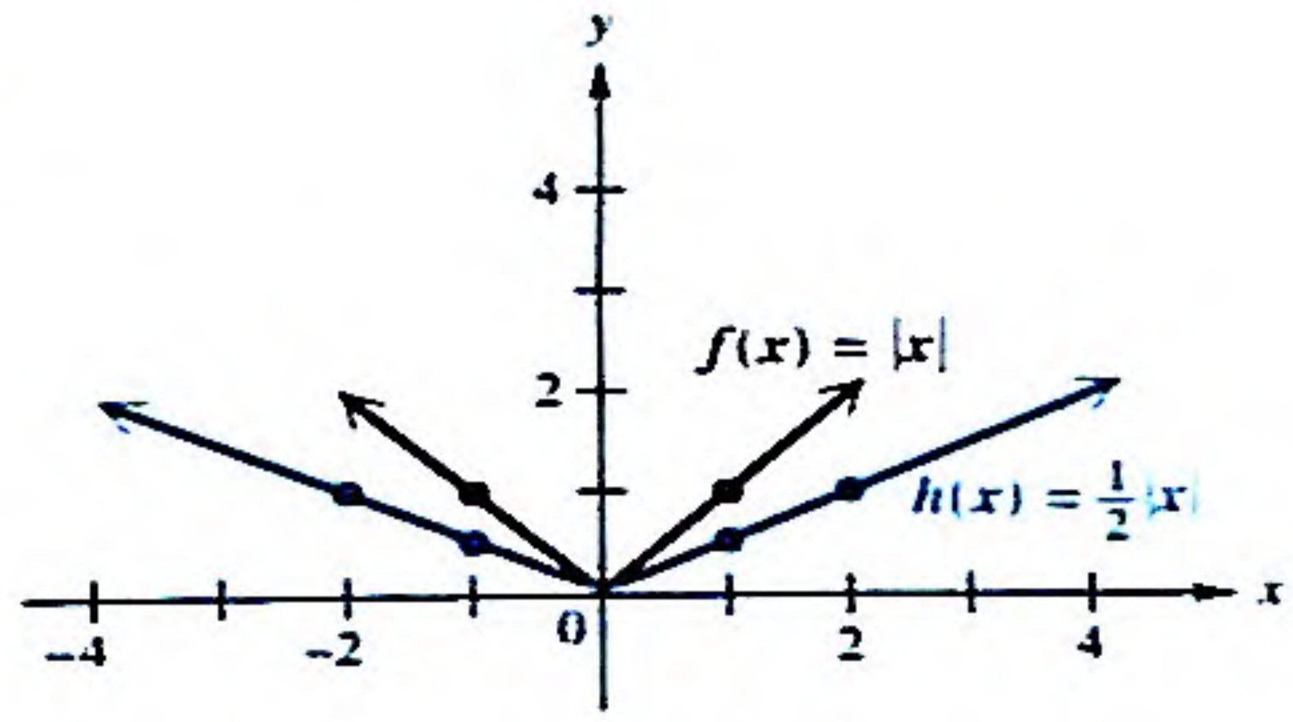


Figure 69

(c) Use Property 2 of absolute value ($|ab| = |a| \cdot |b|$) to rewrite $|2x|$.

$$k(x) = |2x| = |2| \cdot |x| = 2|x| \quad \text{Property 2 (Section R.2)}$$

Therefore, the graph of $k(x) = |2x|$ is the same as the graph of $g(x) = 2|x|$ in part (a). This is a *horizontal shrink* of the graph of $f(x) = |x|$. See **Figure 68** on the previous page.

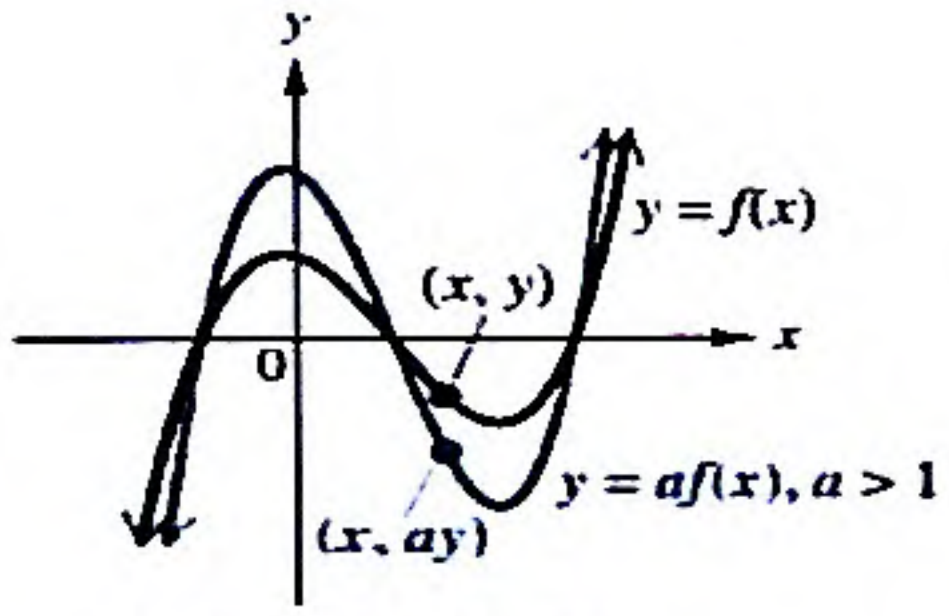
✔ *Now Try Exercises 7 and 9.*

Vertical Stretching or Shrinking of the Graph of a Function

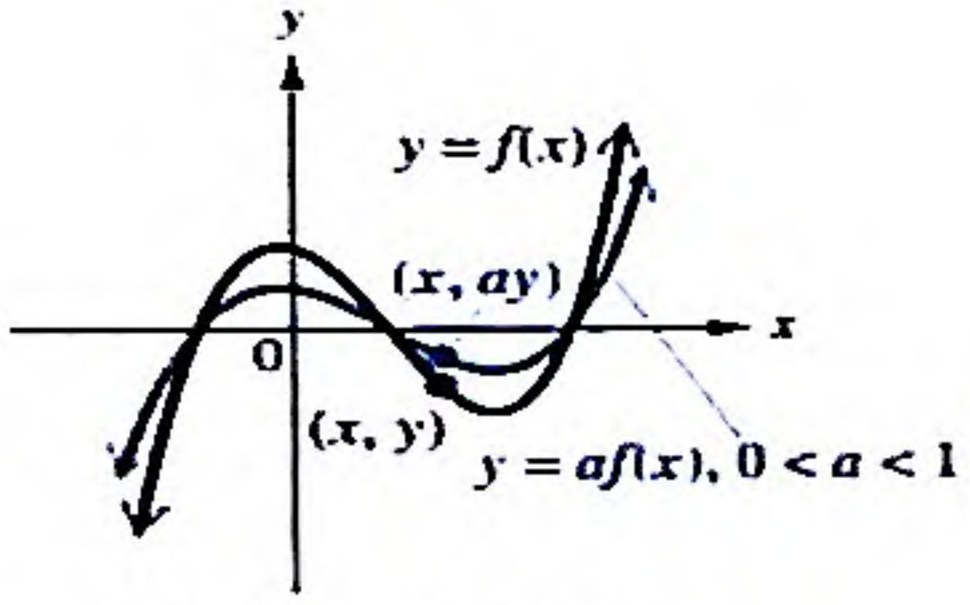
Suppose that $a > 0$. If a point (x, y) lies on the graph of $y = f(x)$, then the point (x, ay) lies on the graph of $y = af(x)$.

(a) If $a > 1$, then the graph of $y = af(x)$ is a **vertical stretching** of the graph of $y = f(x)$.

(b) If $0 < a < 1$, then the graph of $y = af(x)$ is a **vertical shrinking** of the graph of $y = f(x)$.



Vertical stretching
$a > 1$



Vertical shrinking
$0 < a < 1$

Figure 70

Figure 70 shows graphical interpretations of vertical stretching and shrinking. Notice that in both cases, the x-intercepts of the graph remain the same but the y-intercepts *are* affected.

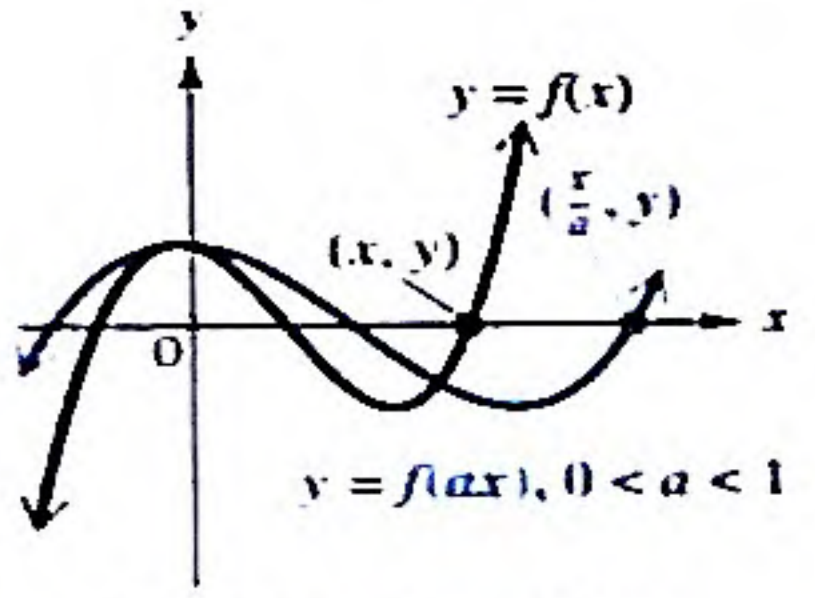
Graphs of functions can also be stretched and shrunk horizontally.

Horizontal Stretching or Shrinking of the Graph of a Function

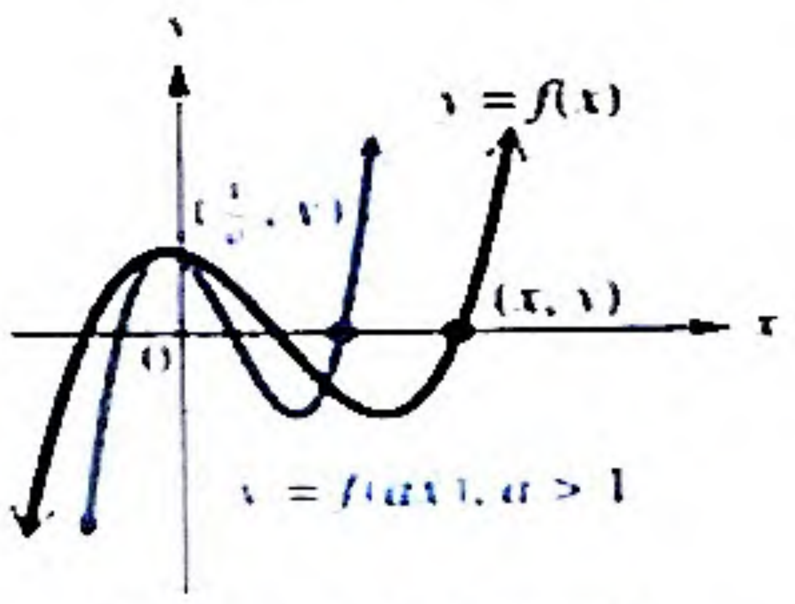
Suppose that $a > 0$. If a point (x, y) lies on the graph of $y = f(x)$, then the point $\left(\frac{x}{a}, y\right)$ lies on the graph of $y = f(ax)$.

(a) If $0 < a < 1$, then the graph of $y = f(ax)$ is a **horizontal stretching** of the graph of $y = f(x)$.

(b) If $a > 1$, then the graph of $y = f(ax)$ is a **horizontal shrinking** of the graph of $y = f(x)$.



Horizontal stretching
$0 < a < 1$



Horizontal shrinking
$a > 1$

Figure 71

See **Figure 71** for graphical interpretations of horizontal stretching and shrinking. Notice that in both cases, the y-intercept remains the same but the x-intercepts *are* affected.

Reflecting Forming the mirror image of a graph across a line is called **reflecting the graph across the line.**

EXAMPLE 2 Reflecting a Graph across an Axis

Graph each function.

(a) $g(x) = -\sqrt{x}$ (b) $h(x) = \sqrt{-x}$

SOLUTION

(a) The tables of values for $g(x) = -\sqrt{x}$ and $f(x) = \sqrt{x}$ are shown with their graphs in **Figure 72**. As the tables suggest, every y-value of the graph of $g(x) = -\sqrt{x}$ is the negative of the corresponding y-value of $f(x) = \sqrt{x}$. This has the effect of reflecting the graph across the x-axis.

x	$f(x) = \sqrt{x}$	$g(x) = -\sqrt{x}$
0	0	0
1	1	-1
4	2	-2

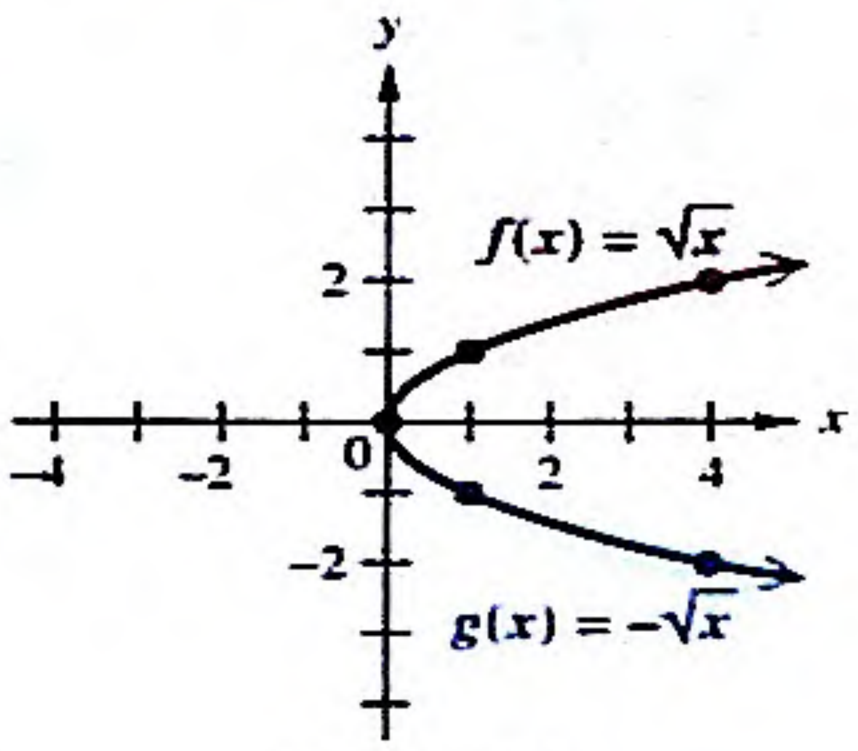


Figure 72

(b) The domain of $h(x) = \sqrt{-x}$ is $x \le 0$, while the domain of $f(x) = \sqrt{x}$ is $x \ge 0$. Choosing x-values for $h(x)$ that are negatives of those used for $f(x)$, we see that corresponding y-values are the same. The graph of h is a reflection of the graph of f across the y-axis. See **Figure 73**.

x	$f(x) = \sqrt{x}$	$h(x) = \sqrt{-x}$
-4	undefined	2
-1	undefined	1
0	0	0
1	1	undefined
4	2	undefined

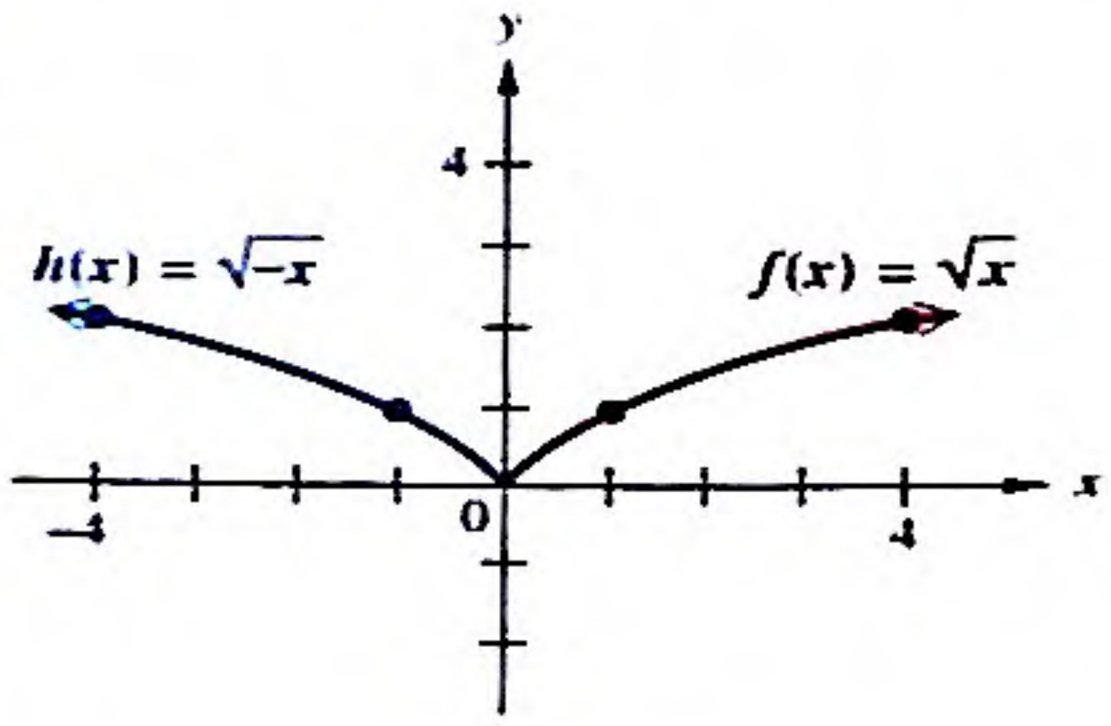


Figure 73

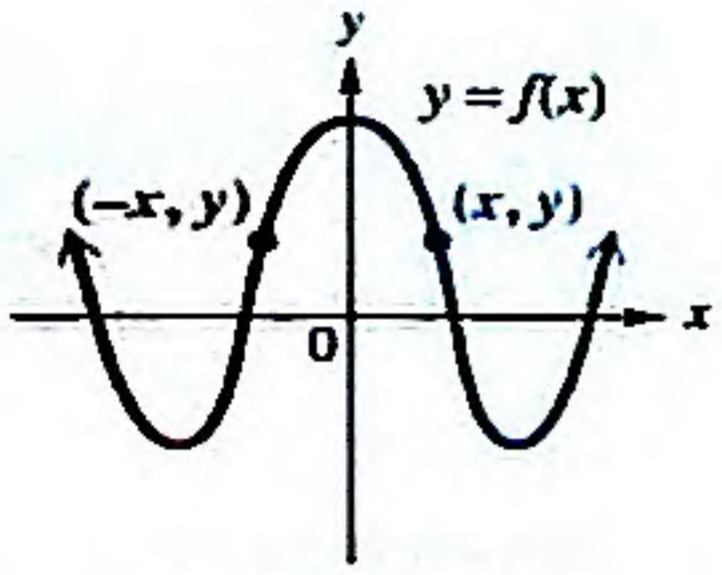
✔ *Now Try Exercises 17 and 23.*

The graphs in **Example 2** suggest the following generalizations.

Reflecting across an Axis

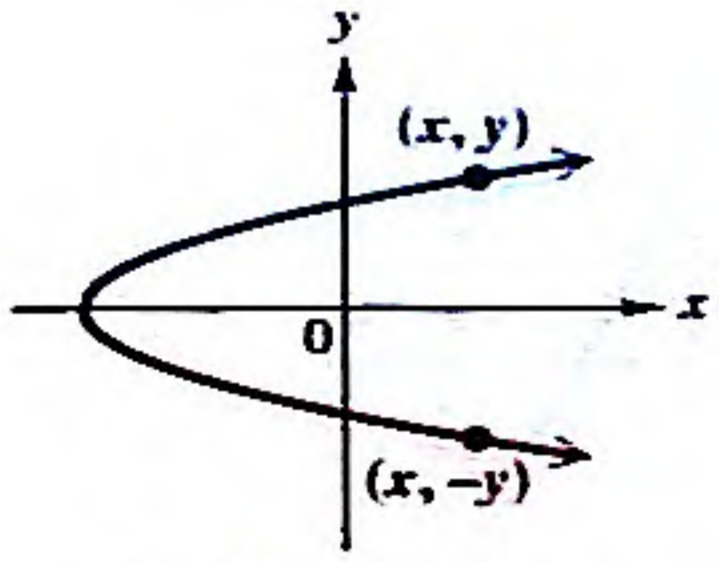
The graph of $y = -f(x)$ is the same as the graph of $y = f(x)$ reflected across the x-axis. (If a point (x, y) lies on the graph of $y = f(x)$, then $(x, -y)$ lies on this reflection.)

The graph of $y = f(-x)$ is the same as the graph of $y = f(x)$ reflected across the y-axis. (If a point (x, y) lies on the graph of $y = f(x)$, then $(-x, y)$ lies on this reflection.)



y-axis symmetry

(a)



x-axis symmetry

(b)

Figure 74

Symmetry The graph of f shown in **Figure 74(a)** is cut in half by the y-axis with each half the mirror image of the other half. Such a graph is *symmetric with respect to the y-axis. The point* $(-x, y)$ *is on the graph whenever the point* (x, y) *is on the graph.*

Similarly, if the graph in **Figure 74(b)** were folded in half along the x-axis, the portion at the top would exactly match the portion at the bottom. Such a graph is *symmetric with respect to the x-axis. The point* $(x, -y)$ *is on the graph whenever the point* (x, y) *is on the graph.*

Symmetry with Respect to an Axis

The graph of an equation is **symmetric with respect to the y-axis** if the replacement of x with $-x$ results in an equivalent equation.

The graph of an equation is **symmetric with respect to the x-axis** if the replacement of y with $-y$ results in an equivalent equation.

In **Section 2.6** we introduced graphs of basic functions. The squaring function and the absolute value function are examples of functions that are symmetric with respect to the y-axis.

EXAMPLE 3 **Testing for Symmetry with Respect to an Axis**

Test for symmetry with respect to the x-axis and the y-axis.

(a) $y = x^2 + 4$ **(b)** $x = y^2 - 3$ **(c)** $x^2 + y^2 = 16$ **(d)** $2x + y = 4$

SOLUTION

(a) In $y = x^2 + 4$, replace x with $-x$.

$$y = x^2 + 4$$

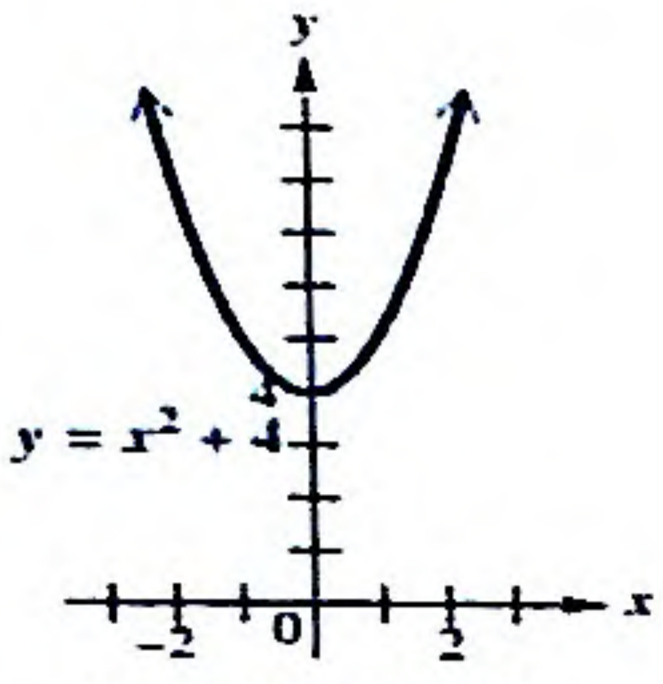
Use parentheses around $-x$.

$$y = (-x)^2 + 4$$

$$y = x^2 + 4$$

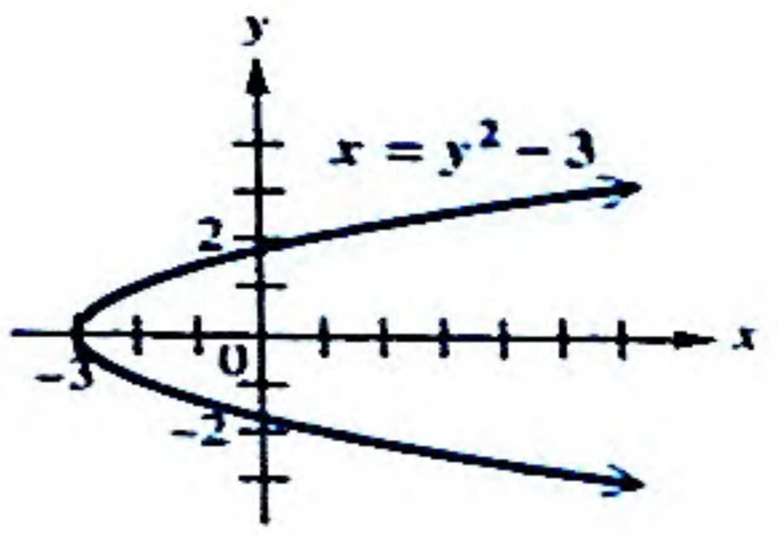
Equivalent

The result is the same as the original equation, so the graph, shown in **Figure 75**, is symmetric with respect to the y-axis. Substituting $-y$ for y does not result in an equivalent equation, and thus the graph is *not* symmetric with respect to the x-axis.



y-axis symmetry

Figure 75



x-axis symmetry

Figure 76

(b) In $x = y^2 - 3$, replace y with $-y$.

$$x = (-y)^2 - 3 = y^2 - 3 \quad \text{Same as the original equation}$$

The graph is symmetric with respect to the x-axis, as shown in **Figure 76.** It is *not* symmetric with respect to the y-axis.

EXAMPLE 9 Graphing Translations of a Given Graph

A graph of a function defined by $y = f(x)$ is shown in **Figure 90.** Use this graph to sketch each of the following graphs.

(a) $g(x) = f(x) + 3$ (b) $h(x) = f(x + 3)$

(c) $k(x) = f(x - 2) + 3$

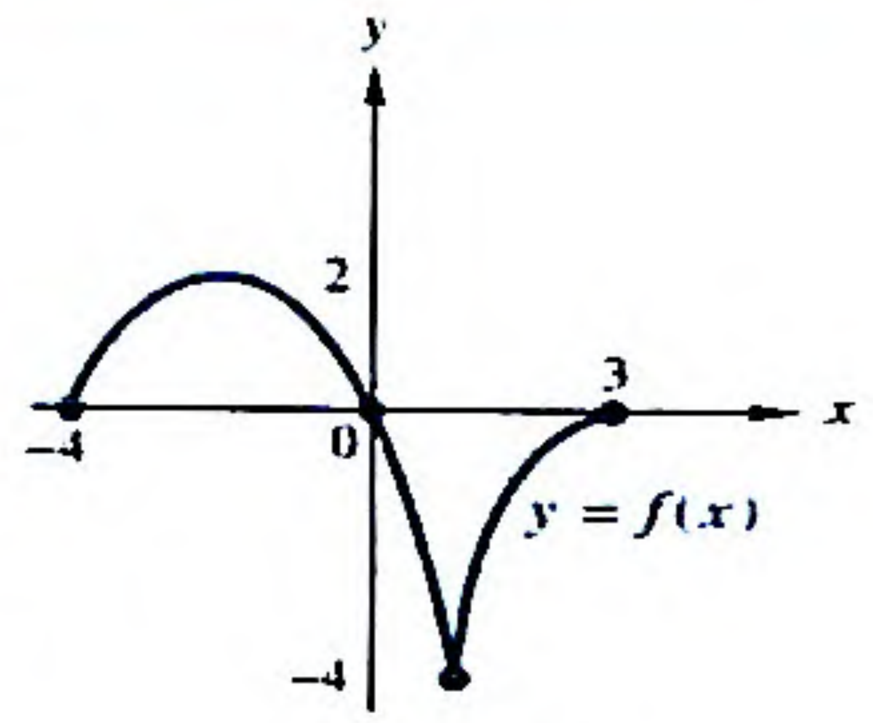
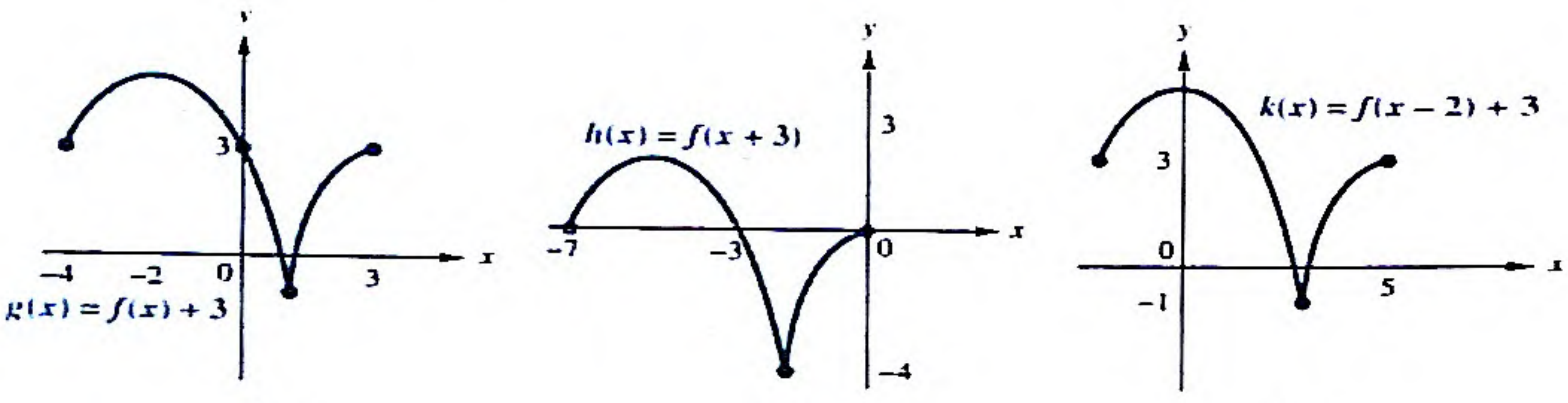


Figure 90

SOLUTION In each part, pay close attention to how the plotted points in **Figure 90** are translated.

(a) The graph of $g(x) = f(x) + 3$ is the same as the graph in **Figure 90,** translated 3 units up. See **Figure 91(a).**

(b) To get the graph of $h(x) = f(x + 3)$, the graph of $y = f(x)$ must be translated 3 units to the left since $x + 3 = 0$ if $x = -3$. See **Figure 91(b).**



(a) (b) (c)

Figure 91

(c) The graph of $k(x) = f(x - 2) + 3$ will look like the graph of $f(x)$ translated 2 units to the right and 3 units up, as shown in **Figure 91(c).**

✓ *Now Try Exercise 77.*

Summary of Graphing Techniques

In the descriptions that follow, assume that $a > 0$, $h > 0$, and $k > 0$. In comparison with the graph of $y = f(x)$:

1. The graph of $y = f(x) + k$ is translated k units up.
2. The graph of $y = f(x) - k$ is translated k units down.
3. The graph of $y = f(x + h)$ is translated h units to the left.
4. The graph of $y = f(x - h)$ is translated h units to the right.
5. The graph of $y = af(x)$ is a vertical stretching of the graph of $y = f(x)$ if $a > 1$. It is a vertical shrinking if $0 < a < 1$.
6. The graph of $y = f(ax)$ is a horizontal stretching of the graph of $y = f(x)$ if $0 < a < 1$. It is a horizontal shrinking if $a > 1$.
7. The graph of $y = -f(x)$ is reflected across the x-axis.
8. The graph of $y = f(-x)$ is reflected across the y-axis.

2.7 Exercises

1. ***Concept Check*** Match each equation in Column I with a description of its graph from Column II as it relates to the graph of $y = x^2$.

I	II
(a) $y = (x - 7)^2$	**A.** a translation 7 units to the left
(b) $y = x^2 - 7$	**B.** a translation 7 units to the right
(c) $y = 7x^2$	**C.** a translation 7 units up
(d) $y = (x + 7)^2$	**D.** a translation 7 units down
(e) $y = x^2 + 7$	**E.** a vertical stretching by a factor of 7

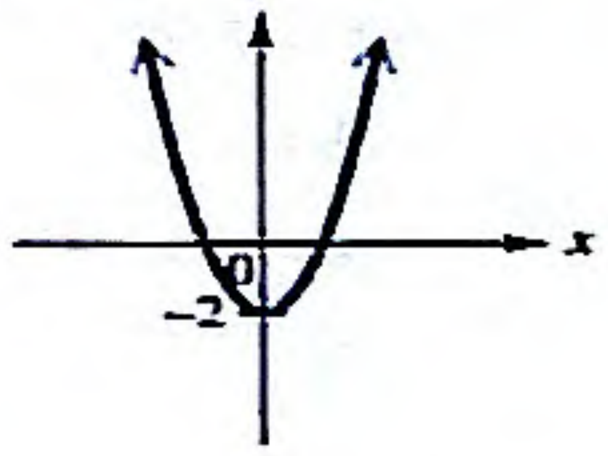
2. ***Concept Check*** Match each equation in Column I with a description of its graph from Column II as it relates to the graph of $y = \sqrt[3]{x}$.

I	II
(a) $y = 4\sqrt[3]{x}$	**A.** a translation 4 units to the right
(b) $y = -\sqrt[3]{x}$	**B.** a translation 4 units down
(c) $y = \sqrt[3]{-x}$	**C.** a reflection across the x-axis
(d) $y = \sqrt[3]{x - 4}$	**D.** a reflection across the y-axis
(e) $y = \sqrt[3]{x} - 4$	**E.** a vertical stretching by a factor of 4

3. ***Concept Check*** Match each equation in parts (a)–(i) with the sketch of its graph.

(a) $y = x^2 + 2$
(b) $y = x^2 - 2$
(c) $y = (x + 2)^2$
(d) $y = (x - 2)^2$
(e) $y = 2x^2$
(f) $y = -x^2$
(g) $y = (x - 2)^2 + 1$
(h) $y = (x + 2)^2 + 1$
(i) $y = (x + 2)^2 - 1$

A.



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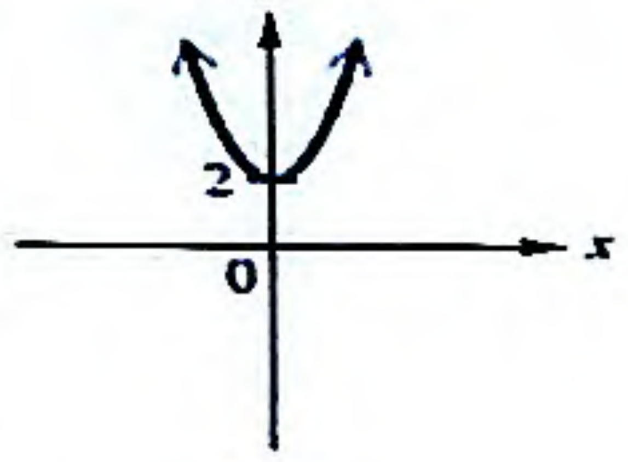

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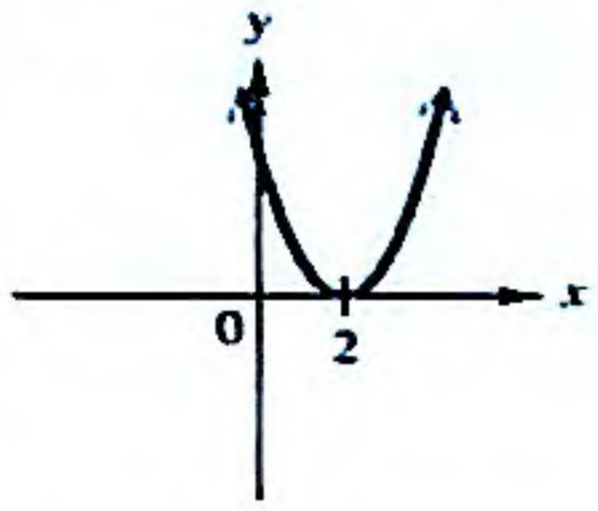

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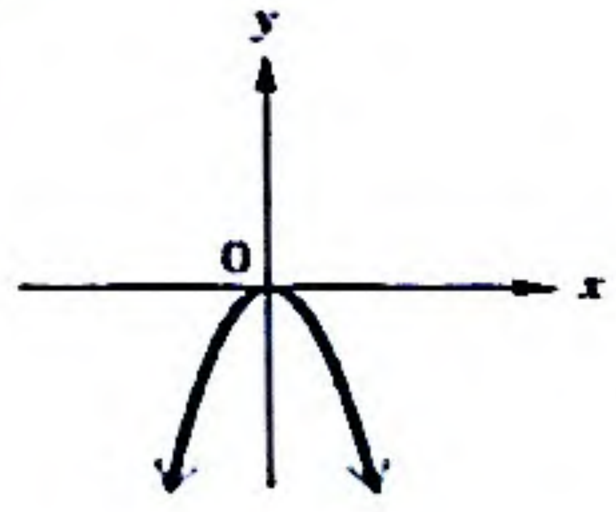

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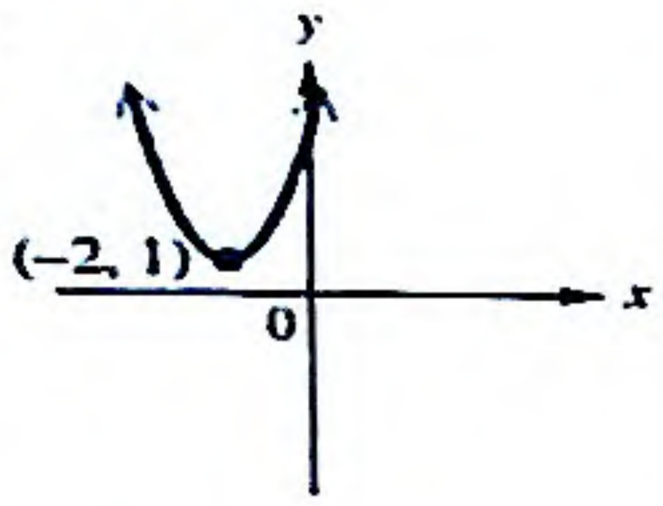

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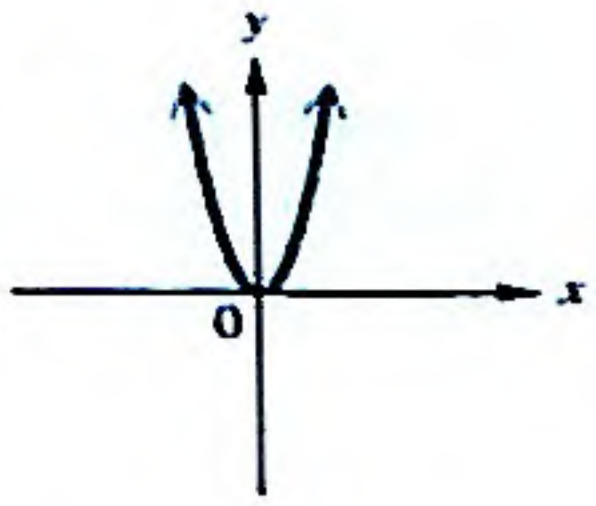

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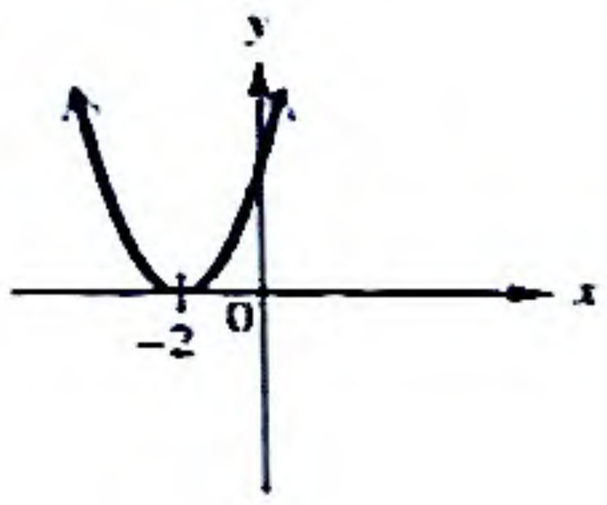

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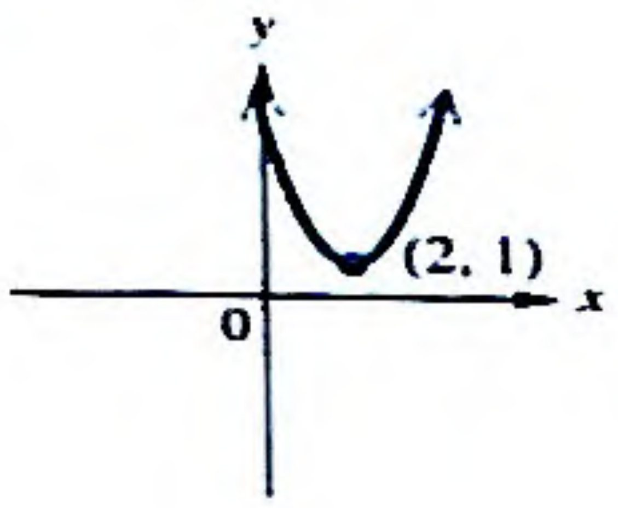
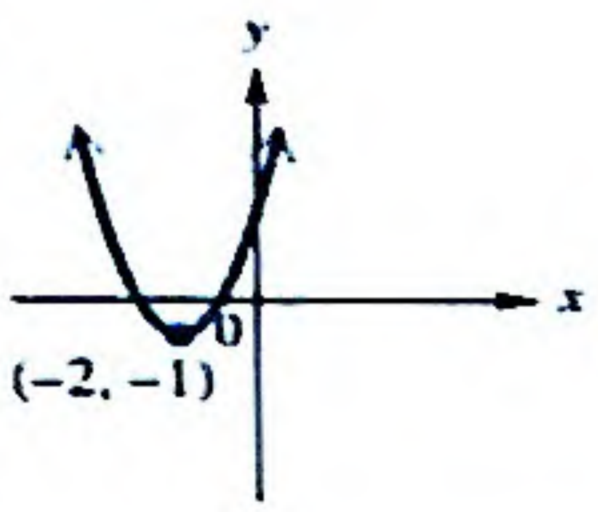


I.



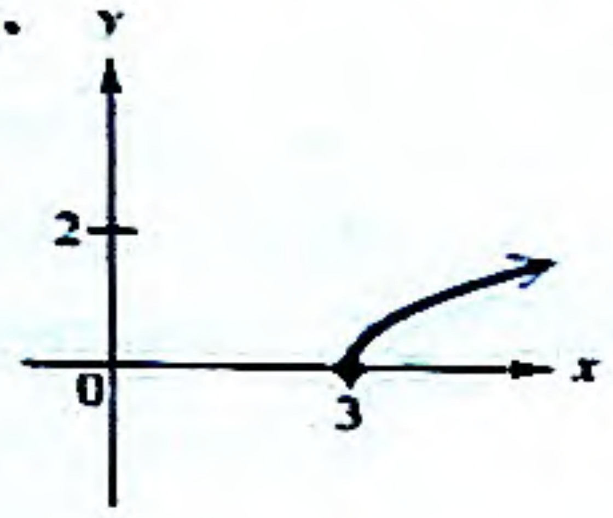
4. ***Concept Check*** Match each equation in parts (a)–(i) with the sketch of its graph.

(a) $y = \sqrt{x + 3}$ **(b)** $y = \sqrt{x} - 3$ **(c)** $y = \sqrt{x} + 3$

(d) $y = 3\sqrt{x}$ **(e)** $y = -\sqrt{x}$ **(f)** $y = \sqrt{x - 3}$

(g) $y = \sqrt{x - 3} + 2$ **(h)** $y = \sqrt{x + 3} + 2$ **(i)** $y = \sqrt{x - 3} - 2$

A.



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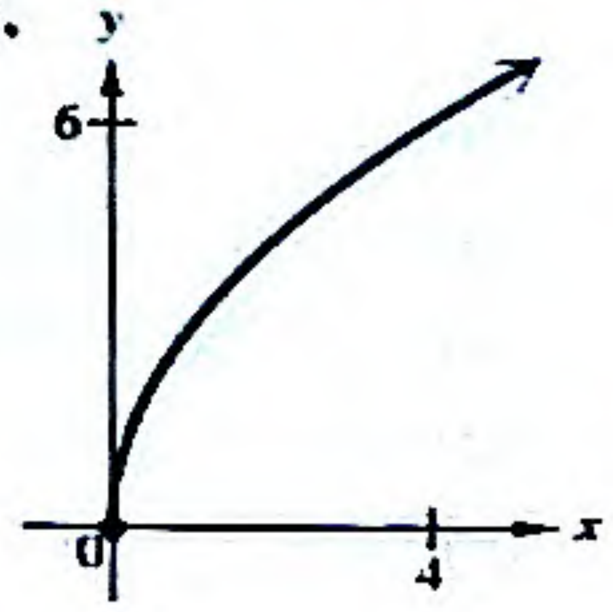

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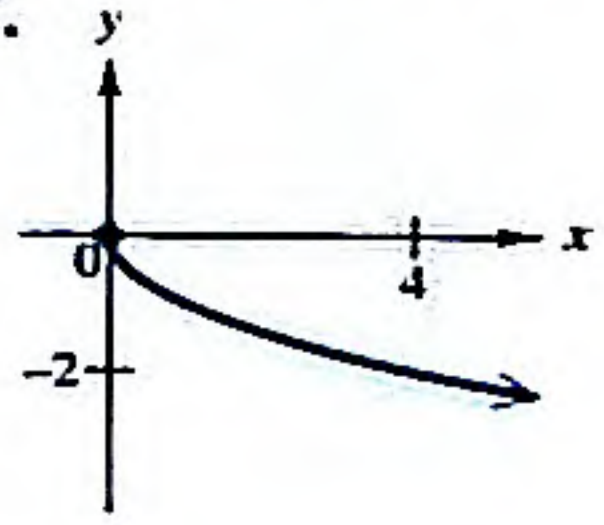

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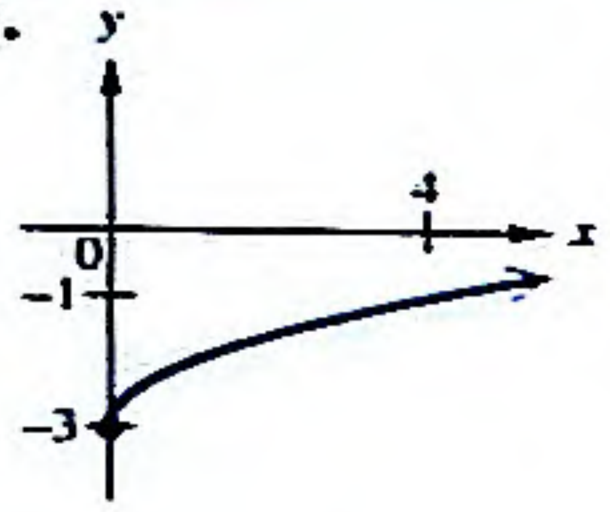

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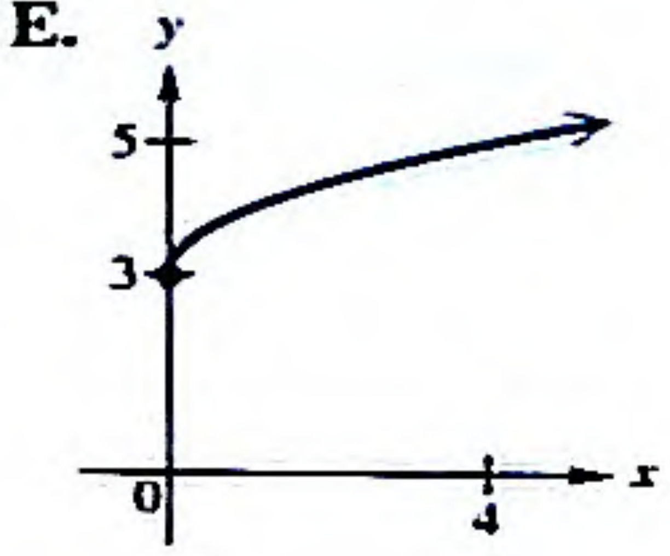

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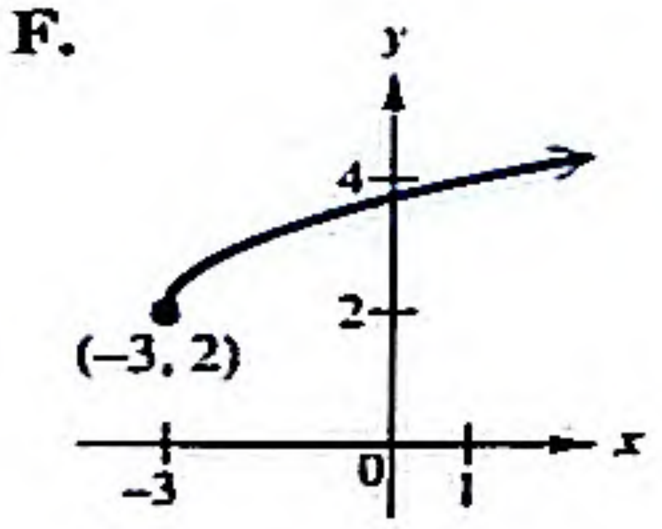

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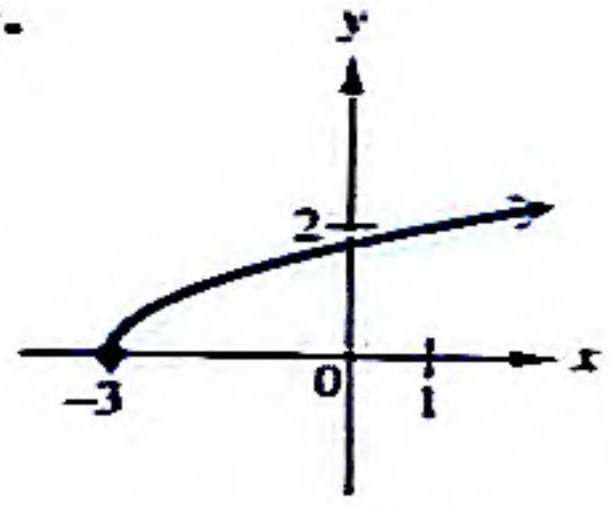

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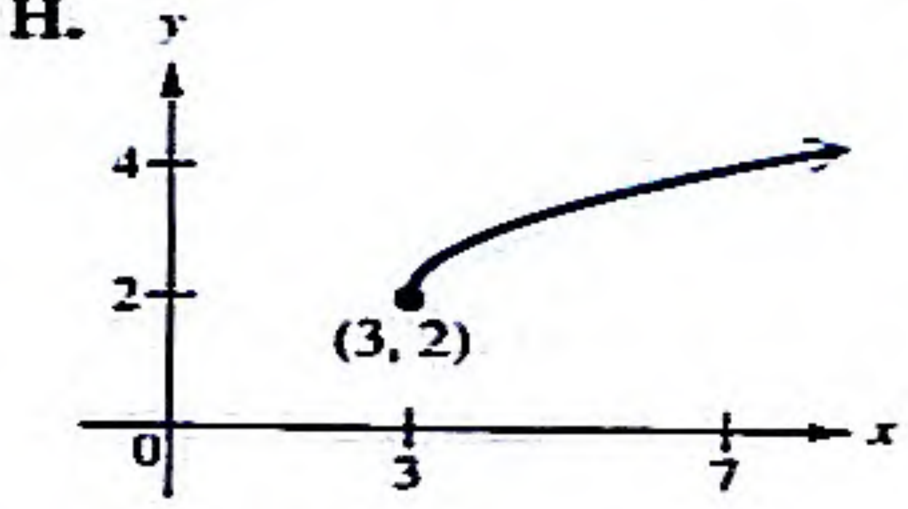

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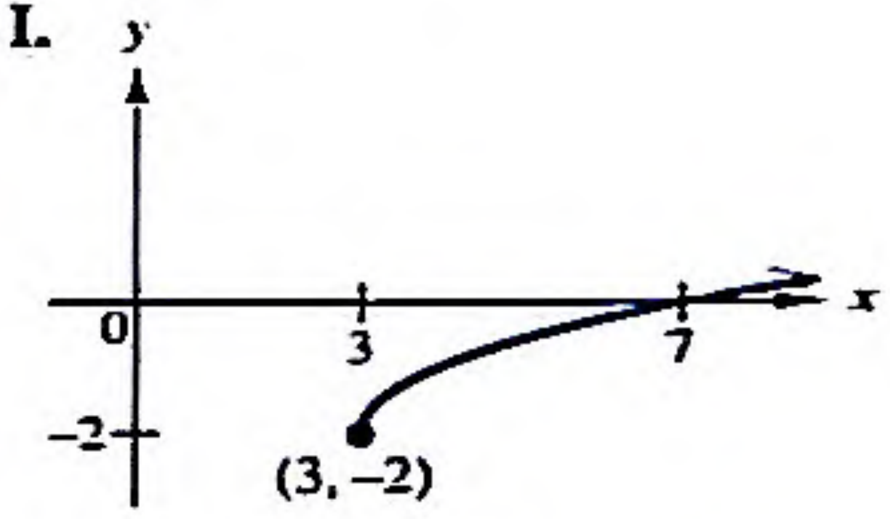


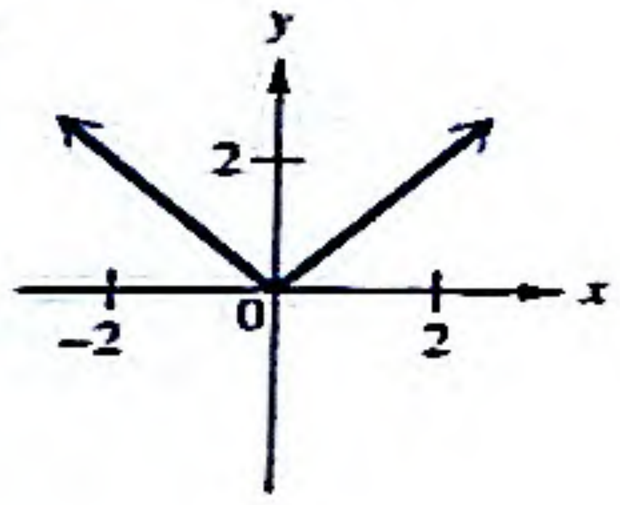
5. ***Concept Check*** Match each equation in parts (a)–(i) with the sketch of its graph.

(a) $y = |x - 2|$ **(b)** $y = |x| - 2$ **(c)** $y = |x| + 2$

(d) $y = 2|x|$ **(e)** $y = -|x|$ **(f)** $y = |-x|$

(g) $y = -2|x|$ **(h)** $y = |x - 2| + 2$ **(i)** $y = |x + 2| - 2$

A.



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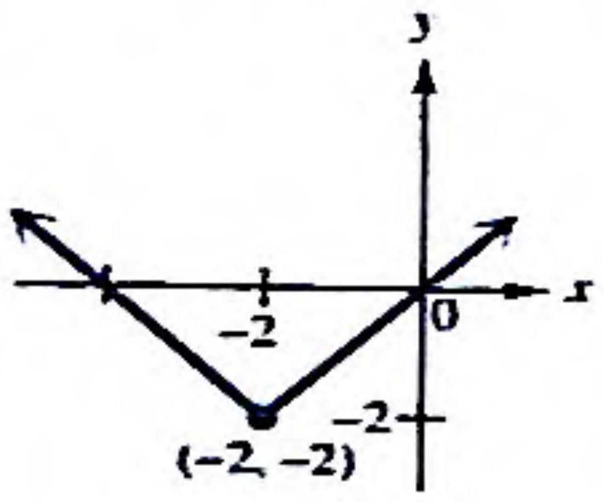

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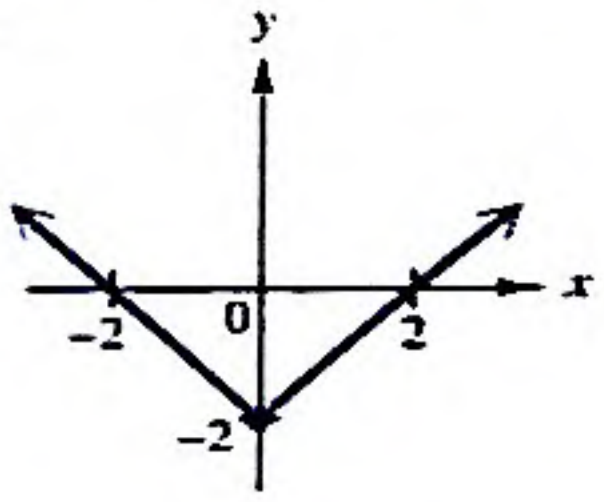

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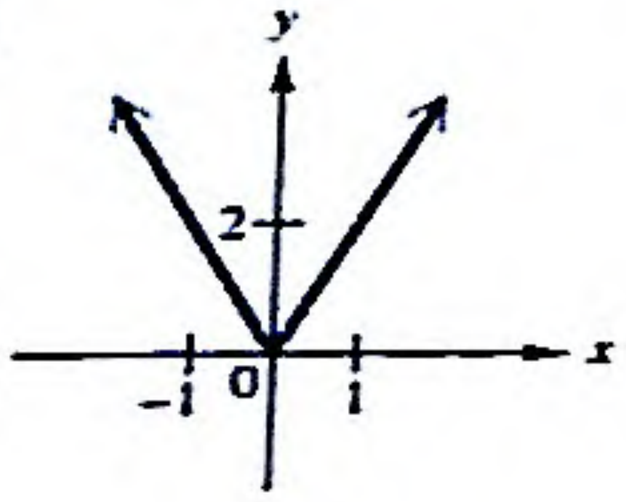

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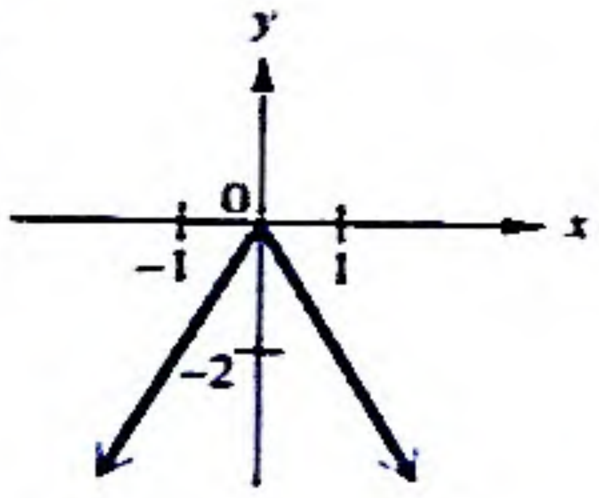

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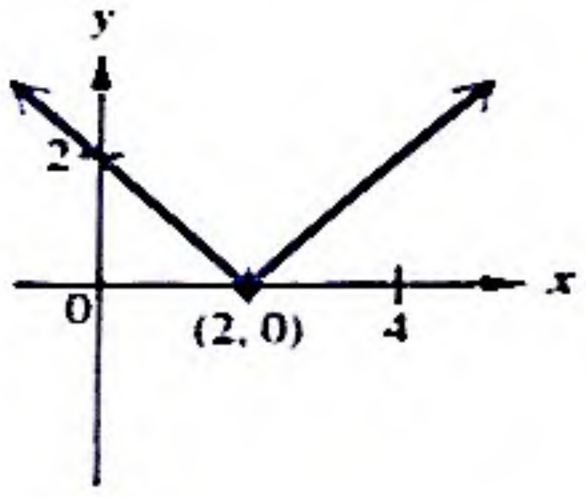

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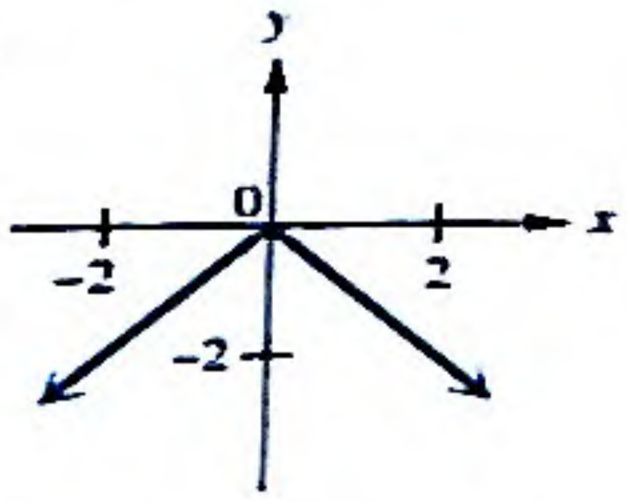

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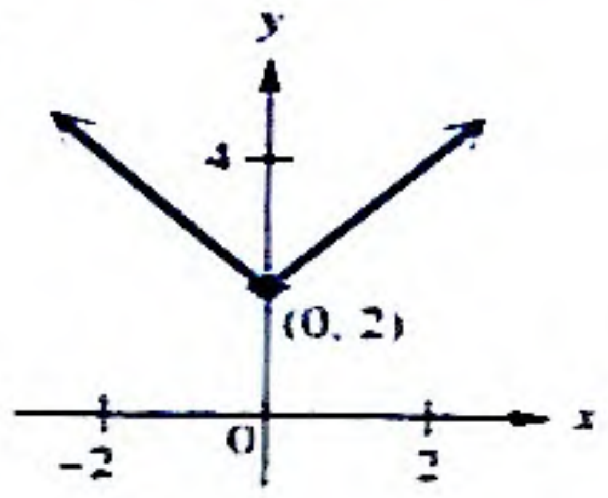

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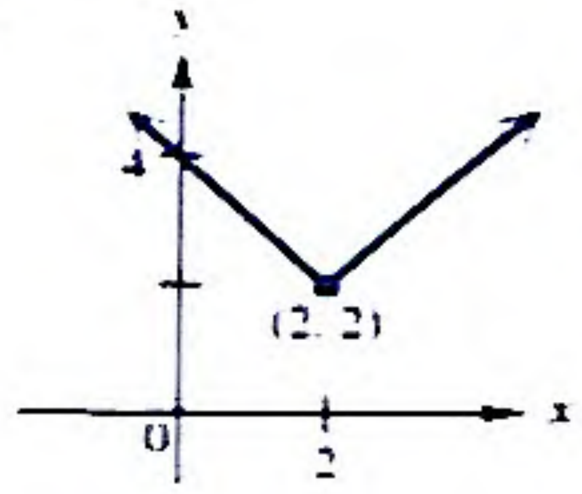


6. **(a)** Suppose the equation $y = F(x)$ is changed to $y = c \cdot F(x)$, for some constant c. What is the effect on the graph of $y = F(x)$? Discuss the effect depending on whether $c > 0$ or $c < 0$, and on whether $|c| > 1$ or $|c| < 1$.

(b) Suppose $y = F(x)$ is changed to $y = F(x + h)$. How are the graphs of these equations related? Is the graph of $y = F(x) + h$ the same as the graph of $y = F(x + h)$? If not, how do they differ?

Graph each function. See Examples 1 and 2.

7. $y = 3|x|$
8. $y = 4|x|$
9. $y = \frac{2}{3}|x|$
10. $y = \frac{3}{4}|x|$
11. $y = 2x^2$
12. $y = 3x^2$
13. $y = \frac{1}{2}x^2$
14. $y = \frac{1}{3}x^2$
15. $y = -\frac{1}{2}x^2$
16. $y = -\frac{1}{3}x^2$
17. $y = -3|x|$
18. $y = -2|x|$
19. $y = \left|-\frac{1}{2}x\right|$
20. $y = \left|-\frac{1}{3}x\right|$
21. $y = \sqrt{4x}$
22. $y = \sqrt{9x}$
23. $y = -\sqrt{-x}$
24. $y = -|-x|$

Concept Check In Exercises 25–28, suppose the point $(8, 12)$ is on the graph of $y = f(x)$. Find a point on the graph of each function.

25. **(a)** $y = f(x + 4)$ **(b)** $y = f(x) + 4$
26. **(a)** $y = \frac{1}{4}f(x)$ **(b)** $y = 4f(x)$
27. **(a)** $y = f(4x)$ **(b)** $y = f\left(\frac{1}{4}x\right)$

28. **(a)** the reflection of the graph of $y = f(x)$ across the x-axis
(b) the reflection of the graph of $y = f(x)$ across the y-axis

Concept Check Plot each point, and then plot the points that are symmetric to the given point with respect to the (a) x-axis, (b) y-axis, and (c) origin.

29. $(5, -3)$
30. $(-6, 1)$
31. $(-4, -2)$
32. $(-8, 0)$

33. *Concept Check* The graph of $y = |x - 2|$ is symmetric with respect to a vertical line. What is the equation of that line?

34. *Concept Check* Repeat Exercise 33 for $y = -|x + 1|$.

Without graphing, determine whether each equation has a graph that is symmetric with respect to the x-axis, the y-axis, the origin, or none of these. See Examples 3 and 4.

35. $y = x^2 + 5$
36. $y = 2x^4 - 3$
37. $x^2 + y^2 = 12$
38. $y^2 - x^2 = -6$
39. $y = -4x^3 + x$
40. $y = x^3 - x$
41. $y = x^2 - x + 8$
42. $y = x + 15$

Determine whether each function is even, odd, or neither. See Example 5.

43. $f(x) = -x^3 + 2x$
44. $f(x) = x^5 - 2x^3$
45. $f(x) = 0.5x^4 - 2x^2 + 6$
46. $f(x) = 0.75x^2 + |x| + 4$
47. $f(x) = x^3 - x + 9$
48. $f(x) = x^4 - 5x + 8$

Graph each function. See Examples 6–8.

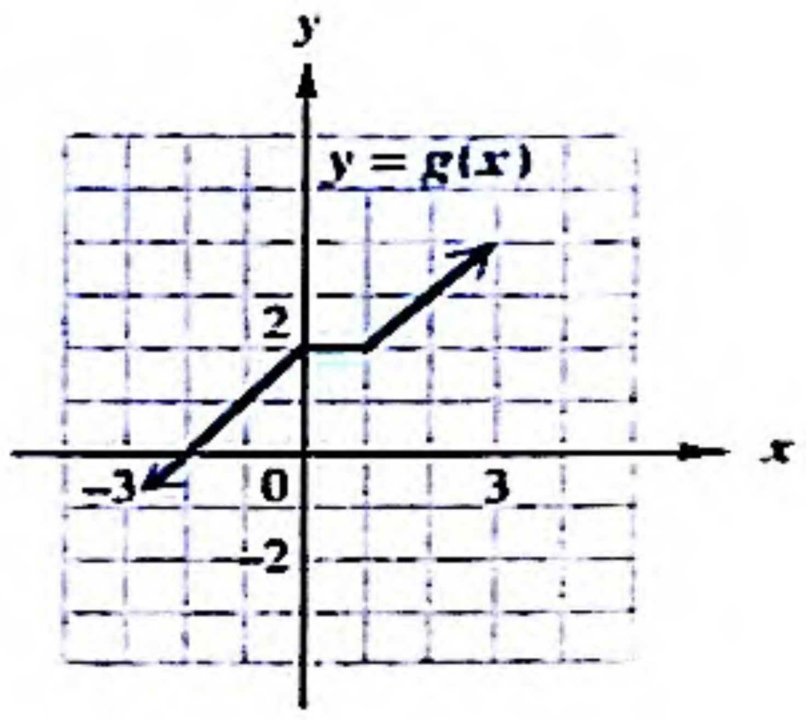
49. $y = x^2 - 1$
50. $y = x^2 - 2$
51. $y = x^2 + 2$
52. $y = x^2 + 3$
53. $y = (x - 4)^2$
54. $y = (x - 2)^2$
55. $y = (x + 2)^2$
56. $y = (x + 3)^2$
57. $y = |x| - 1$
58. $y = |x + 3| + 2$
59. $y = -(x + 1)^3$
60. $y = -(x - 1)^3$

61. $y = 2x^2 - 1$
62. $y = 3x^2 - 2$
63. $f(x) = 2(x - 2)^2 - 4$
64. $f(x) = -3(x - 2)^2 + 1$
65. $f(x) = \sqrt{x + 2}$
66. $f(x) = \sqrt{x - 3}$
67. $f(x) = -\sqrt{x}$
68. $f(x) = \sqrt{x} - 2$
69. $f(x) = 2\sqrt{x} + 1$
70. $y = 3\sqrt{x} - 2$
71. $y = \frac{1}{2}x^3 - 4$
72. $y = \frac{1}{2}x^3 + 2$
73. $y = (x + 3)^3$
74. $y = (x - 2)^3$
75. $y = \frac{2}{3}(x - 2)^2$

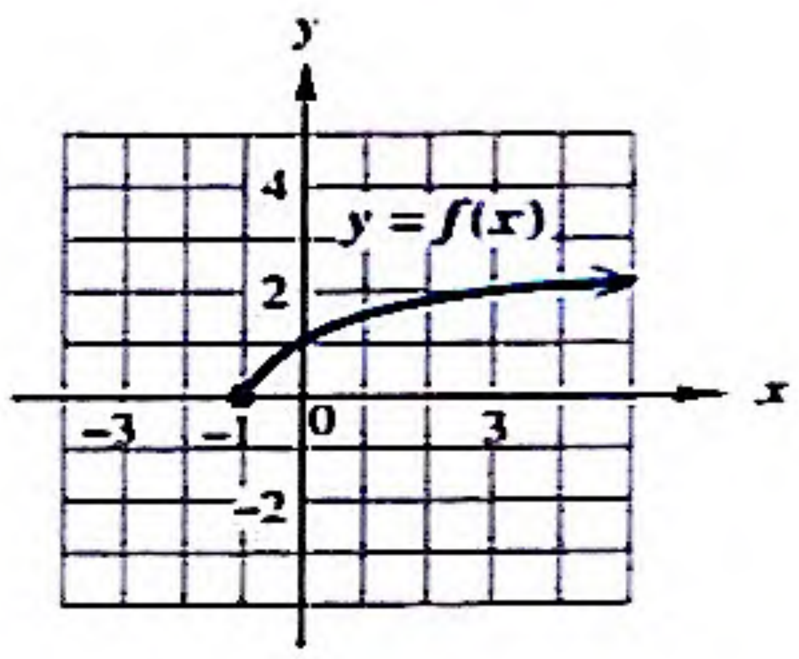
76. **Concept Check** What is the relationship between the graphs of $f(x) = |x|$ and $g(x) = |-x|$?

For Exercises 77 and 78, see Example 9.

77. Given the graph of $y = g(x)$ in the figure, sketch the graph of each function, and explain how it is obtained from the graph of $y = g(x)$.
(a) $y = g(-x)$
(b) $y = g(x - 2)$
(c) $y = -g(x) + 2$

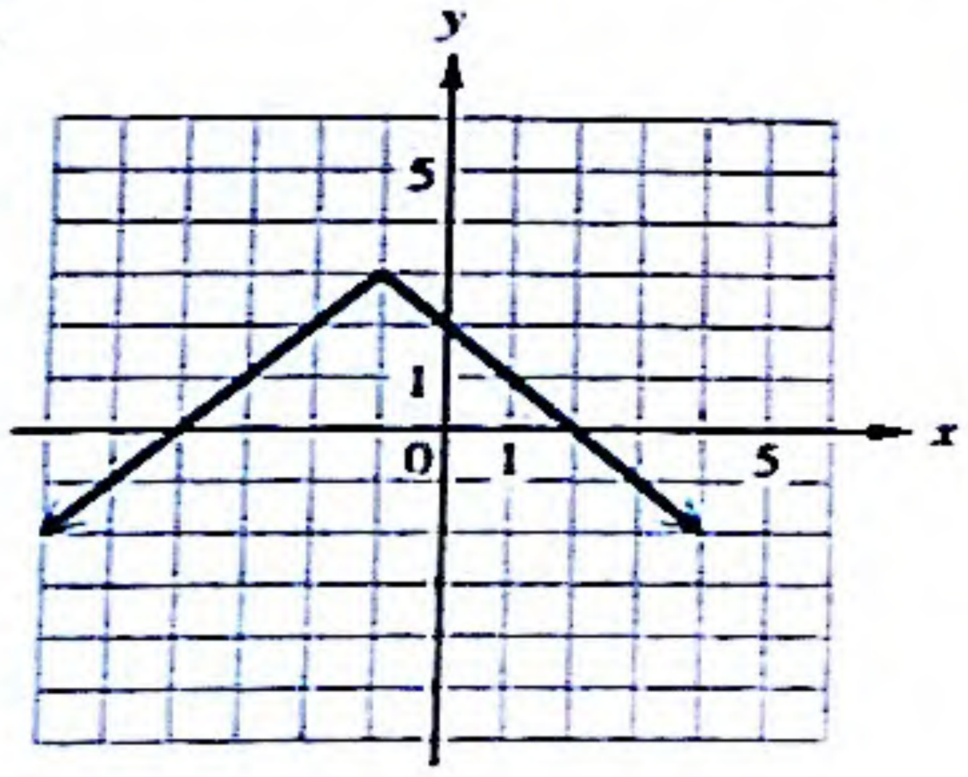


78. Given the graph of $y = f(x)$ in the figure, sketch the graph of each function, and explain how it is obtained from the graph of $y = f(x)$.
(a) $y = -f(x)$
(b) $y = 2f(x)$
(c) $y = f(-x)$



Connecting Graphs with Equations *Each of the following graphs is obtained from the graph of $f(x) = |x|$ or $g(x) = \sqrt{x}$ by applying several of the transformations discussed in this section. Describe the transformations and give the equation for the graph.*

79.



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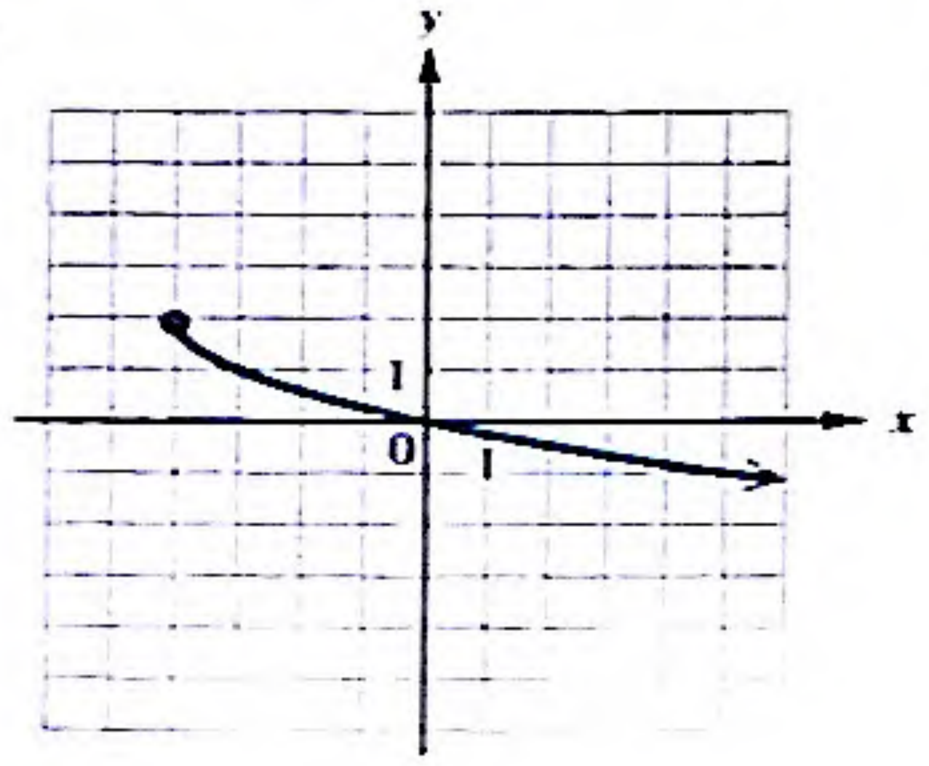

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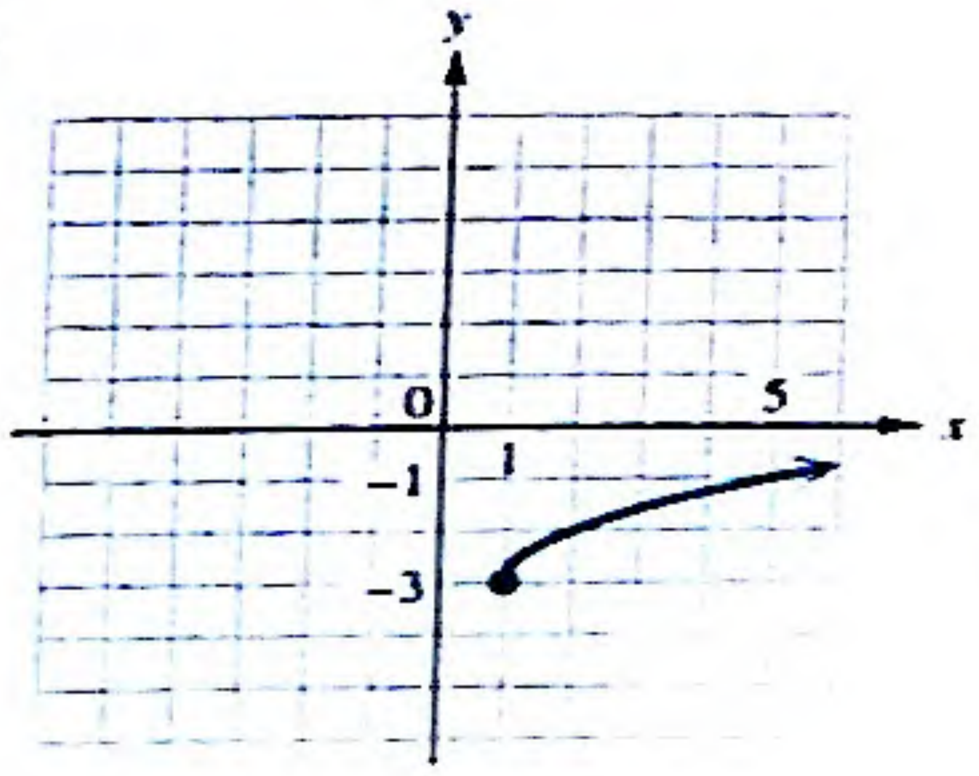

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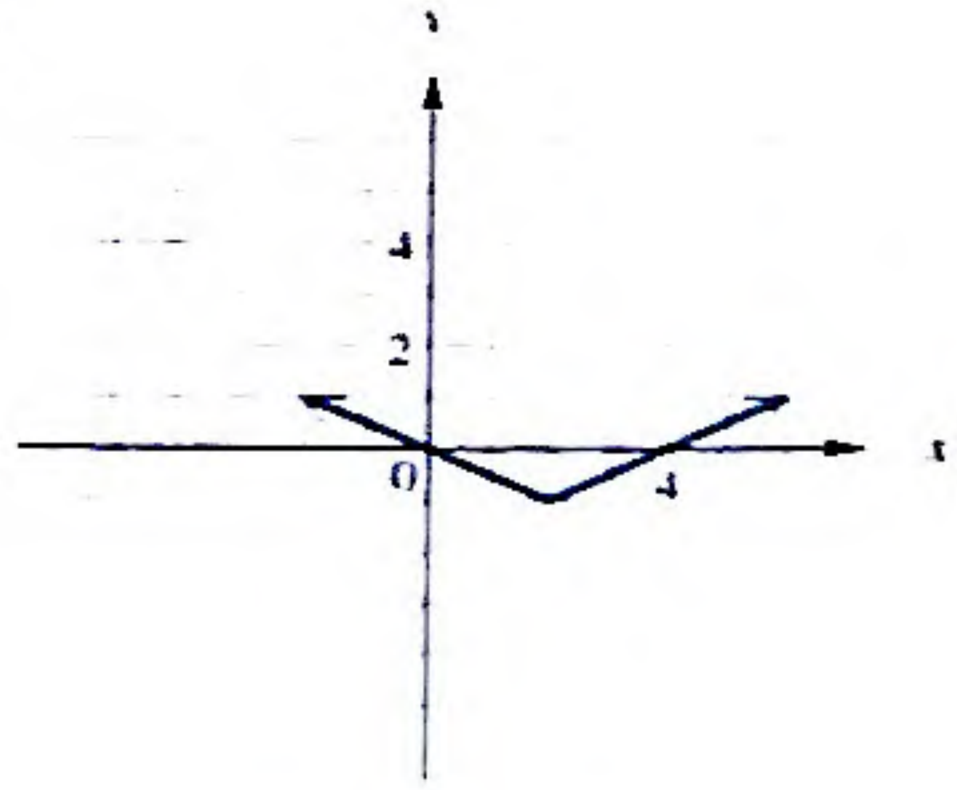

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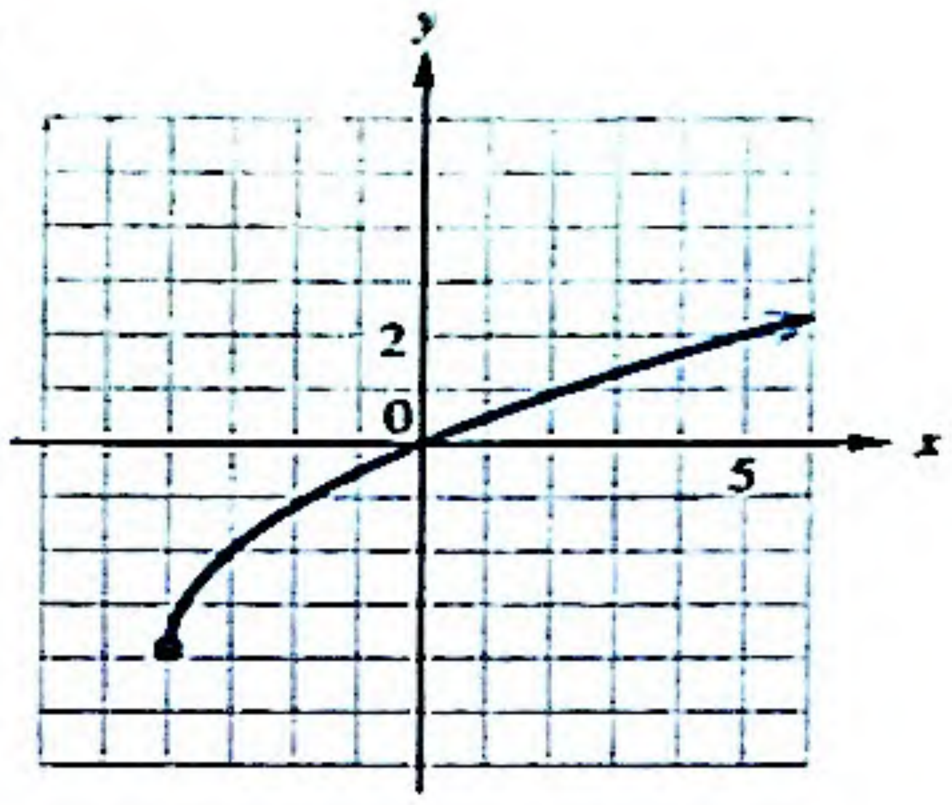

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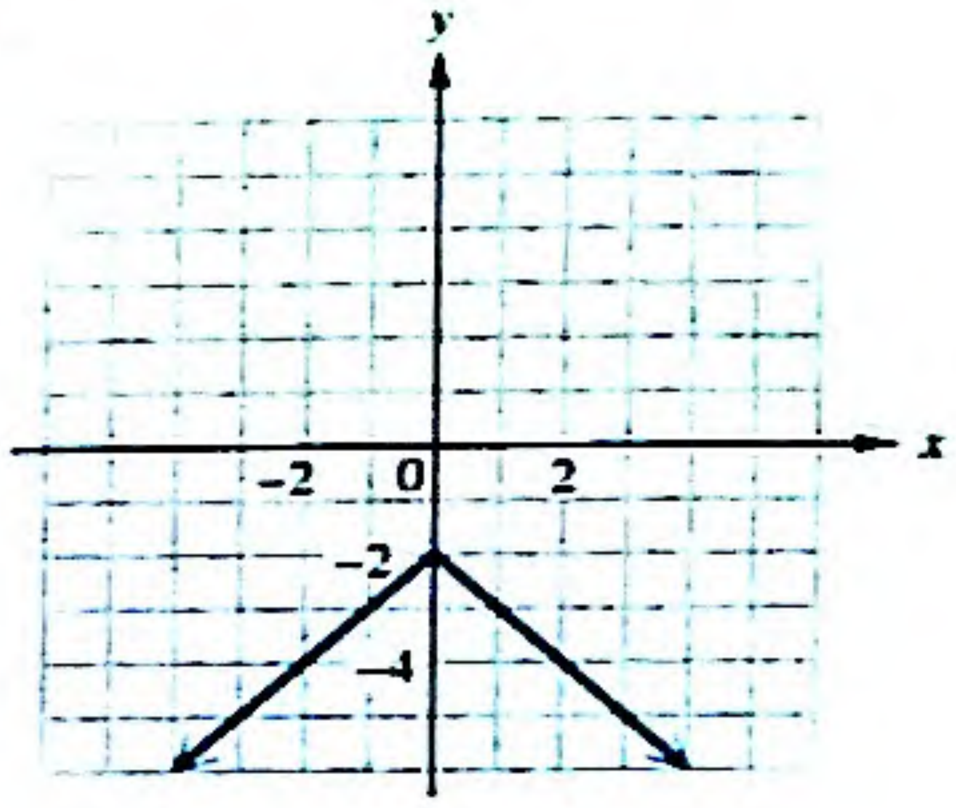


Concept Check *Suppose that for a function f, $f(3) = 6$. For the given assumptions in Exercises 85–90, find another function value.*

85. The graph of $y = f(x)$ is symmetric with respect to the origin.

86. The graph of $y = f(x)$ is symmetric with respect to the y-axis.

87. The graph of $y = f(x)$ is symmetric with respect to the line $x = 6$.

88. For all x, $f(-x) = f(x)$.

89. For all x, $f(-x) = -f(x)$.

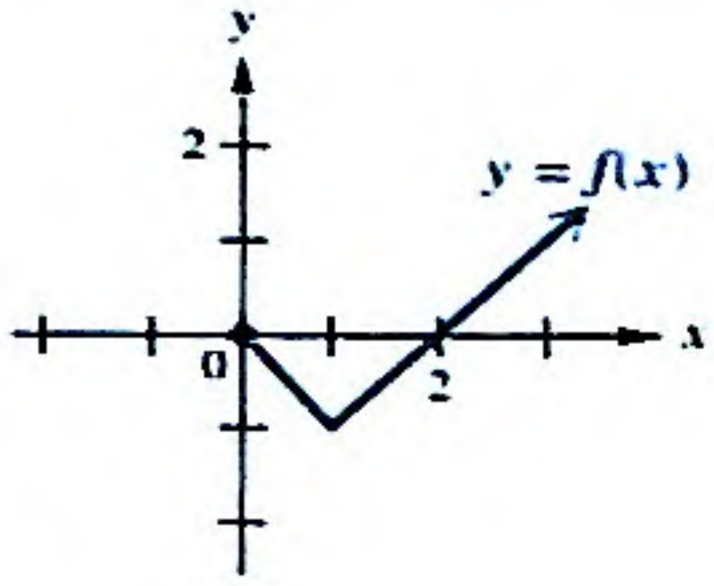
90. f is an odd function.

91. Find the function $g(x) = ax + b$ whose graph can be obtained by translating the graph of $f(x) = 2x + 5$ up 2 units and to the left 3 units.

92. Find the function $g(x) = ax + b$ whose graph can be obtained by translating the graph of $f(x) = 3 - x$ down 2 units and to the right 3 units.

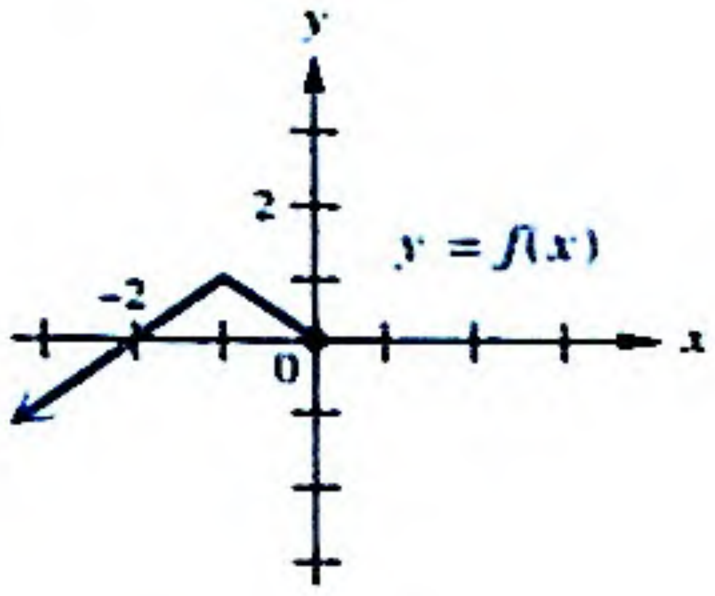
93. *Concept Check* Complete the left half of the graph of $y = f(x)$ in the figure for each condition.

(a) $f(-x) = f(x)$ **(b)** $f(-x) = -f(x)$



94. *Concept Check* Complete the right half of the graph of $y = f(x)$ in the figure for each condition.

(a) f is odd. **(b)** f is even.



Chapter 2 Quiz (Sections 2.5–2.7)

1. For the line passing through the points $(-3, 5)$ and $(-1, 9)$, find the following:

(a) the slope-intercept form of its equation **(b)** its x-intercept.

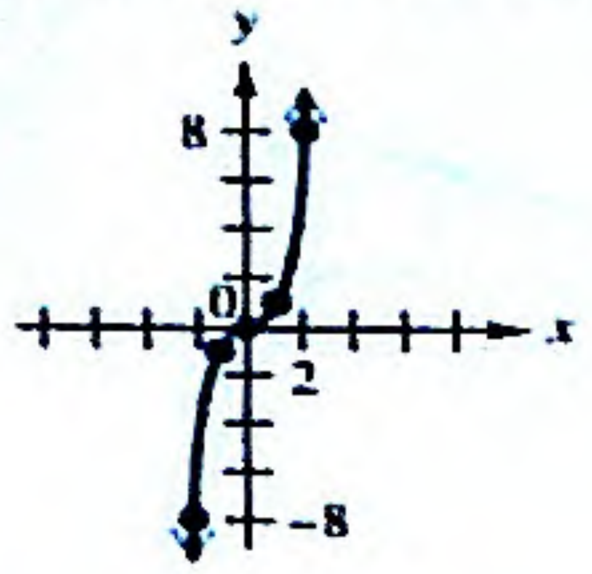
2. Find the slope-intercept form of the equation of the line passing through the point $(-6, 4)$ and perpendicular to the graph of $3x - 2y = 6$.

3. Suppose that P has coordinates $(-8, 5)$. Find the equation of the line through P that is

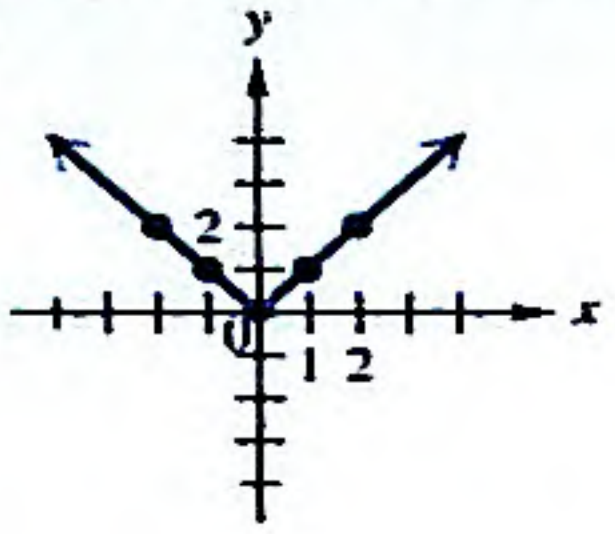
(a) vertical **(b)** horizontal

4. For each basic function graphed, give the name of the function, the domain, the range, and intervals over which it is decreasing, increasing, or constant.

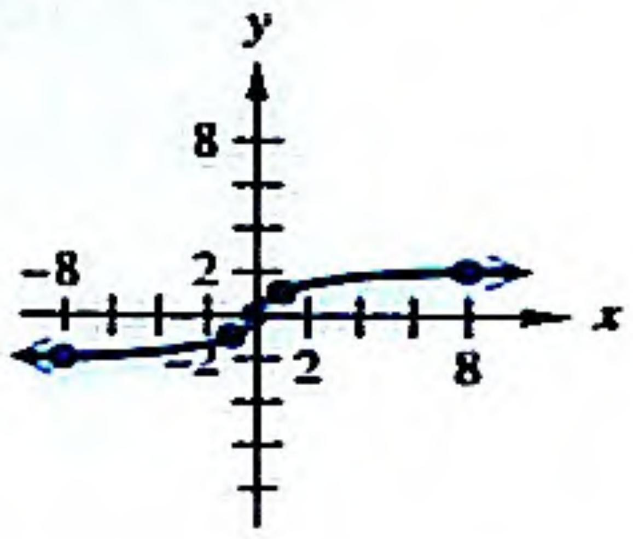
(a)



(b)



(c)



5. *(Modeling) Long-Distance Call Charges* A certain long-distance carrier provides service between Podunk and Nowheresville. If x represents the number of minutes for the call, where $x > 0$, then the function

$$f(x) = 0.40[\![x]\!] + 0.75$$

gives the total cost of the call in dollars. Find the cost of a 5.5-min call.

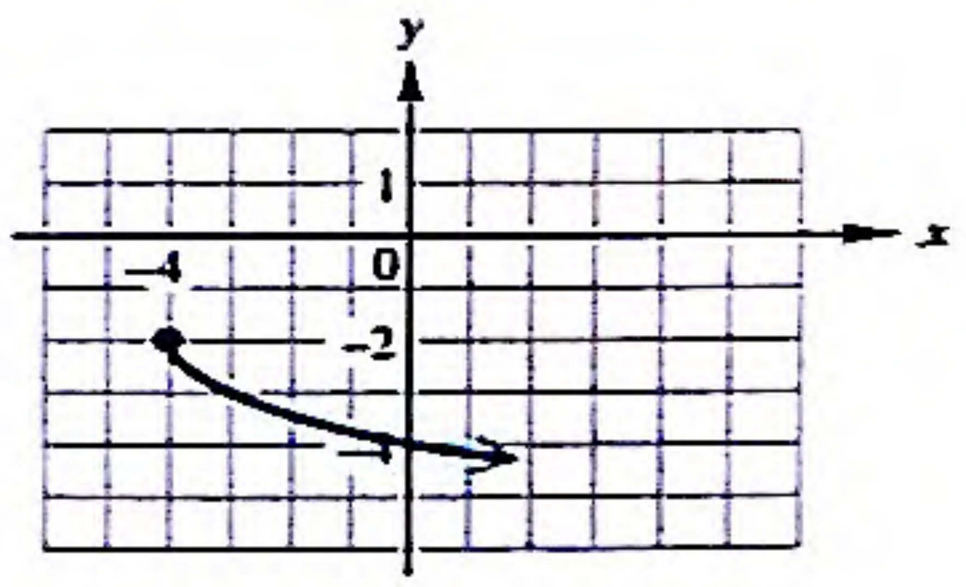
Graph each function.

6. $f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ 2x + 3 & \text{if } x < 0 \end{cases}$

7. $f(x) = -x^3 + 1$

8. $f(x) = 2|x - 1| + 3$

9. *Connecting Graphs with Equations* The function graphed here is obtained by stretching, shrinking, reflecting, and/or translating the graph of $f(x) = \sqrt{x}$. Give the equation that defines this function.



10. Determine whether each function is *even*, *odd*, or *neither*.

(a) $f(x) = x^2 - 7$ (b) $f(x) = x^3 - x - 1$ (c) $f(x) = x^{101} - x^{99}$

2.8 Function Operations and Composition

- Arithmetic Operations on Functions
- The Difference Quotient
- Composition of Functions and Domain

Arithmetic Operations on Functions **Figure 92** shows the situation for a company that manufactures DVDs. The two lines are the graphs of the linear functions for revenue $R(x) = 168x$ and cost $C(x) = 118x + 800$, where x is the number of DVDs produced and sold, and x, $R(x)$, and $C(x)$ are given in thousands. When 30,000 (that is, 30 thousand) DVDs are produced and sold, profit is found as follows.

$$\begin{aligned} P(x) &= R(x) - C(x) && \text{Profit function (Section 2.4)} \\ P(30) &= R(30) - C(30) && \text{Let } x = 30 \\ &= 5040 - 4340 && R(30) = 168(30),\ C(30) = 118(30) + 800 \\ P(30) &= 700 && \text{Subtract} \end{aligned}$$

Thus, the profit from the sale of 30,000 DVDs is $700,000.

The profit function is found by *subtracting* the cost function from the revenue function. New functions can be formed by using other operations as well.

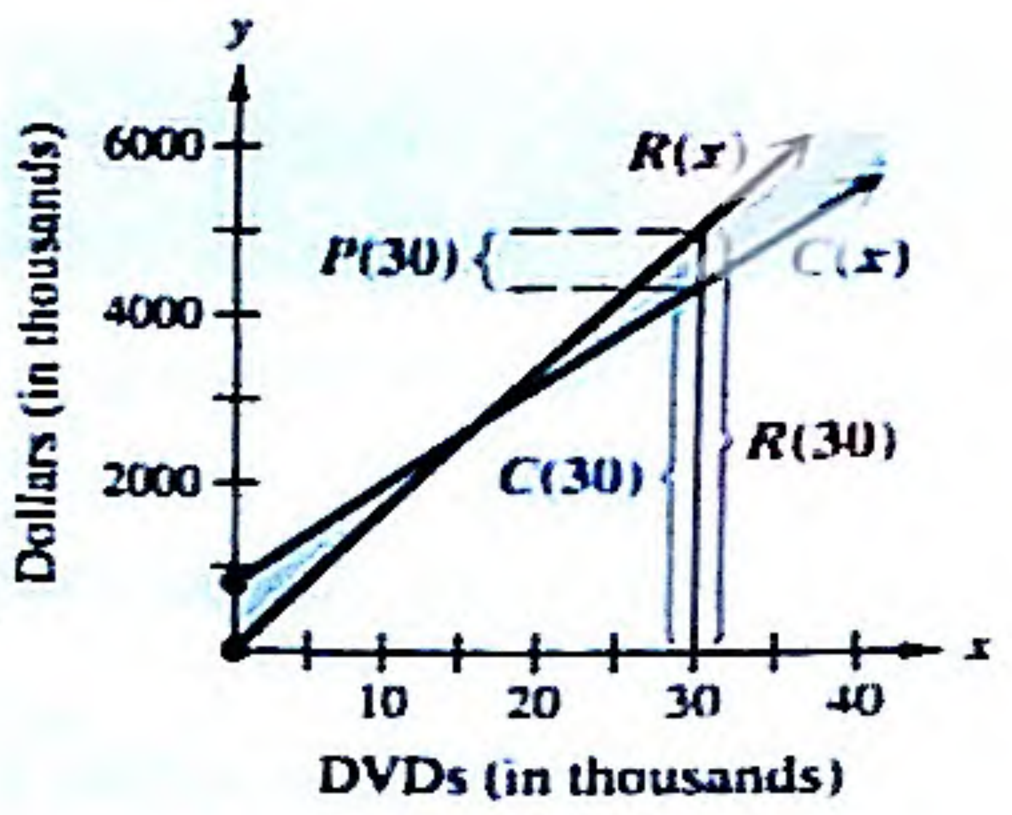


Figure 92

Operations on Functions and Domains

Given two functions f and g, then for all values of x for which both $f(x)$ and $g(x)$ are defined, the functions $f+g$, $f-g$, fg, and $\frac{f}{g}$ are defined as follows.

$$(f+g)(x) = f(x) + g(x) \qquad \text{Sum}$$

$$(f-g)(x) = f(x) - g(x) \qquad \text{Difference}$$

$$(fg)(x) = f(x) \cdot g(x) \qquad \text{Product}$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0 \qquad \text{Quotient}$$

The domains of $f+g$, $f-g$, and fg include all real numbers in the intersection of the domains of f and g, while the domain of $\frac{f}{g}$ includes those real numbers in the intersection of the domains of f and g for which $g(x) \neq 0$.

NOTE The condition $g(x) \neq 0$ in the definition of the quotient means that the domain of $\left(\frac{f}{g}\right)(x)$ is restricted to all values of x for which $g(x)$ is not 0. The condition does not mean that $g(x)$ is a function that is never 0.

EXAMPLE 1 Using Operations on Functions

Let $f(x) = x^2 + 1$ and $g(x) = 3x + 5$. Find each of the following.

(a) $(f+g)(1)$ **(b)** $(f-g)(-3)$ **(c)** $(fg)(5)$ **(d)** $\left(\frac{f}{g}\right)(0)$

SOLUTION

(a) First determine $f(1) = 2$ and $g(1) = 8$. Then use the definition.

$$\begin{aligned} (f+g)(1) &= f(1) + g(1) && (f+g)(x) = f(x) + g(x) \\ &= 2 + 8 && f(1) = 1^2 + 1,\ g(1) = 3(1) + 5 \\ &= 10 && \text{Add.} \end{aligned}$$

(b)

$$\begin{aligned} (f-g)(-3) &= f(-3) - g(-3) && (f-g)(x) = f(x) - g(x) \\ &= 10 - (-4) && f(-3) = (-3)^2 + 1,\ g(-3) = 3(-3) + 5 \\ &= 14 && \text{Subtract.} \end{aligned}$$

(c)

$$\begin{aligned} (fg)(5) &= f(5) \cdot g(5) \\ &= (5^2 + 1)(3 \cdot 5 + 5) \\ &= 26 \cdot 20 \\ &= 520 \end{aligned}$$

(d)

$$\begin{aligned} \left(\frac{f}{g}\right)(0) &= \frac{f(0)}{g(0)} && \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \\ &= \frac{0^2 + 1}{3(0) + 5} && f(x) = x^2 + 1,\ g(x) = 3x + 5 \\ &= \frac{1}{5} && \text{Simplify.} \end{aligned}$$

✔ *Now Try Exercises 1, 3, 5, and 7.*

As Example 8 shows, *it is not always true that $f \circ g = g \circ f$*. In fact, the composite functions $f \circ g$ and $g \circ f$ are equal only for a special class of functions, discussed in Section 4.1.

In calculus it is sometimes necessary to treat a function as a composition of two functions. The next example shows how this can be done.

EXAMPLE 9 Finding Functions That Form a Given Composite

Find functions f and g such that

$$(f \circ g)(x) = (x^2 - 5)^3 - 4(x^2 - 5) + 3.$$

SOLUTION Note the repeated quantity $x^2 - 5$. If we choose $g(x) = x^2 - 5$ and $f(x) = x^3 - 4x + 3$, then we have the following.

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) && \text{By definition} \\ &= f(x^2 - 5) && g(x) = x^2 - 5 \\ &= (x^2 - 5)^3 - 4(x^2 - 5) + 3 && \text{Use the rule for } f. \end{aligned}$$

There are other pairs of functions f and g that also satisfy these conditions. Here is another such pair.

$$f(x) = (x - 5)^3 - 4(x - 5) + 3 \quad \text{and} \quad g(x) = x^2$$

✓ ***Now Try Exercise 89.***

2.8 Exercises

Let $f(x) = x^2 + 3$ and $g(x) = -2x + 6$. Find each of the following. See Example 1.

1. $(f + g)(3)$
2. $(f + g)(-5)$
3. $(f - g)(-1)$
4. $(f - g)(4)$
5. $(fg)(4)$
6. $(fg)(-3)$
7. $\left(\frac{f}{g}\right)(-1)$
8. $\left(\frac{f}{g}\right)(5)$

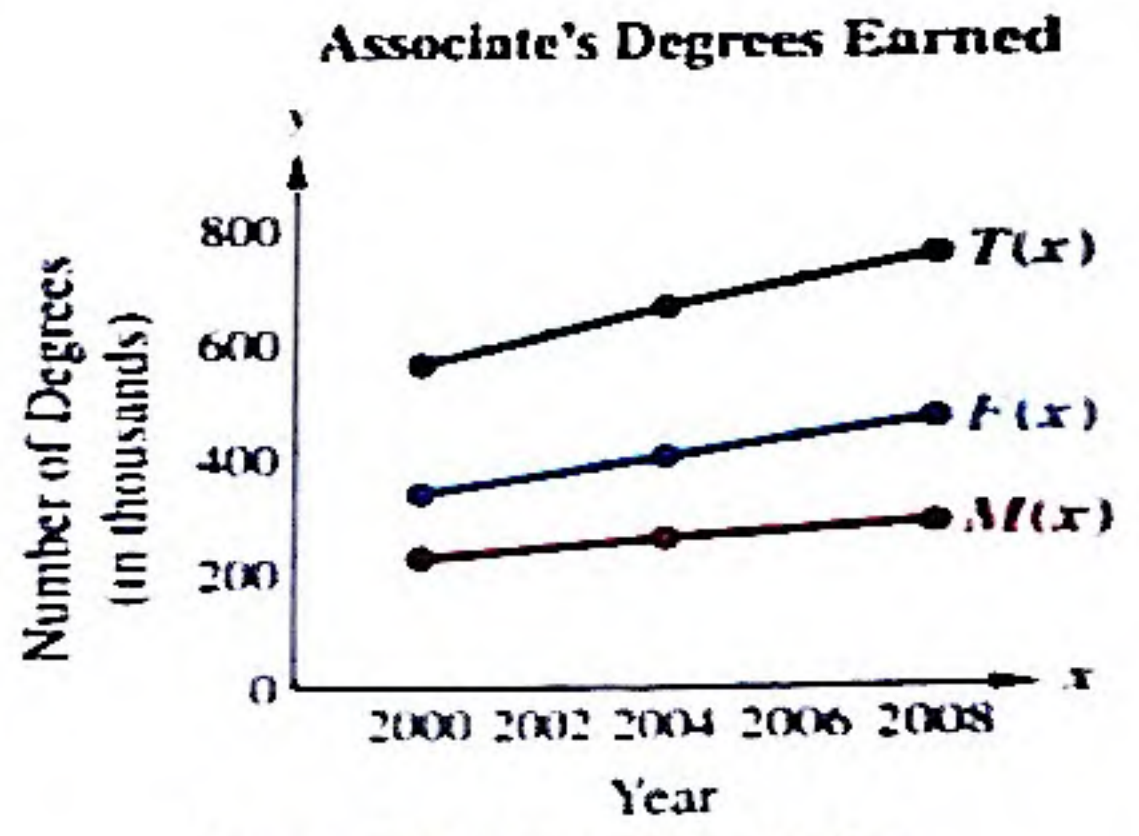
For the pair of functions defined, find $(f + g)(x)$, $(f - g)(x)$, $(fg)(x)$, and $\left(\frac{f}{g}\right)(x)$. Give the domain of each. See Example 2.

9. $f(x) = 3x + 4$, $g(x) = 2x - 5$
10. $f(x) = 6 - 3x$, $g(x) = -4x + 1$
11. $f(x) = 2x^2 - 3x$, $g(x) = x^2 - x + 3$
12. $f(x) = 4x^2 + 2x$, $g(x) = x^2 - 3x + 2$
13. $f(x) = \sqrt{4x - 1}$, $g(x) = \frac{1}{x}$
14. $f(x) = \sqrt{5x - 4}$, $g(x) = -\frac{1}{x}$

Associate's Degrees Earned *The graph shows the number of associate's degrees earned (in thousands) in the United States from 2000 through 2008. $M(x)$ gives the number of degrees earned by males, $F(x)$ gives the number earned by females, and $T(x)$ gives the total number for both groups. Use the graph in Exercises 15–18.*

15. Estimate $M(2004)$ and $F(2004)$, and use your results to estimate $T(2004)$.

16. Estimate $M(2008)$ and $F(2008)$, and use your results to estimate $T(2008)$.



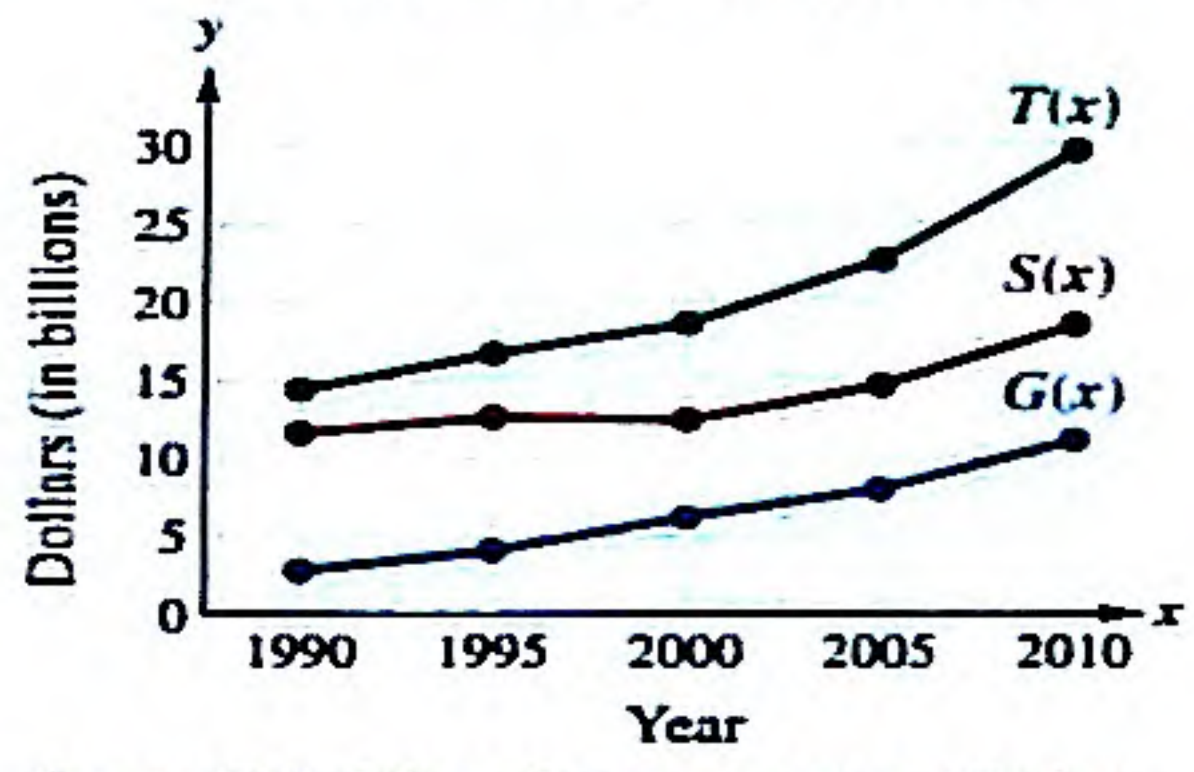
Source: U.S. National Center for Education Statistics

17. Use the slopes of the line segments to decide in which period (2000–2004 or 2004–2008) the total number of associate's degrees earned increased more rapidly.

18. *Concept Check* Refer to the graph of Associate's Degrees Earned on the previous page. If $2000 \le k \le 2008$, $T(k) = r$, and $F(k) = s$, then $M(k) =$ ______.

Science and Space/Technology Spending The graph shows dollars (in billions) spent for general science and for space/other technologies in selected years. $G(x)$ represents the dollars spent for general science, and $S(x)$ represents the dollars spent for space and other technologies. $T(x)$ represents the total expenditures for these two categories. Use the graph in Exercises 19–22.

Science and Space Spending



Source: U.S. Office of Management and Budget.

19. Estimate $(T - S)(2000)$. What does this function represent?

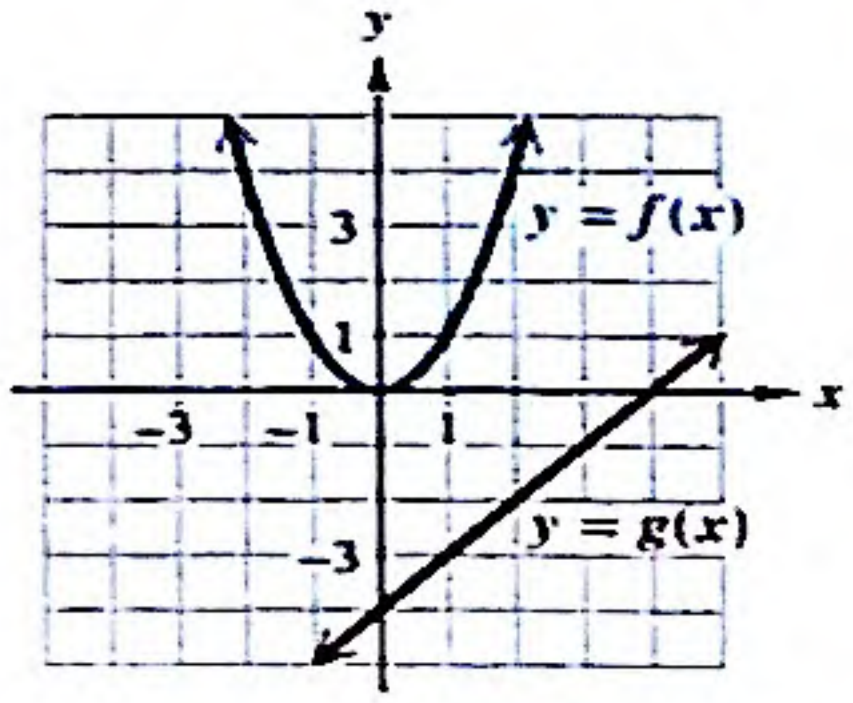
20. Estimate $(T - G)(2010)$. What does this function represent?

21. In which of the categories was spending almost static for several years? In which years did this occur?

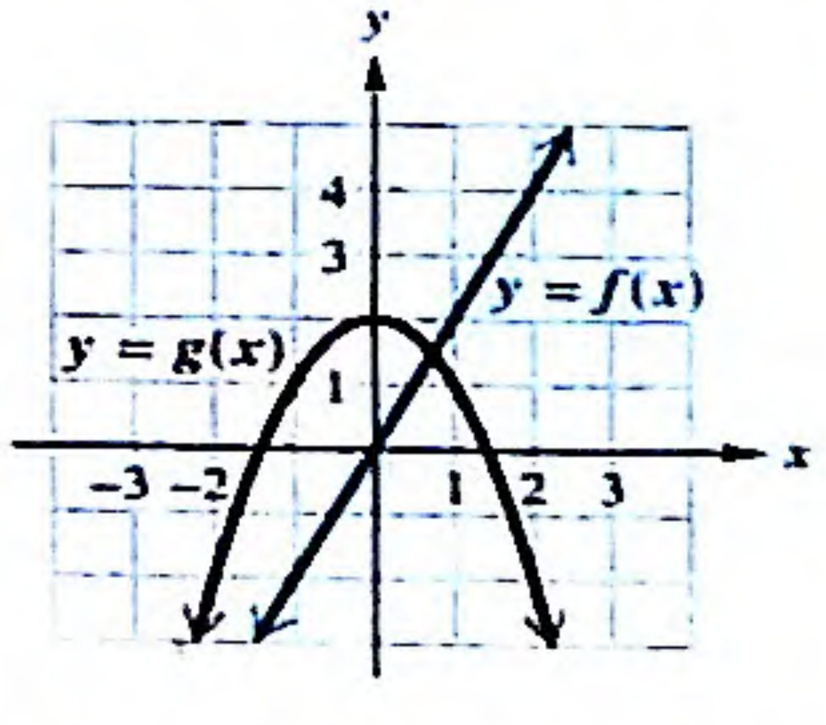
22. In which period and which category does spending for $G(x)$ or $S(x)$ increase most?

Use the graph to evaluate each expression. See Example 3(a).

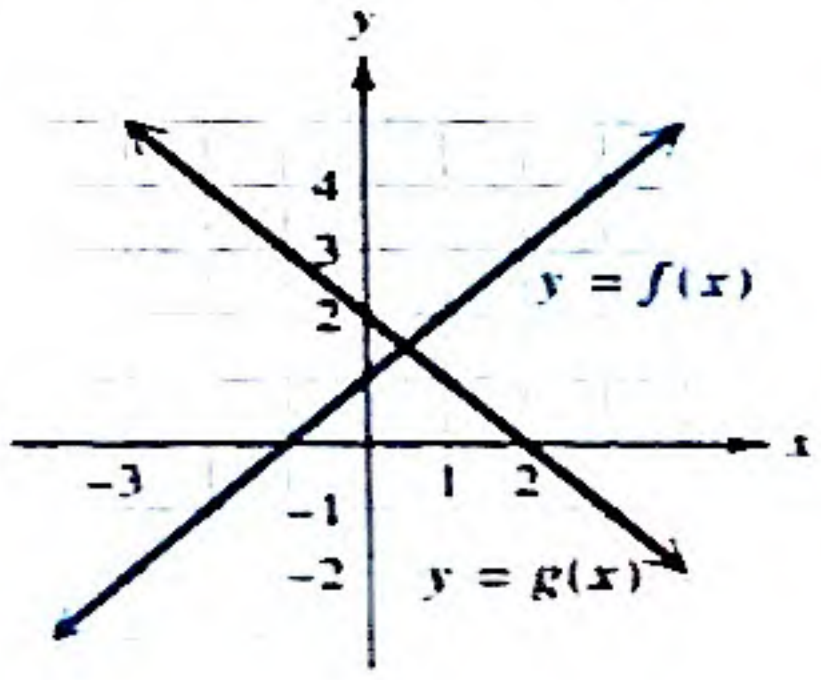
23. (a) $(f + g)(2)$ (b) $(f - g)(1)$
(c) $(fg)(0)$ (d) $\left(\frac{f}{g}\right)(1)$



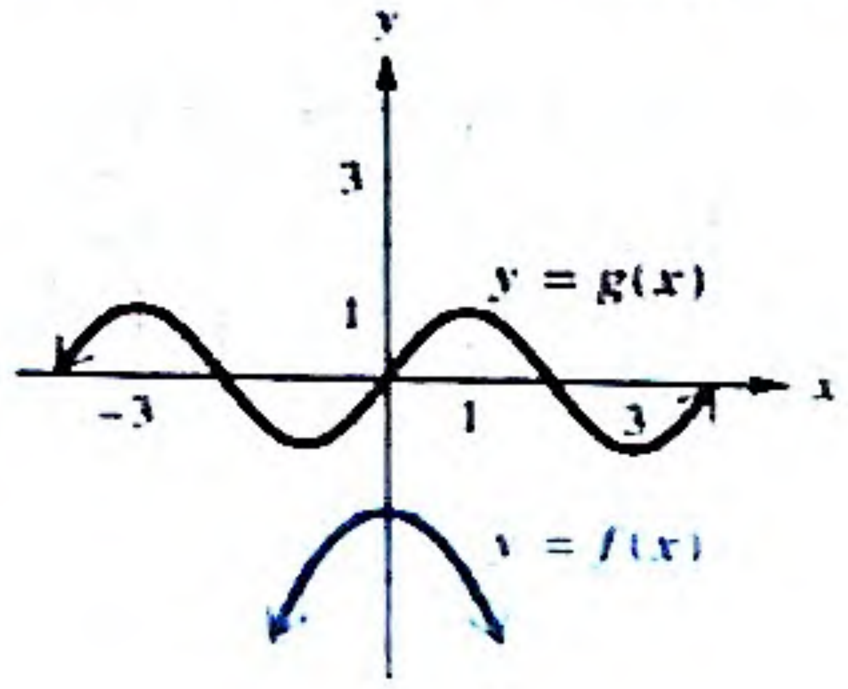
24. (a) $(f + g)(0)$ (b) $(f - g)(-1)$
(c) $(fg)(1)$ (d) $\left(\frac{f}{g}\right)(2)$



25. (a) $(f + g)(-1)$ (b) $(f - g)(-2)$
(c) $(fg)(0)$ (d) $\left(\frac{f}{g}\right)(2)$



26. (a) $(f + g)(1)$ (b) $(f - g)(0)$
(c) $(fg)(-1)$ (d) $\left(\frac{f}{g}\right)(1)$



In Exercises 27 and 28, use the table to evaluate each expression in parts (a)–(d), if possible. See Example 3(b).

(a) $(f+g)(2)$ (b) $(f-g)(4)$ (c) $(fg)(-2)$ (d) $\left(\frac{f}{g}\right)(0)$

27.

x	$f(x)$	$g(x)$
-2	0	6
0	5	0
2	7	-2
4	10	5

28.

x	$f(x)$	$g(x)$
-2	-4	2
0	8	-1
2	5	4
4	0	0

29. Use the table in Exercise 27 to complete the following table.

x	$(f+g)(x)$	$(f-g)(x)$	$(fg)(x)$	$\left(\frac{f}{g}\right)(x)$
-2				
0				
2				
4				

30. Use the table in Exercise 28 to complete the following table.

x	$(f+g)(x)$	$(f-g)(x)$	$(fg)(x)$	$\left(\frac{f}{g}\right)(x)$
-2				
0				
2				
4				

31. How is the difference quotient related to slope?

32. Refer to **Figure 93.** How is the secant line PQ related to the tangent line to a curve at point P?

For each of the functions in Exercises 33–46, find (a) $f(x+h)$, *(b)* $f(x+h)-f(x)$, *and (c)* $\dfrac{f(x+h)-f(x)}{h}$. *See Example 4.*

33. $f(x)=2-x$

34. $f(x)=1-x$

35. $f(x)=6x+2$

36. $f(x)=4x+11$

37. $f(x)=-2x+5$

38. $f(x)=-4x+2$

39. $f(x)=\dfrac{1}{x}$

40. $f(x)=\dfrac{1}{x^2}$

41. $f(x)=x^2$

42. $f(x)=-x^2$

43. $f(x)=1-x^2$

44. $f(x)=1+2x^2$

45. $f(x)=x^2+3x+1$

46. $f(x)=x^2-4x+2$

Let $f(x)=2x-3$ *and* $g(x)=-x+3$. *Find each function value. See Example 5.*

47. $(f\circ g)(4)$

48. $(f\circ g)(2)$

49. $(f\circ g)(-2)$

50. $(g\circ f)(3)$

51. $(g\circ f)(0)$

52. $(g\circ f)(-2)$

53. $(f\circ f)(2)$

54. $(g\circ g)(-2)$

Concept Check *The tables give some selected ordered pairs for functions f and g.*

x	3	4	6
$f(x)$	1	3	9

x	2	7	1	9
$g(x)$	3	6	9	12

Find each of the following.

55. $(f \circ g)(2)$

56. $(f \circ g)(7)$

57. $(g \circ f)(3)$

58. $(g \circ f)(6)$

59. $(f \circ f)(4)$

60. $(g \circ g)(1)$

61. ***Concept Check*** Why can you not determine $(f \circ g)(1)$ given the information in the tables for Exercises 55–60?

62. ***Concept Check*** Extend the concept of composition of functions to evaluate $(g \circ (f \circ g))(7)$ using the tables for Exercises 55–60.

Given functions f and g, find (a) $(f \circ g)(x)$ and its domain, and (b) $(g \circ f)(x)$ and its domain. See Examples 6 and 7.

63. $f(x) = -6x + 9, \quad g(x) = 5x + 7$

64. $f(x) = 8x + 12, \quad g(x) = 3x - 1$

65. $f(x) = \sqrt{x}, \quad g(x) = x + 3$

66. $f(x) = \sqrt{x}, \quad g(x) = x - 1$

67. $f(x) = x^3, \quad g(x) = x^2 + 3x - 1$

68. $f(x) = x + 2, \quad g(x) = x^4 + x^2 - 4$

69. $f(x) = \sqrt{x - 1}, \quad g(x) = 3x$

70. $f(x) = \sqrt{x - 2}, \quad g(x) = 2x$

71. $f(x) = \frac{2}{x}, \quad g(x) = x + 1$

72. $f(x) = \frac{4}{x}, \quad g(x) = x + 4$

73. $f(x) = \sqrt{x + 2}, \quad g(x) = -\frac{1}{x}$

74. $f(x) = \sqrt{x + 4}, \quad g(x) = -\frac{2}{x}$

75. $f(x) = \sqrt{x}, \quad g(x) = \frac{1}{x + 5}$

76. $f(x) = \sqrt{x}, \quad g(x) = \frac{3}{x + 6}$

77. $f(x) = \frac{1}{x - 2}, \quad g(x) = \frac{1}{x}$

78. $f(x) = \frac{1}{x + 4}, \quad g(x) = -\frac{1}{x}$

79. ***Concept Check*** Fill in the missing entries in the table.

x	$f(x)$	$g(x)$	$g(f(x))$
1	3	2	7
2	1	5	
3	2		

80. ***Concept Check*** Suppose $f(x)$ is an odd function and $g(x)$ is an even function. Fill in the missing entries in the table.

x	-2	-1	0	1	2
$f(x)$			0	-2	
$g(x)$	0	2	1		
$(f \circ g)(x)$		1	-2		

81. Show that $(f \circ g)(x)$ is not equivalent to $(g \circ f)(x)$ for

$$f(x) = 3x - 2 \quad \text{and} \quad g(x) = 2x - 3.$$

82. Describe the steps required to find the composite function $f \circ g$, given

$$f(x) = 2x - 5 \quad \text{and} \quad g(x) = x^2 + 3$$

For certain pairs of functions f and g, $(f \circ g)(x) = x$ *and* $(g \circ f)(x) = x$*. Show that this is true for each pair in **Exercises 83–86**.*

83. $f(x) = 4x + 2,\ g(x) = \frac{1}{4}(x - 2)$

84. $f(x) = -3x,\ g(x) = -\frac{1}{3}x$

85. $f(x) = \sqrt[3]{5x + 4},\ g(x) = \frac{1}{5}x^3 - \frac{4}{5}$

86. $f(x) = \sqrt[3]{x + 1},\ g(x) = x^3 - 1$

Find functions f and g such that $(f \circ g)(x) = h(x)$*. (There are many possible ways to do this.) See **Example 9**.*

87. $h(x) = (6x - 2)^2$

88. $h(x) = (11x^2 + 12x)^2$

89. $h(x) = \sqrt{x^2 - 1}$

90. $h(x) = (2x - 3)^3$

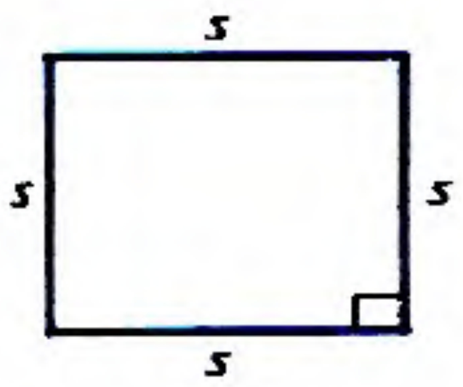
91. $h(x) = \sqrt{6x} + 12$

92. $h(x) = \sqrt[3]{2x + 3} - 4$

Solve each problem.

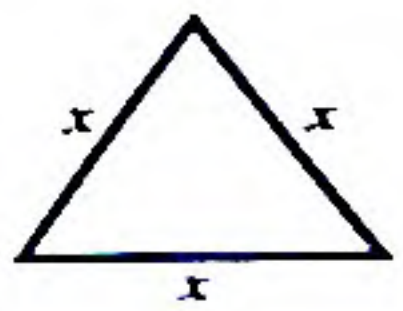
93. ***Relationship of Measurement Units*** The function defined by $f(x) = 12x$ computes the number of inches in x feet, and the function defined by $g(x) = 5280x$ computes the number of feet in x miles. What does $(f \circ g)(x)$ compute?

94. ***Perimeter of a Square*** The perimeter x of a square with side of length s is given by the formula $x = 4s$.



(a) Solve for s in terms of x.
(b) If y represents the area of this square, write y as a function of the perimeter x.
(c) Use the composite function of part (b) to find the area of a square with perimeter 6.

95. ***Area of an Equilateral Triangle*** The area of an equilateral triangle with sides of length x is given by the function defined by $\mathscr{A}(x) = \frac{\sqrt{3}}{4}x^2$.



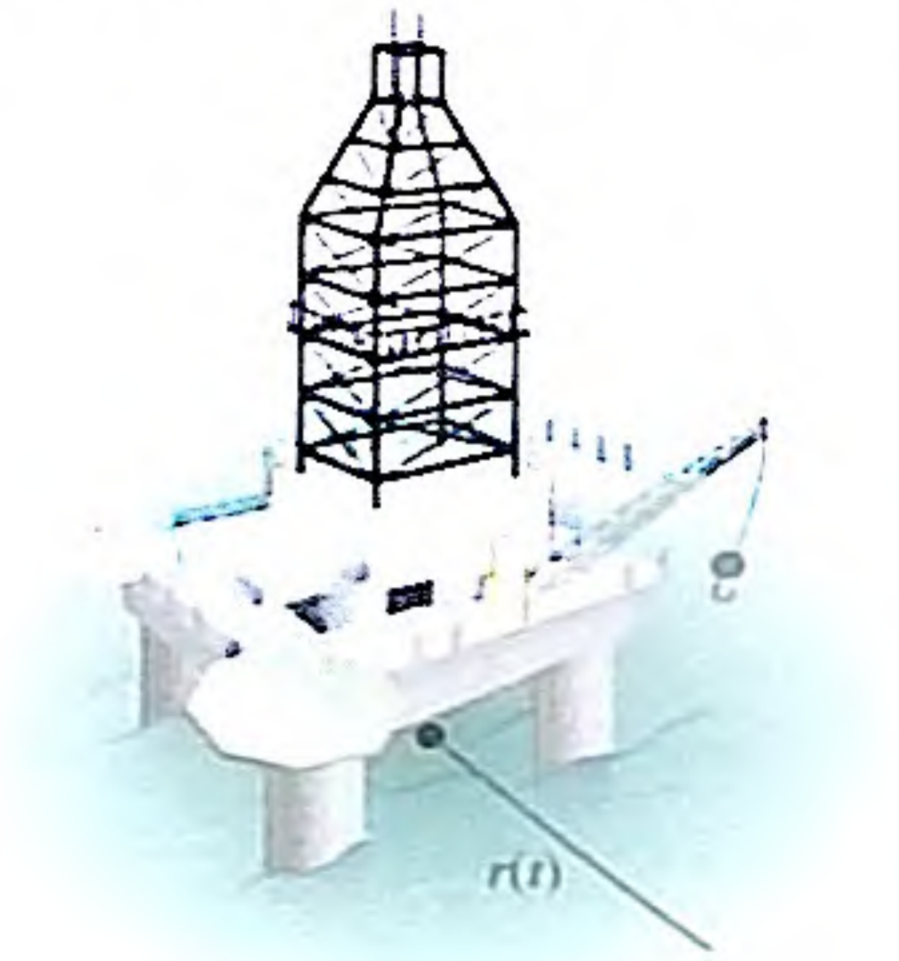
(a) Find $\mathscr{A}(2x)$, the function representing the area of an equilateral triangle with sides of length twice the original length.
(b) Find the area of an equilateral triangle with side length 16. Use the formula $\mathscr{A}(2x)$ found in part (a).

96. ***Software Author Royalties*** A software author invests his royalties in two accounts for 1 yr.

(a) The first account pays 4% simple interest. If he invests x dollars in this account, write an expression for y_1 in terms of x, where y_1 represents the amount of interest earned.
(b) He invests in a second account \$500 more than he invested in the first account. This second account pays 2.5% simple interest. Write an expression for y_2, where y_2 represents the amount of interest earned.
(c) What does $y_1 + y_2$ represent?
(d) How much interest will he receive if \$250 is invested in the first account?

97. ***Oil Leak*** An oil well off the Gulf Coast is leaking, with the leak spreading oil over the water's surface as a circle. At any time t, in minutes, after the beginning of the leak, the radius of the circular oil slick on the surface is $r(t) = 4t$ feet. Let $\mathscr{A}(r) = \pi r^2$ represent the area of a circle of radius r.

(a) Find $(\mathscr{A} \circ r)(t)$.
(b) Interpret $(\mathscr{A} \circ r)(t)$.
(c) What is the area of the oil slick after 3 min?



98. ***Emission of Pollutants*** When a thermal inversion layer is over a city (as happens in Los Angeles), pollutants cannot rise vertically but are trapped below the layer and must disperse horizontally. Assume that a factory smokestack begins emitting a pollutant at 8 A.M. Assume that the pollutant disperses horizontally over a circular area. If t represents the time, in hours, since the factory began emitting pollutants ($t = 0$ represents 8 A.M.), assume that the radius of the circle of pollutants at time t is $r(t) = 2t$ miles. Let $\mathcal{A}(r) = \pi r^2$ represent the area of a circle of radius r.

(a) Find $(\mathcal{A} \circ r)(t)$. (b) Interpret $(\mathcal{A} \circ r)(t)$.

(c) What is the area of the circular region covered by the layer at noon?

99. ***(Modeling) Catering Cost*** The cost to hire a caterer for a party depends on the number of guests attending. If 100 people attend, the cost per person will be \$20. For each person less than 100, the cost will increase by \$5. Assume that no more than 100 people will attend. Let x represent the number less than 100 who do not attend. For example, if 95 attend, $x = 5$.

(a) Write a function defined by $N(x)$ giving the number of guests.

(b) Write a function defined by $G(x)$ giving the cost per guest.

(c) Write a function defined by $N(x) \cdot G(x)$ for the total cost, $C(x)$.

(d) What is the total cost if 80 people attend?

100. ***Area of a Square*** The area of a square is x^2 square inches. Suppose that 3 in. is added to one dimension and 1 in. is subtracted from the other dimension. Express the area $\mathcal{A}(x)$ of the resulting rectangle as a product of two functions.

Chapter 2 Test Prep

Key Terms

2.1 ordered pair
origin
x-axis
y-axis
rectangular (Cartesian) coordinate system
coordinate plane (xy-plane)
quadrants
coordinates
collinear
graph of an equation
x-intercept
y-intercept

2.2 circle
radius
center of a circle

2.3 dependent variable
independent variable
relation
function
domain
range
function notation
increasing function
decreasing function
constant function

2.4 linear function
standard form
relatively prime
change in x
change in y
slope
average rate of change
linear cost function
cost
fixed cost
revenue function
profit function

2.5 point-slope form
slope-intercept form
scatter diagram

2.6 continuous function
parabola
vertex
piecewise-defined function
step function

2.7 symmetry
even function
odd function
vertical translation
horizontal translation

2.8 difference quotient
composite function (composition)

New Symbols

(a, b)	ordered pair
$f(x)$	function f evaluated at x (read "f of x")
Δx	change in x
Δy	change in y
m	slope
$[\![x]\!]$	the greatest integer less than or equal to x
$g \circ f$	composite function

(continued)

Concepts	Examples

2.6 Graphs of Basic Functions

Basic Functions

Identity Function $f(x) = x$

Squaring Function $f(x) = x^2$

Cubing Function $f(x) = x^3$

Square Root Function $f(x) = \sqrt{x}$

Cube Root Function $f(x) = \sqrt[3]{x}$

Absolute Value Function $f(x) = |x|$

Greatest Integer Function $f(x) = [\![x]\!]$

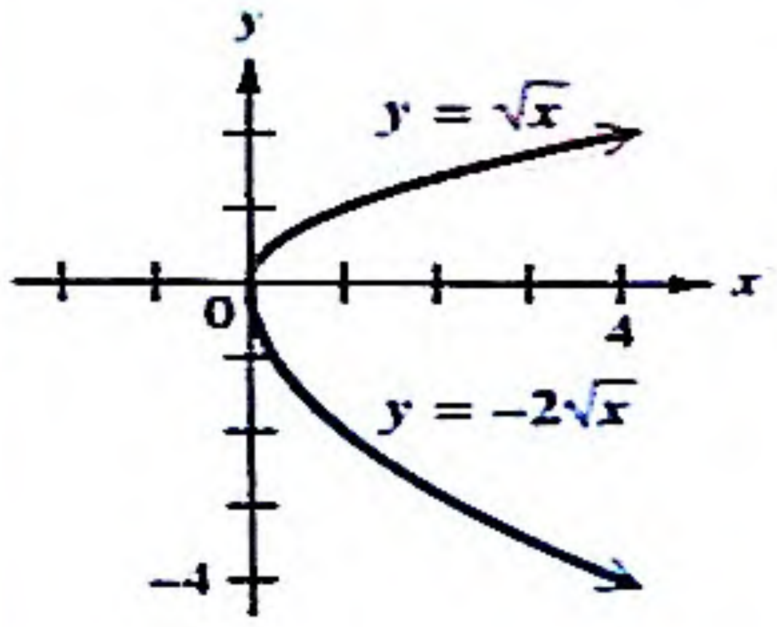
Refer to the function boxes in Section 2.6. Graphs of the basic functions are also shown on the back inside cover.

2.7 Graphing Techniques

Stretching and Shrinking

If $a > 1$, then the graph of $y = af(x)$ is a vertical stretching of the graph of $y = f(x)$. If $0 < a < 1$, then the graph of $y = af(x)$ is a vertical shrinking of the graph of $y = f(x)$.

If $0 < a < 1$, then the graph of $y = f(ax)$ is a horizontal stretching of the graph of $y = f(x)$. If $a > 1$, then the graph of $y = f(ax)$ is a horizontal shrinking of the graph of $y = f(x)$.



The graph of $y = -2\sqrt{x}$ is the graph of $y = \sqrt{x}$ stretched vertically by a factor of 2 and reflected across the x-axis.

Reflection Across an Axis

The graph of $y = -f(x)$ is the same as the graph of $y = f(x)$ reflected across the x-axis.

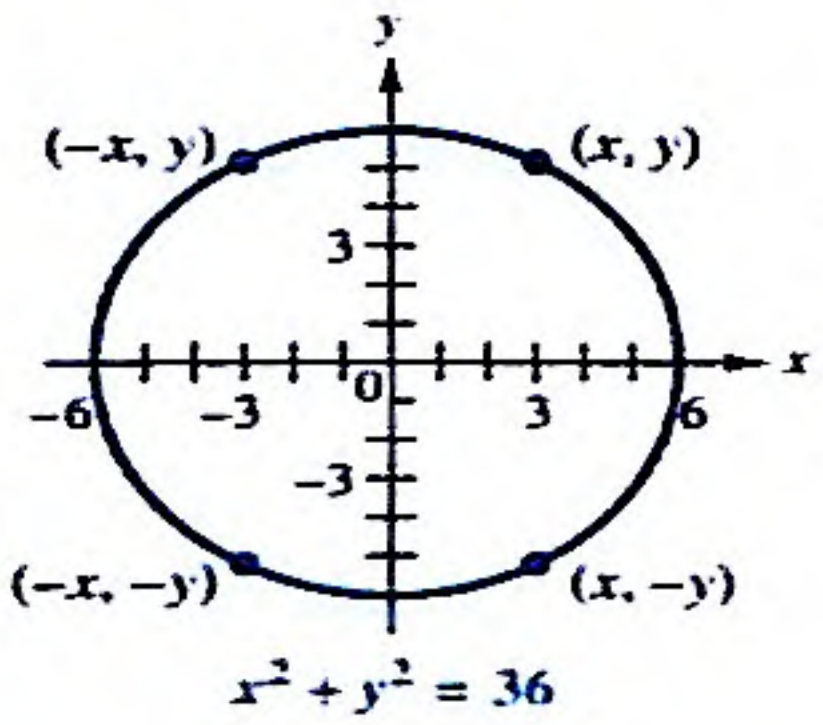
The graph of $y = f(-x)$ is the same as the graph of $y = f(x)$ reflected across the y-axis.

Symmetry

The graph of an equation is symmetric with respect to the y-axis if the replacement of x with $-x$ results in an equivalent equation.

The graph of an equation is symmetric with respect to the x-axis if the replacement of y with $-y$ results in an equivalent equation.

The graph of an equation is symmetric with respect to the origin if the replacement of both x with $-x$ and y with $-y$ at the same time results in an equivalent equation.

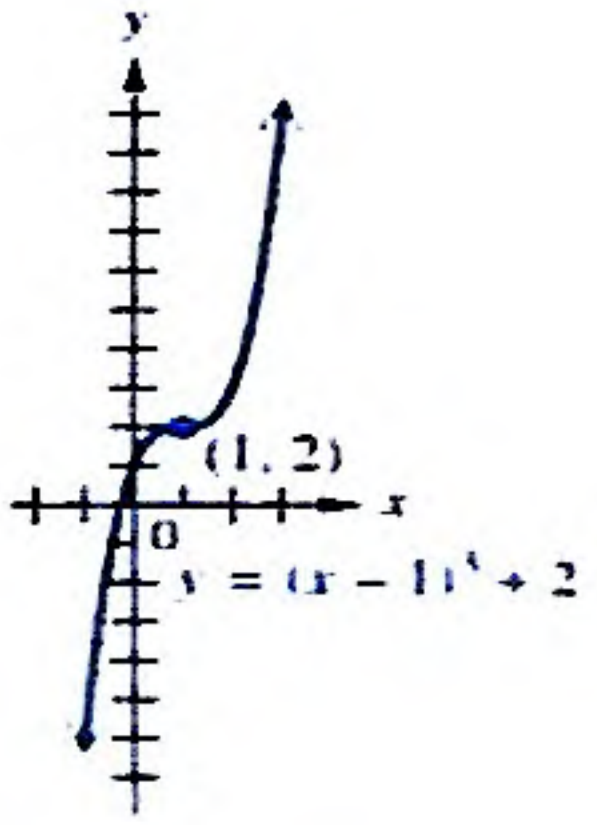


The graph of $x^2 + y^2 = 36$ is symmetric with respect to the y-axis, the x-axis, and the origin.

Translations

Let f be a function and c be a positive number.

To Graph:	Shift the Graph of $y = f(x)$ by c Units:
$y = f(x) + c$	up
$y = f(x) - c$	down
$y = f(x + c)$	left
$y = f(x - c)$	right



The graph of

$$y = (x - 1)^3 + 2$$

is the graph of $y = x^3$ translated 1 unit to the right and 2 units up.

Concepts	Examples
2.8 Function Operations and Composition	
Operations on Functions Given two functions f and g, then for all values of x for which both $f(x)$ and $g(x)$ are defined, the following operations are defined. $(f+g)(x) = f(x) + g(x)$ Sum $(f-g)(x) = f(x) - g(x)$ Difference $(fg)(x) = f(x) \cdot g(x)$ Product $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},\ g(x) \neq 0$ Quotient	Let $f(x) = 2x - 4$ and $g(x) = \sqrt{x}$. $\left.\begin{aligned}(f+g)(x) &= 2x - 4 + \sqrt{x}\\ (f-g)(x) &= 2x - 4 - \sqrt{x}\\ (fg)(x) &= (2x-4)\sqrt{x}\end{aligned}\right\}$ The domain is $[0, \infty)$. $\left.\left(\frac{f}{g}\right)(x) = \frac{2x-4}{\sqrt{x}}\right\}$ The domain is $(0, \infty)$.
Difference Quotient The line joining $P(x, f(x))$ and $Q(x+h, f(x+h))$ has slope $m = \frac{f(x+h) - f(x)}{h},\ h \neq 0.$	Refer to Example 4 in Section 2.8.
Composition of Functions If f and g are functions, then the composite function, or composition, of g and f is defined by $(g \circ f)(x) = g(f(x)).$ The domain of $g \circ f$ is the set of all x in the domain of f such that $f(x)$ is in the domain of g.	Using f and g as defined above, $(g \circ f)(x) = \sqrt{2x-4}.$ The domain is all x such that $2x - 4 \geq 0$. Solving this inequality gives the interval $[2, \infty)$.

Chapter 2 Review Exercises

Find the distance between each pair of points, and give the coordinates of the midpoint of the segment joining them.

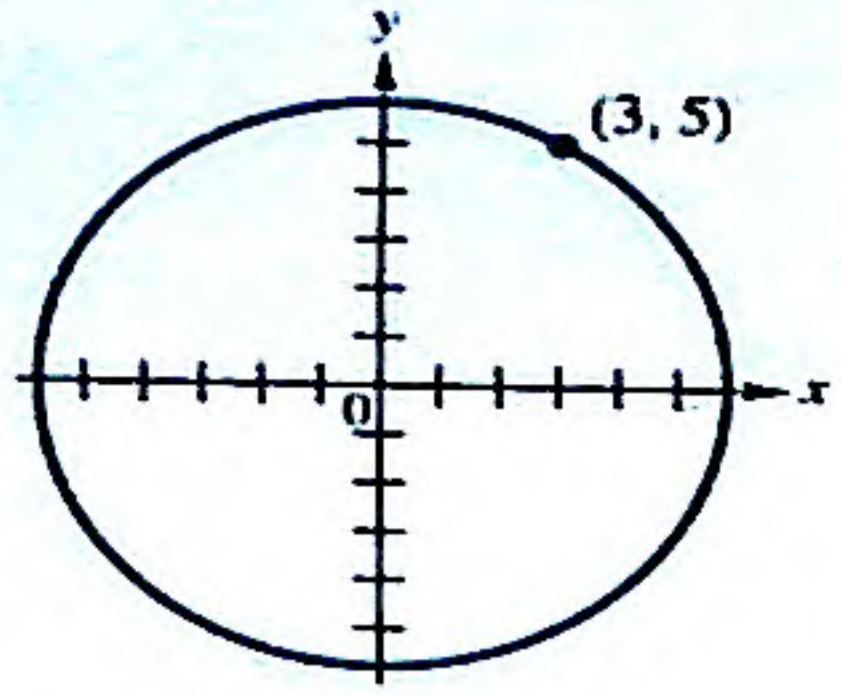
1. $P(3, -1)$, $Q(-4, 5)$
2. $M(-8, 2)$, $N(3, -7)$
3. $A(-6, 3)$, $B(-6, 8)$
4. Are the points $(5, 7)$, $(3, 9)$, and $(6, 8)$ the vertices of a right triangle? If so, at what point is the right angle?
5. Determine the coordinates of B for segment AB, given that A has coordinates $(-6, 10)$ and the coordinates of its midpoint M are $(8, 2)$.
6. Use the distance formula to determine whether the points $(-2, -5)$, $(1, 7)$, and $(3, 15)$ are collinear

Find the center-radius form of the equation for each circle described.

7. center $(-2, 3)$, radius 15
8. center $(\sqrt{5}, -\sqrt{7})$, radius $\sqrt{3}$
9. center $(-8, 1)$, passing through $(0, 16)$
10. center $(3, -6)$, tangent to the x-axis

Connecting Graphs with Equations ***Use each graph to determine the equation of the circle. Express in center-radius form.***

11.



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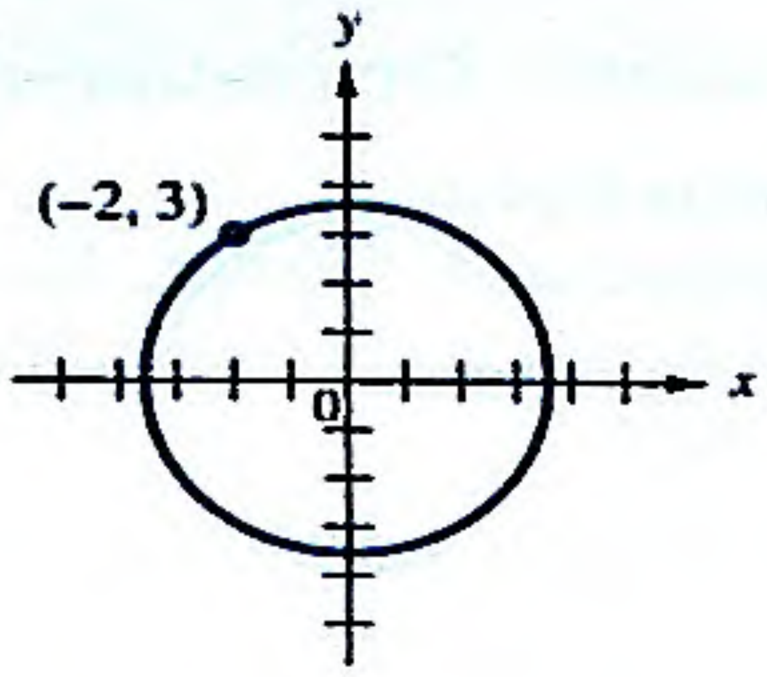

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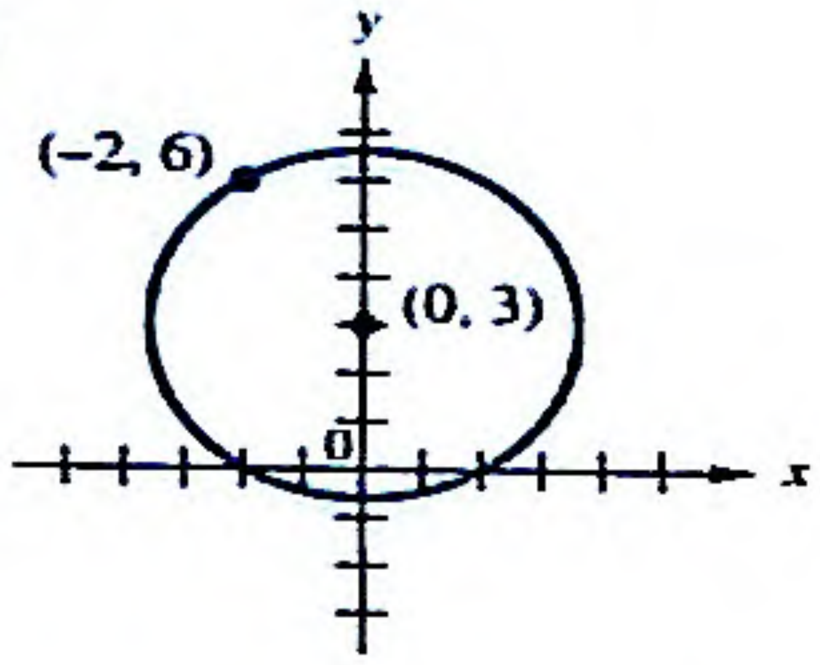

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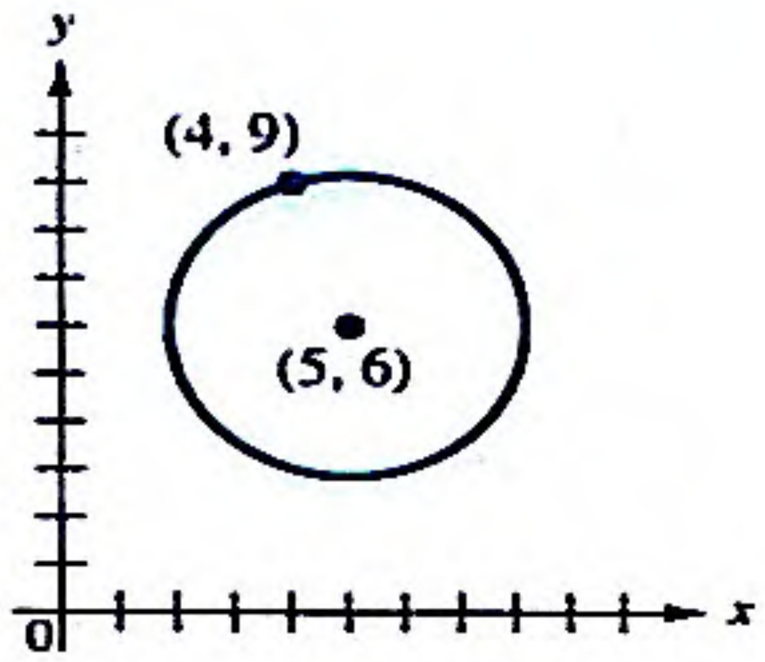


Find the center and radius of each circle.

15. $x^2 + y^2 - 4x + 6y + 12 = 0$

16. $x^2 + y^2 - 6x - 10y + 30 = 0$

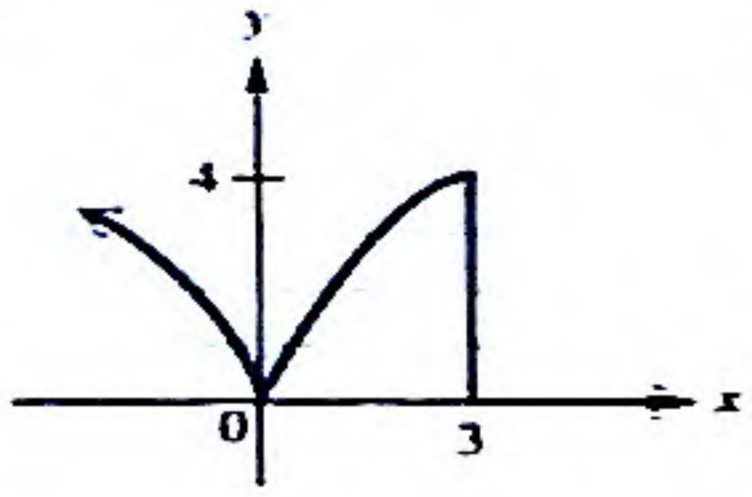
17. $2x^2 + 2y^2 + 14x + 6y + 2 = 0$

18. $3x^2 + 3y^2 + 33x - 15y = 0$

19. Find all possible values of x so that the distance between $(x, -9)$ and $(3, -5)$ is 6

For each graph, determine whether y is a function of x. Give the domain and range of each relation.

20.



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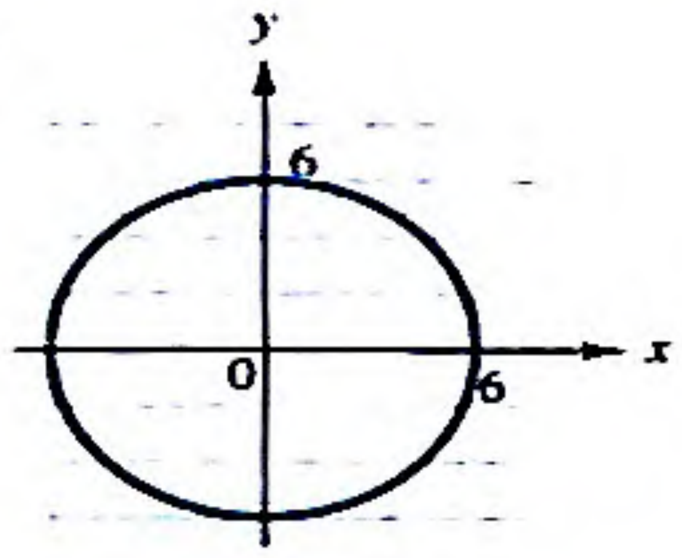

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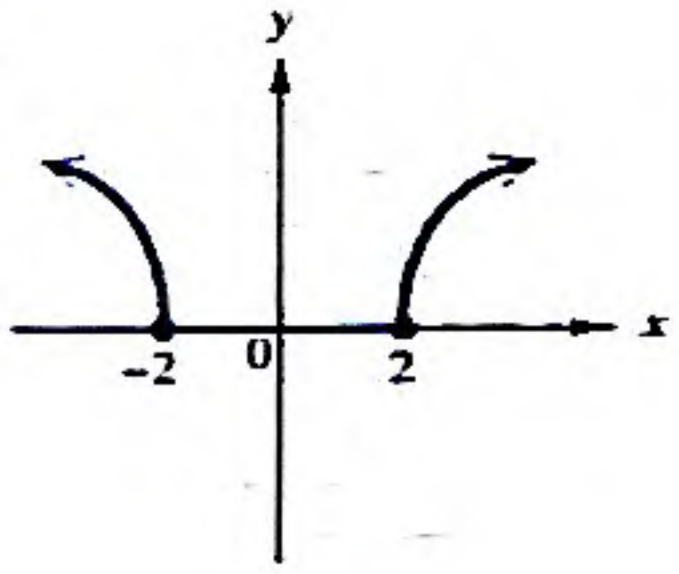

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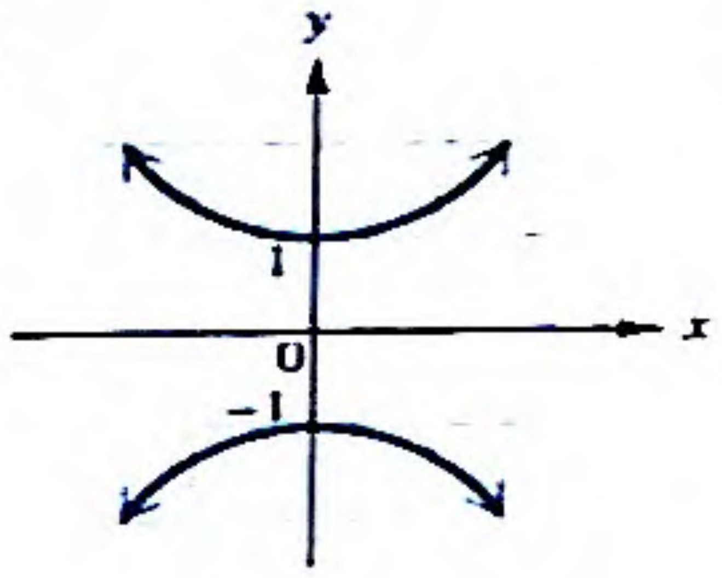

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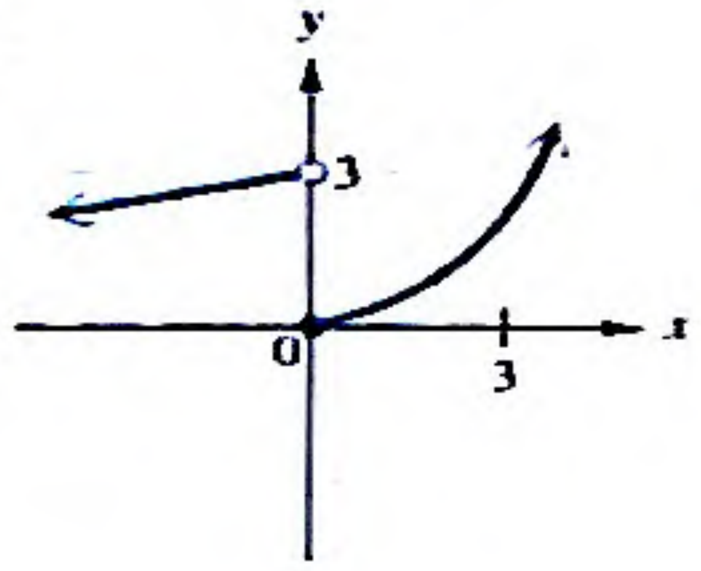

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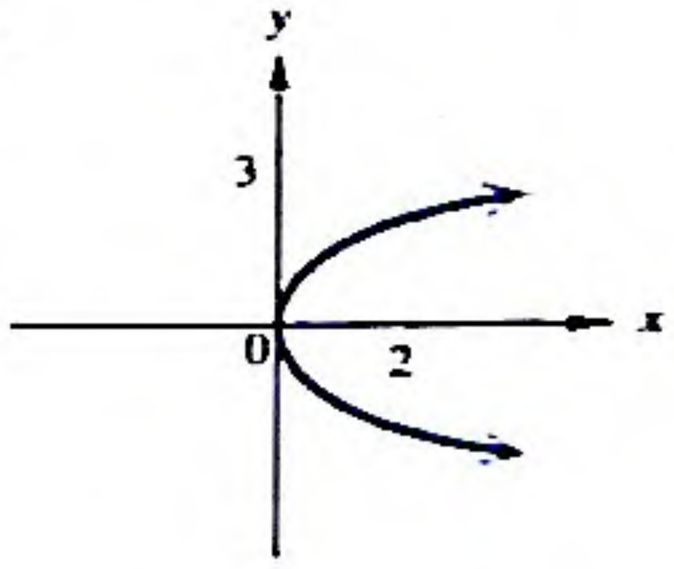


Determine whether each equation defines y as a function of x.

26. $x = \frac{1}{3}y^2$

27. $y = 6 - x^2$

28. $y = -\frac{4}{x}$

29. $y = \pm\sqrt{x - 2}$

Give the domain of each function.

30. $f(x) = -4 + |x|$

31. $f(x) = \frac{8 + x}{8 - x}$

32. $f(x) = \sqrt{6 - 3x}$

33. For the function graphed in **Exercise 22,** give the largest interval over which it is **(a)** increasing, **(b)** decreasing, and **(c)** constant.

34. Suppose the graph in **Exercise 24** is that of $y = f(x)$. What is $f(0)$?

Given $f(x) = -2x^2 + 3x - 6$, *find each function value or expression.*

35. $f(3)$

36. $f(-0.5)$

37. $f(k)$

Graph each equation.

38. $3x + 7y = 14$

39. $2x - 5y = 5$

40. $3y = x$

41. $2x + 5y = 20$

42. $x - 4y = 8$

43. $f(x) = x$

44. $f(x) = 3$

45. $x = -5$

46. $y + 2 = 0$

47. *Concept Check* The equation of the line that lies along the x-axis is ____.

Graph the line satisfying the given conditions.

48. through $(2, -4)$, $m = \frac{3}{4}$

49. through $(0, 5)$, $m = -\frac{2}{3}$

Find the slope for each line, provided that it has a slope.

50. through $(8, 7)$ and $\left(\frac{1}{2}, -2\right)$

51. through $(2, -2)$ and $(3, -4)$

52. through $(5, 6)$ and $(5, -2)$

53. through $(0, -7)$ and $(3, -7)$

54. $9x - 4y = 2$

55. $11x + 2y = 3$

56. $x - 5y = 0$

57. $x - 2 = 0$

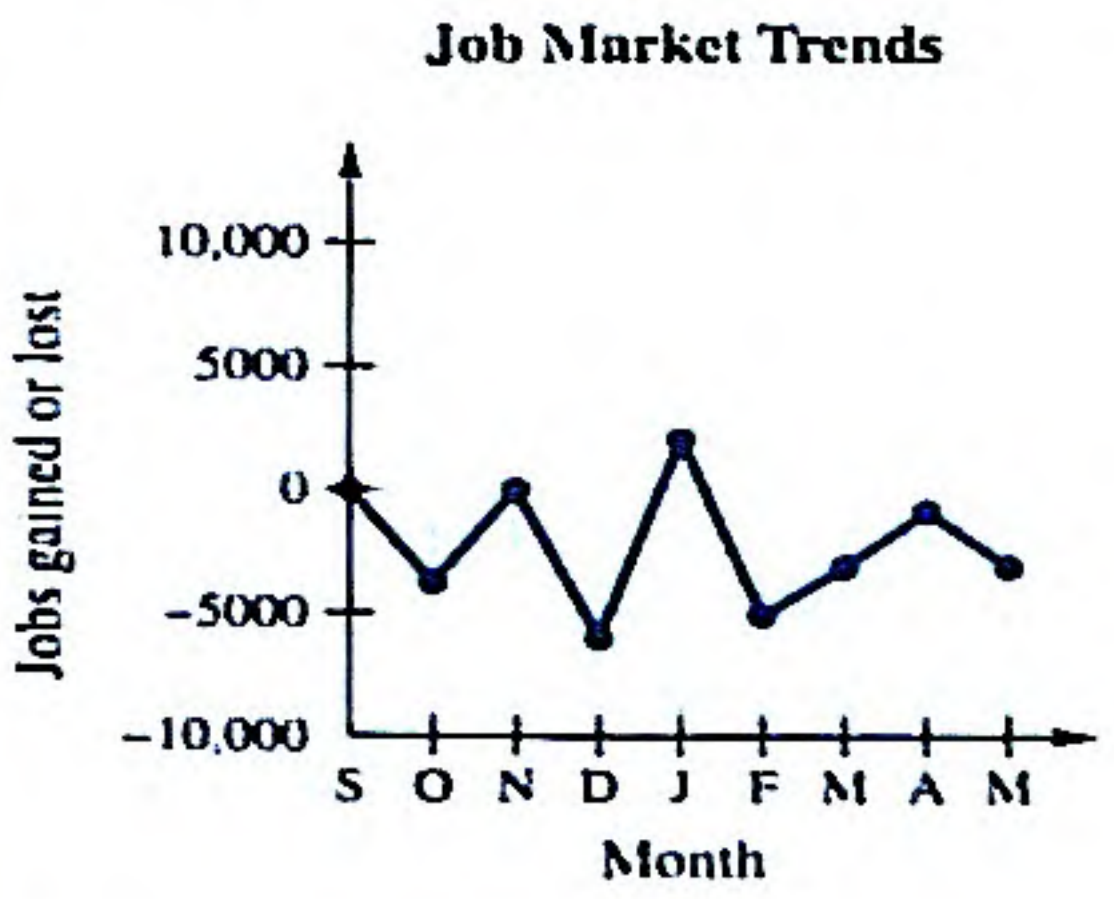
58. *(Modeling) Job Market* The figure shows the number of jobs gained or lost in a recent period from September to May.

(a) Is this the graph of a function?

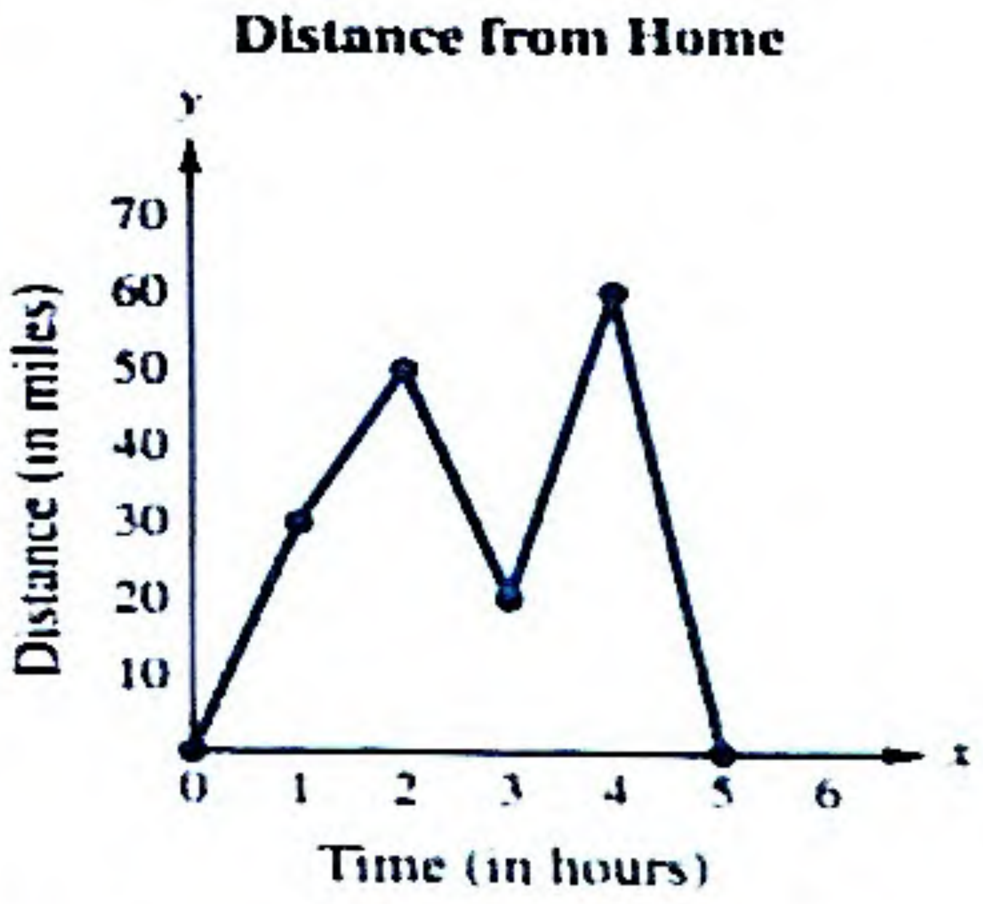
(b) In what month were the most jobs lost? the most gained?

(c) What was the largest number of jobs lost? of jobs gained?

(d) Do these data show an upward or a downward trend? If so, which is it?



59. *(Modeling) Distance from Home* The graph depicts the distance y that a person driving a car on a straight road is from home after x hours. Interpret the graph. At what speeds did the car travel?



60. *Family Income* Family income in the United States has steadily increased for many years. In 1980 the median family income was about $21,000 per year. In 2008 it was about $61,500 per year. Find the average annual rate of change of median family income to the nearest dollar over that period. (*Source:* U.S. Census Bureau.)

61. *(Modeling) E-Filing Tax Returns* The percent of tax returns filed electronically in 2001 was 30.7%. In 2009, the figure was 67.2%. (*Source:* Internal Revenue Service.)

(a) Use the information given for the years 2001 and 2009, letting $x = 0$ represent 2001, $x = 8$ represent 2009, and y represent the percent of returns filed electronically, to find a linear equation that models the data. Write the equation in slope-intercept form. Interpret the slope of the graph of this equation.

(b) Use your equation from part (a) to approximate the percent of tax returns that were filed electronically in 2007.

For each line described, write the equation in (a) slope-intercept form, if possible, and (b) standard form.

62. through $(-2, 4)$ and $(1, 3)$

63. through $(3, -5)$ with slope -2

64. x-intercept -3, y-intercept 5

65. through $(2, -1)$, parallel to $3x - y = 1$

66. through $(0, 5)$, perpendicular to $8x + 5y = 3$

67. through $(2, -10)$, perpendicular to a line with undefined slope

68. through $(3, -5)$, parallel to $y = 4$

69. through $(-7, 4)$, perpendicular to $y = 8$

Graph each function.

70. $f(x) = -|x|$

71. $f(x) = |x| - 3$

72. $f(x) = -\sqrt{x} - 2$

73. $f(x) = -(x + 1)^2 + 3$

74. $f(x) = 2\sqrt[3]{x + 1} - 2$

75. $f(x) = [\![x - 3]\!]$

76. $f(x) = \begin{cases} x^2 + 3 & \text{if } x < 2 \\ -x + 4 & \text{if } x \geq 2 \end{cases}$

77. $f(x) = \begin{cases} -4x + 2 & \text{if } x \leq 1 \\ 3x - 5 & \text{if } x > 1 \end{cases}$

78. $f(x) = \begin{cases} |x| & \text{if } x < 3 \\ 6 - x & \text{if } x \geq 3 \end{cases}$

79. *Concept Check* If x represents an integer, then what is the simplest form of the expression

$$[\![x]\!] + x?$$

Concept Check Decide whether each statement is true *or* false. *If false, tell why.*

80. The graph of a nonzero function cannot be symmetric with respect to the x-axis.

81. The graph of an even function is symmetric with respect to the y-axis.

82. The graph of an odd function is symmetric with respect to the origin.

83. If (a, b) is on the graph of an even function, so is $(a, -b)$.

84. If (a, b) is on the graph of an odd function, so is $(-a, b)$.

85. The constant function $f(x) = 0$ is both even and odd.

Decide whether each equation has a graph that is symmetric with respect to the x-axis, the y-axis, the origin, or none of these.

86. $5y^2 + 5x^2 = 30$

87. $x + y^2 = 10$

88. $y^3 = x + 4$

89. $x^2 = y^3$

90. $|y| = -x$

91. $6x + y = 4$

92. $|x| = |y|$

93. $y = 1$

94. $x^2 - y^2 = 0$

Describe how the graph of each function can be obtained from the graph of $f(x) = |x|$.

95. $g(x) = -|x|$

96. $h(x) = |x| - 2$

97. $k(x) = 2|x - 4|$

Let $f(x) = 3x - 4$. Find an equation for each reflection of the graph of $f(x)$.

98. across the x-axis **99.** across the y-axis **100.** across the origin

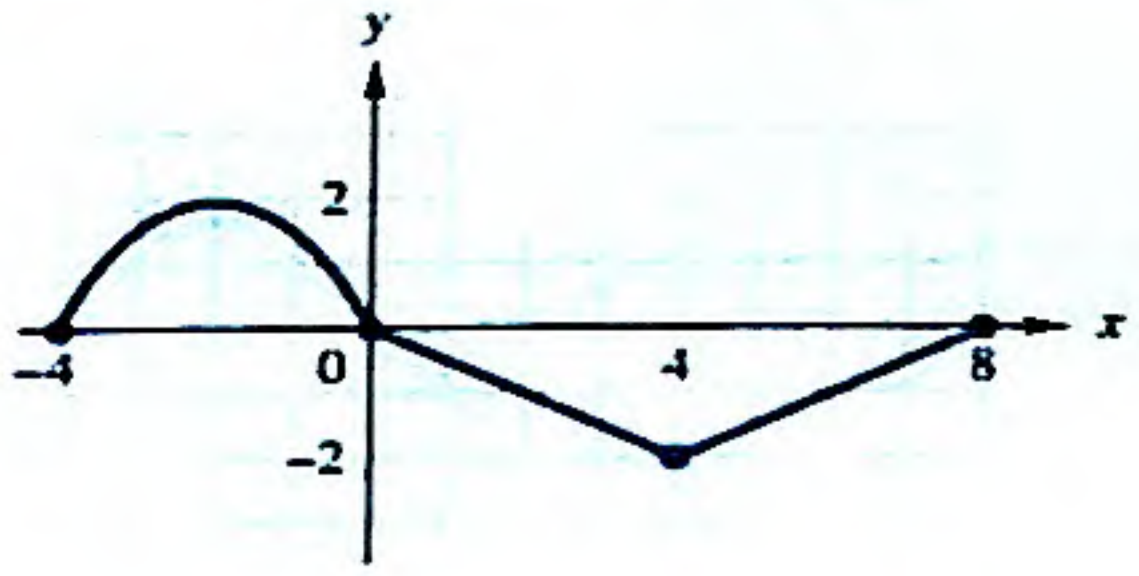
101. *Concept Check* The graph of a function f is shown in the figure. Sketch the graph of each function defined as follows.

(a) $y = f(x) + 3$

(b) $y = f(x - 2)$

(c) $y = f(x + 3) - 2$

(d) $y = |f(x)|$



Let $f(x) = 3x^2 - 4$ and $g(x) = x^2 - 3x - 4$. Find each of the following.

102. $(f + g)(x)$ **103.** $(fg)(x)$ **104.** $(f - g)(4)$

105. $(f + g)(-4)$ **106.** $(f + g)(2k)$ **107.** $\left(\frac{f}{g}\right)(3)$

108. $\left(\frac{f}{g}\right)(-1)$ **109.** the domain of $(fg)(x)$ **110.** the domain of $\left(\frac{f}{g}\right)(x)$

111. Which of the following is *not* equivalent to $(f \circ g)(x)$ for $f(x) = \frac{1}{x}$ and $g(x) = x^2 + 1$? (*Hint:* There may be more than one.)

A. $f(g(x))$ B. $\frac{1}{x^2 + 1}$ C. $\frac{1}{x^2}$ D. $(g \circ f)(x)$

For each function, find and simplify $\frac{f(x + h) - f(x)}{h}$, $h \neq 0$.

112. $f(x) = 2x + 9$ **113.** $f(x) = x^2 - 5x + 3$

Let $f(x) = \sqrt{x - 2}$ and $g(x) = x^2$. Find each of the following, if possible.

114. $(f \circ g)(x)$ **115.** $(g \circ f)(x)$ **116.** $(f \circ g)(-6)$

117. $(g \circ f)(3)$ **118.** $(g \circ f)(-1)$ **119.** the domain of $f \circ g$

Use the table to evaluate each expression, if possible.

120. $(f + g)(1)$

121. $(f - g)(3)$

122. $(fg)(-1)$

123. $\left(\frac{f}{g}\right)(0)$

124. $\left(\frac{f}{g}\right)(3)$

x	$f(x)$	$g(x)$
-1	3	-2
0	5	0
1	7	1
3	9	9

Tables for f and g are given. Use them to evaluate the expressions in Exercises 125 and 126.

125. $(g \circ f)(-2)$

126. $(f \circ g)(3)$

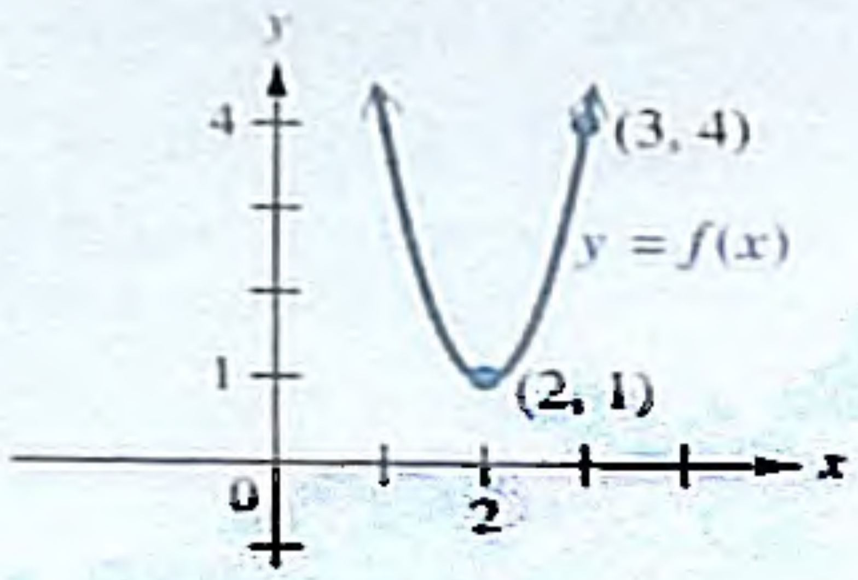
x	$f(x)$
-2	1
0	4
2	3
4	2

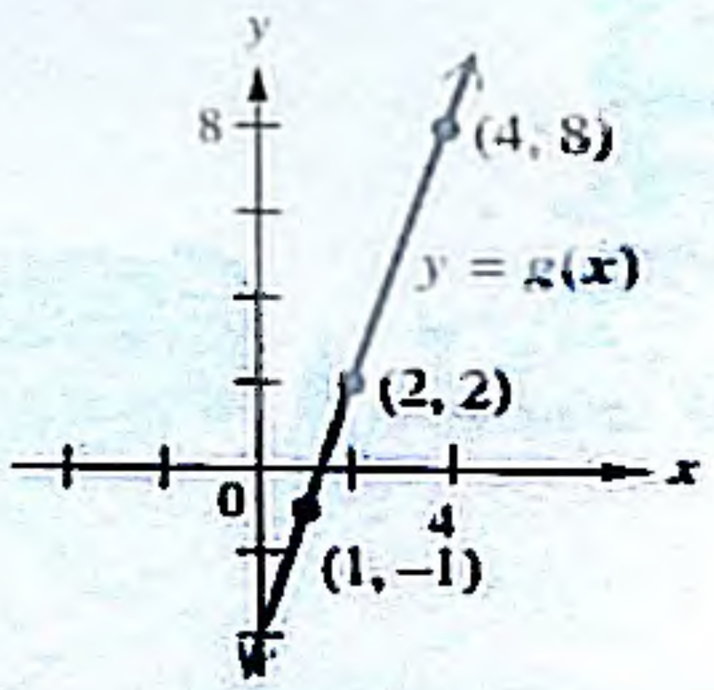
x	$g(x)$
1	2
2	4
3	-2
4	0

Concept Check *The graphs of two functions f and g are shown in the figures.*

127. Find $(f \circ g)(2)$.

128. Find $(g \circ f)(3)$.



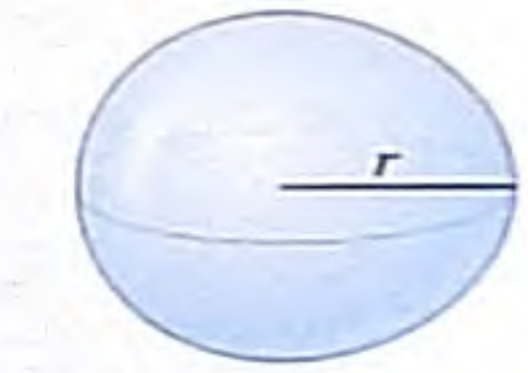


Solve each problem.

129. ***Relationship of Measurement Units*** There are 36 in. in 1 yd, and there are 1760 yd in 1 mi. Express the number of inches in x miles by forming two functions and then considering their composition.

130. ***(Modeling) Perimeter of a Rectangle*** Suppose the length of a rectangle is twice its width. Let x represent the width of the rectangle. Write a formula for the perimeter P of the rectangle in terms of x alone. Then use $P(x)$ notation to describe it as a function. What type of function is this?

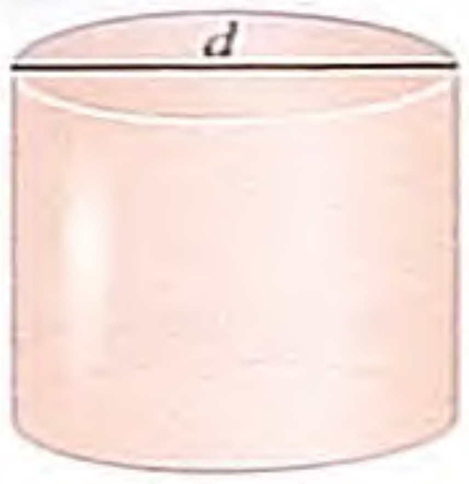
131. ***(Modeling) Volume of a Sphere*** The formula for the volume of a sphere is $V(r) = \frac{4}{3}\pi r^3$, where r represents the radius of the sphere. Construct a model function V representing the amount of volume gained when the radius r (in inches) of a sphere is increased by 3 in.



132. ***(Modeling) Dimensions of a Cylinder*** A cylindrical can makes the most efficient use of materials when its height is the same as the diameter of its top.

(a) Express the volume V of such a can as a function of the diameter d of its top.

(b) Express the surface area S of such a can as a function of the diameter d of its top. (*Hint:* The curved side is made from a rectangle whose length is the circumference of the top of the can.)



Chapter 2 Test

1. Match the set described in Column I with the correct interval notation from Column II. Choices in Column II may be used once, more than once, or not at all.

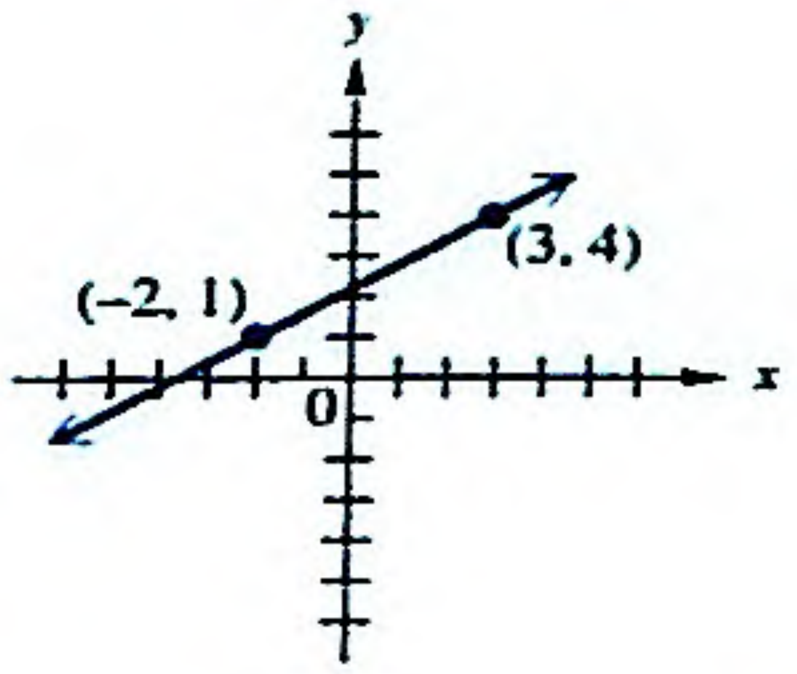
I

(a) Domain of $f(x) = \sqrt{x} + 3$
(b) Range of $f(x) = \sqrt{x - 3}$
(c) Domain of $f(x) = x^2 - 3$
(d) Range of $f(x) = x^2 + 3$
(e) Domain of $f(x) = \sqrt[3]{x - 3}$
(f) Range of $f(x) = \sqrt[3]{x} + 3$
(g) Domain of $f(x) = |x| - 3$
(h) Range of $f(x) = |x + 3|$
(i) Domain of $x = y^2$
(j) Range of $x = y^2$

II

A. $[-3, \infty)$
B. $[3, \infty)$
C. $(-\infty, \infty)$
D. $[0, \infty)$
E. $(-\infty, 3)$
F. $(-\infty, 3]$
G. $(3, \infty)$
H. $(-\infty, 0]$

The graph shows the line that passes through the points $(-2, 1)$ *and* $(3, 4)$. *Refer to it to answer Exercises 2–6.*



2. What is the slope of the line?

3. What is the distance between the two points shown?

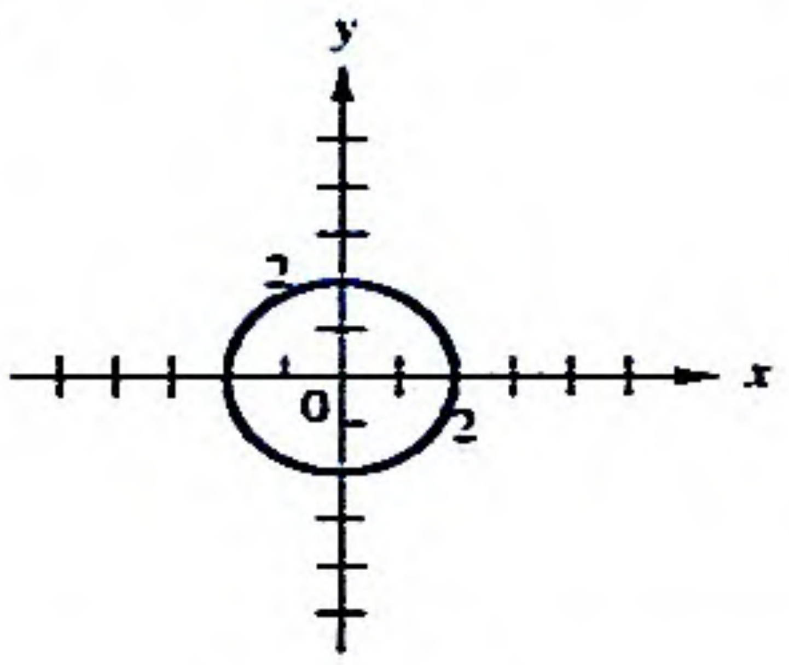
4. What are the coordinates of the midpoint of the segment joining the two points?

5. Find the standard form of the equation of the line.

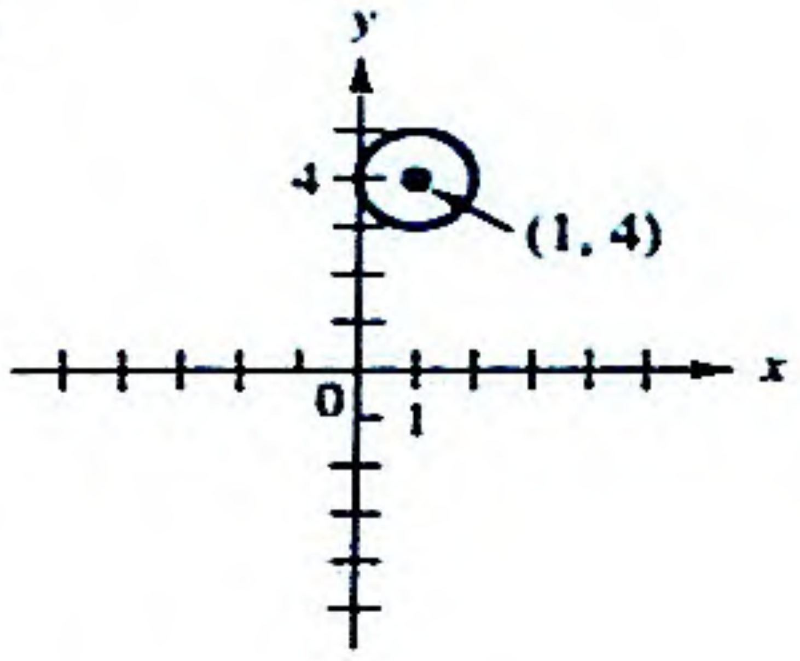
6. Write the linear function $f(x) = ax + b$ that has this line as its graph.

7. ***Connecting Graphs with Equations*** Use each graph to determine the equation of the circle. Express in center-radius form.

(a)



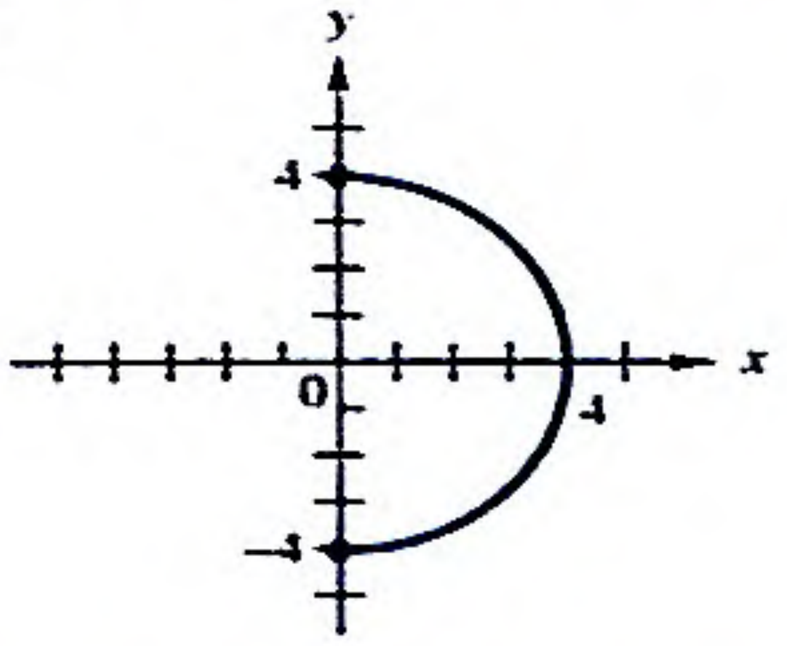
(b)



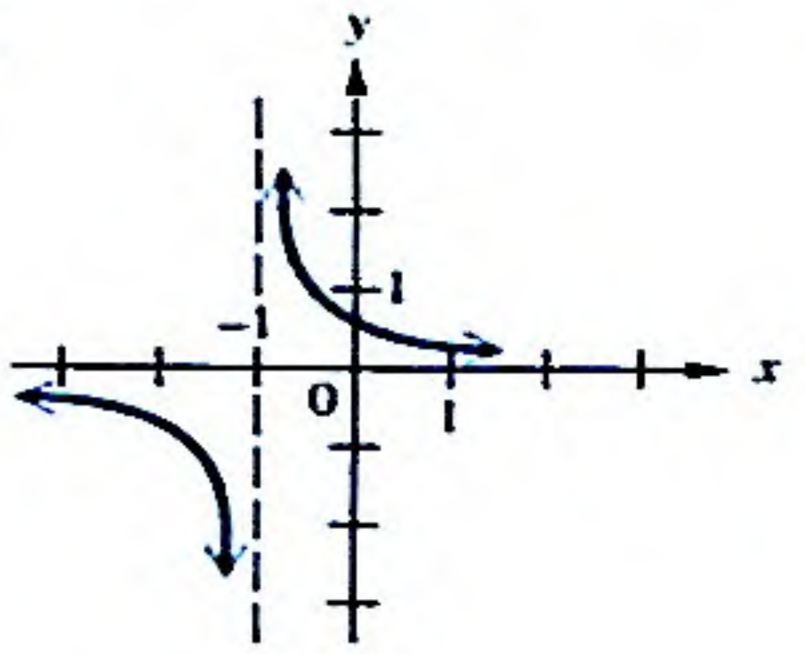
8. Graph the circle with equation $x^2 + y^2 + 4x - 10y + 13 = 0$.

9. In each case, determine whether y is a function of x. Give the domain and range. If it is a function, give the intervals where it is increasing, decreasing, or constant.

(a)



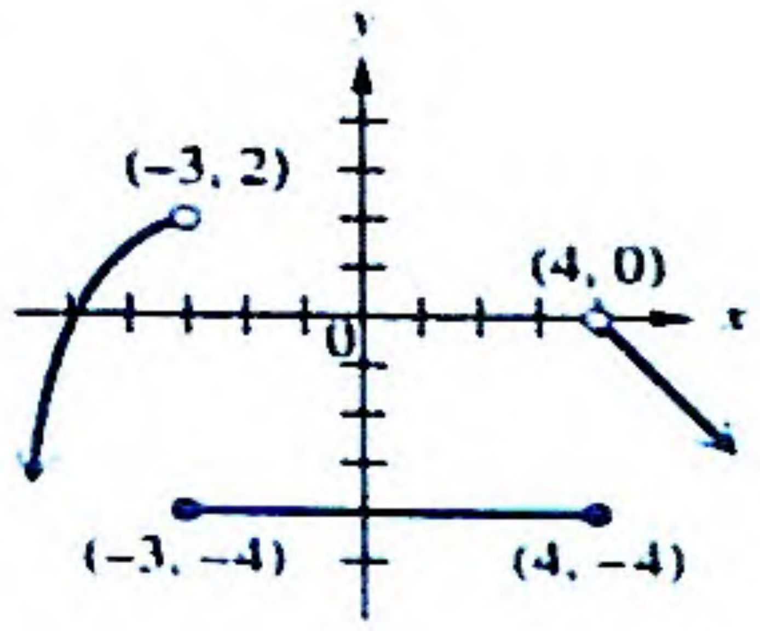
(b)



10. Suppose point A has coordinates $(5, -3)$.
    (a) What is the equation of the vertical line through A?
    (b) What is the equation of the horizontal line through A?

11. Find the slope-intercept form of the equation of the line passing through $(2, 3)$ and
    (a) parallel to the graph of $y = -3x + 2$;
    (b) perpendicular to the graph of $y = -3x + 2$.

12. Consider the graph of the function shown here. Give the interval(s) over which the function is
    (a) increasing,
    (b) decreasing,
    (c) constant,
    (d) continuous.
    (e) What is the domain of this function?
    (f) What is the range of this function?



Graph each function.

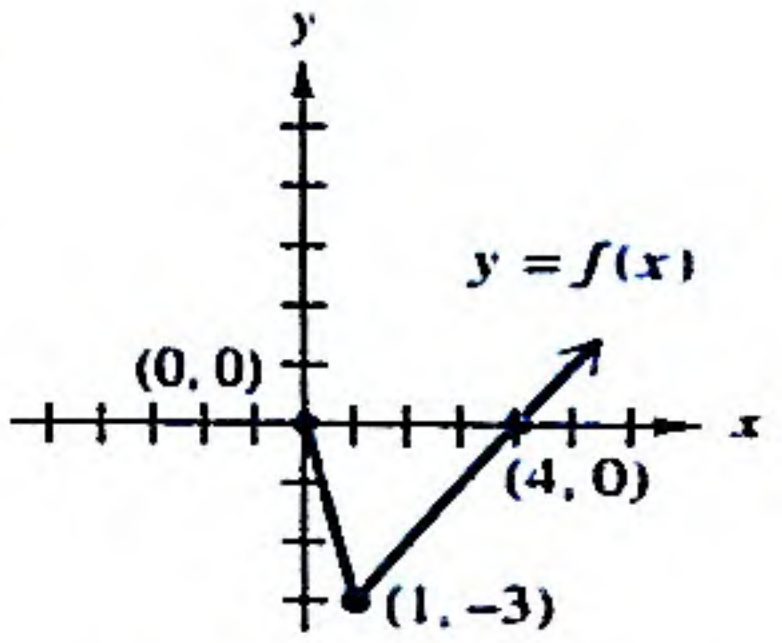
13. $y = |x - 2| - 1$

14. $f(x) = [\![x + 1]\!]$

15. $f(x) = \begin{cases} 3 & \text{if } x < -2 \\ 2 - \frac{1}{2}x & \text{if } x \geq -2 \end{cases}$

16. The graph of $y = f(x)$ is shown here. Sketch the graph of each of the following. Use ordered pairs to indicate three points on the graph.

(a) $y = f(x) + 2$
(b) $y = f(x + 2)$
(c) $y = -f(x)$
(d) $y = f(-x)$
(e) $y = 2f(x)$



17. Describe how the graph of $y = -2\sqrt{x + 2} - 3$ can be obtained from the graph of $y = \sqrt{x}$.

18. Determine whether the graph of $3x^2 - 2y^2 = 3$ is symmetric with respect to

(a) the x-axis, (b) the y-axis, and (c) the origin.

In Exercises 19 and 20, given $f(x) = 2x^2 - 3x + 2$ *and* $g(x) = -2x + 1$, *find each of the following. Simplify the expressions when possible.*

19. (a) $(f - g)(x)$ (b) $\left(\frac{f}{g}\right)(x)$

(c) the domain of $\frac{f}{g}$ (d) $\frac{f(x + h) - f(x)}{h}$ $(h \neq 0)$

20. (a) $(f + g)(1)$ (b) $(fg)(2)$ (c) $(f \circ g)(0)$

In Exercises 21 and 22, let $f(x) = \sqrt{x + 1}$ *and* $g(x) = 2x - 7$.

21. Find $(f \circ g)(x)$ and give its domain.

22. Find $(g \circ f)(x)$ and give its domain.

23. *(Modeling) Cell Phone Charges* A cell phone service provides communication between two cities. If x represents the number of minutes for the call, where $x > 0$, then the function

$$f(x) = 0.75[\![x]\!] + 1.50$$

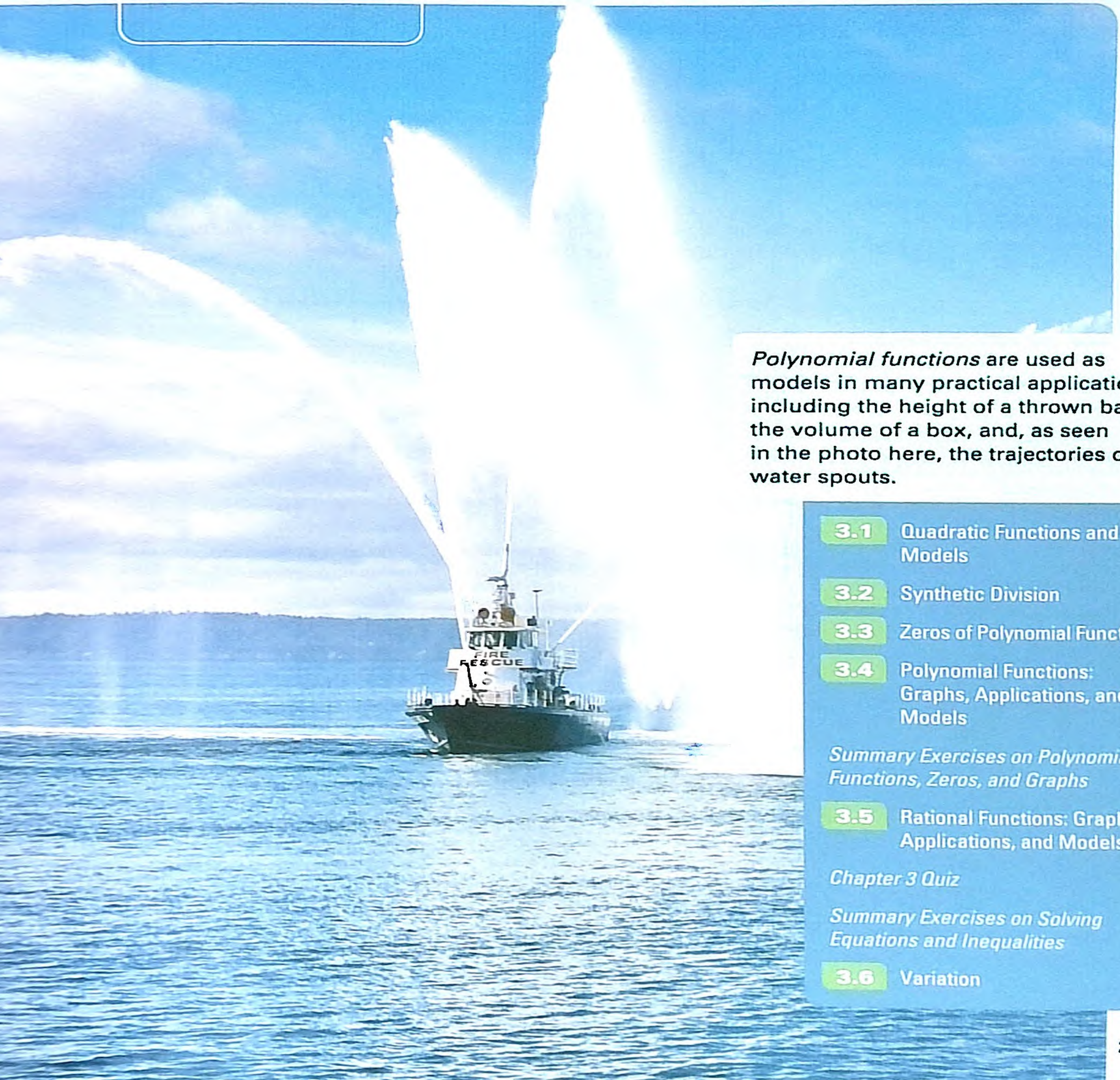
gives the total cost of the call in dollars. Find the cost of a 7.5-min call.

24. *(Modeling) Cost, Revenue, and Profit Analysis* Dotty McGinnis starts up a small business manufacturing bobble-head figures of famous soccer players. Her initial cost is \$3300. Each figure costs \$4.50 to manufacture

(a) Write a cost function C, where x represents the number of figures manufactured.
(b) Find the revenue function R if each figure in part (a) sells for \$10.50.
(c) Give the profit function P.
(d) How many figures must be produced and sold for Dotty to earn a profit?

3

Polynomial and Rational Functions



Polynomial functions are used as models in many practical applications including the height of a thrown ball, the volume of a box, and, as seen in the photo here, the trajectories of water spouts.

- 3.1 Quadratic Functions and Models
- 3.2 Synthetic Division
- 3.3 Zeros of Polynomial Functions
- 3.4 Polynomial Functions: Graphs, Applications, and Models
- *Summary Exercises on Polynomial Functions, Zeros, and Graphs*
- 3.5 Rational Functions: Graphs, Applications, and Models
- *Chapter 3 Quiz*
- *Summary Exercises on Solving Equations and Inequalities*
- 3.6 Variation

In the table on the preceding page, 95 represents 1995, 100 represents 2000, and so on, and the number of outpatient visits is given in millions.

(a) Prepare a scatter diagram, and determine a quadratic model for these data.

(b) Use the model from part (a) to predict the number of visits in 2012.

SOLUTION

(a) In Section 2.5 we used linear regression to determine linear equations that modeled data. With a graphing calculator, we can use **quadratic regression** to find quadratic equations that model data.

The scatter diagram in **Figure 15(a)** suggests that a quadratic function with a negative value of a (so the graph opens down) would be a reasonable model for the data. Using quadratic regression, the quadratic function

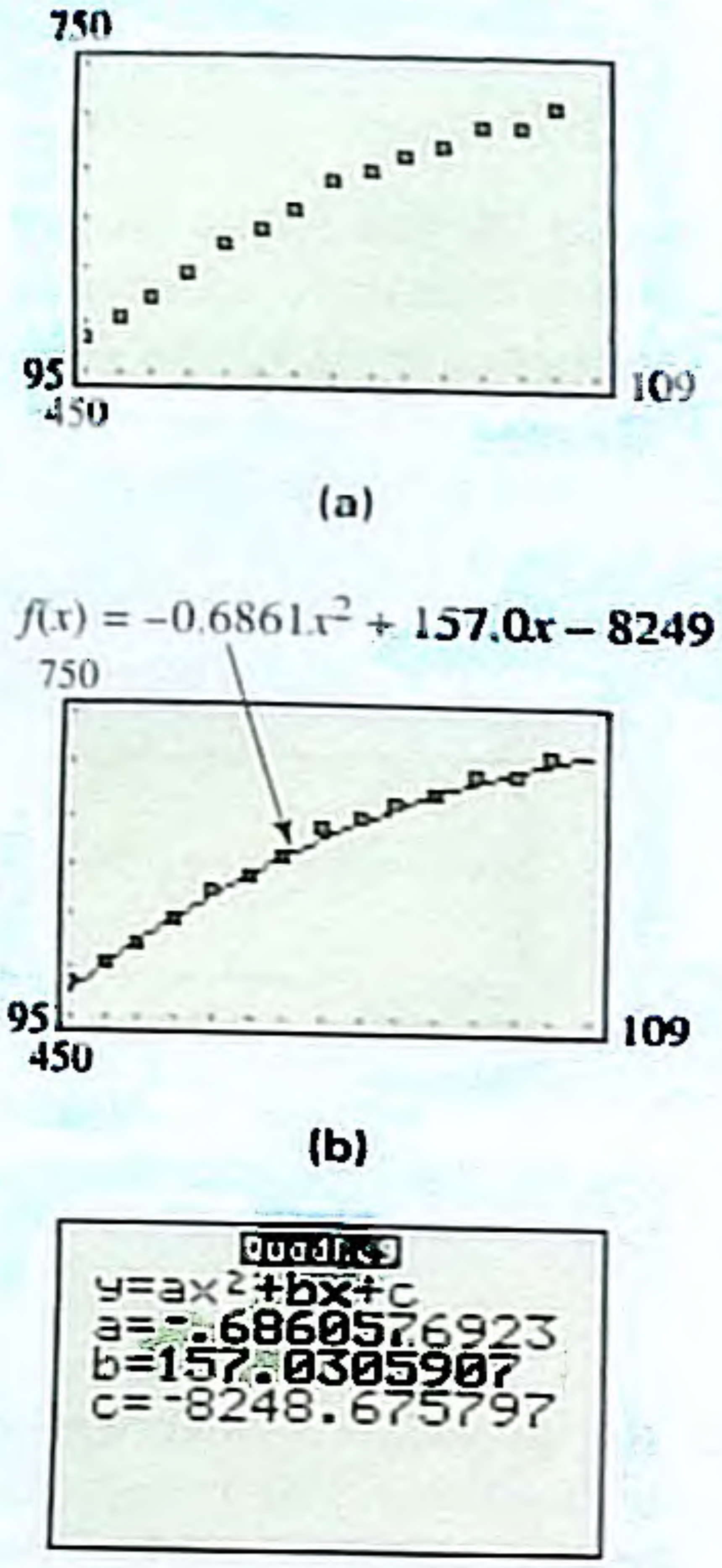
$$f(x) = -0.6861x^2 + 157.0x - 8249$$

approximates the data well. See **Figure 15(b)**. The quadratic regression values of a, b, and c are displayed in **Figure 15(c)**.

(b) The year 2012 corresponds to $x = 112$. The model predicts that there will be 729 million visits in 2012.

$$f(x) = -0.6861x^2 + 157.0x - 8249$$
$$f(112) = -0.6861(112)^2 + 157.0(112) - 8249$$
$$\approx 729 \text{ million}$$

Now Try Exercise 65.



(a)

(b)

```
QuadReg
y=ax²+bx+c
a=-.6860576923
b=157.0305907
c=-8248.675797
```

(c)

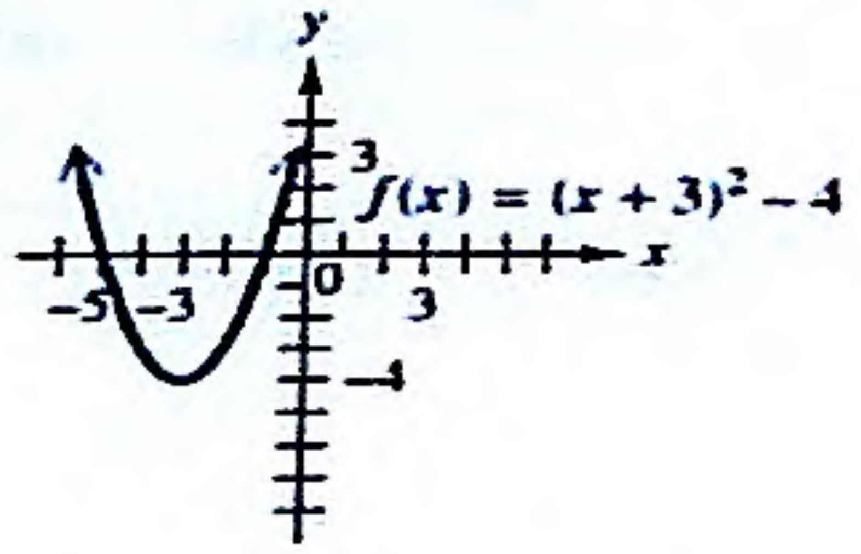
Figure 15

3.1 Exercises

In Exercises 1–4, you are given an equation and the graph of a quadratic function. Do each of the following. See Examples 1–4.

(a) Give the domain and range.
(b) Give the coordinates of the vertex.
(c) Give the equation of the axis.
(d) Find the y-intercept.
(e) Find the x-intercepts.

1.



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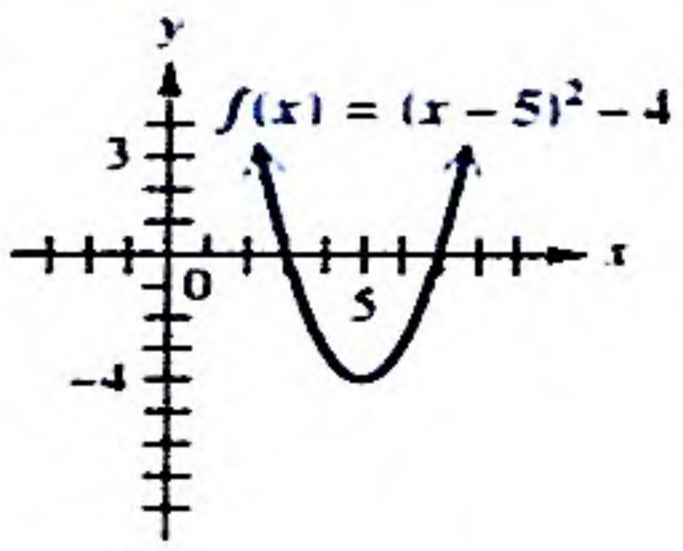

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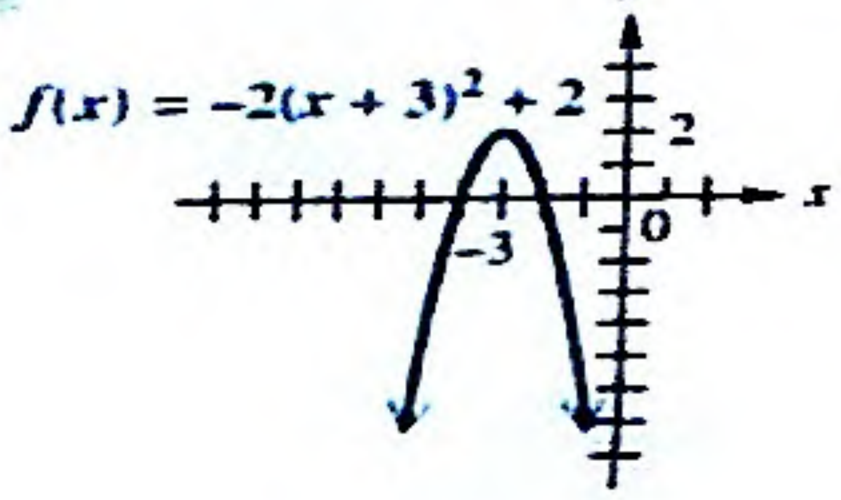

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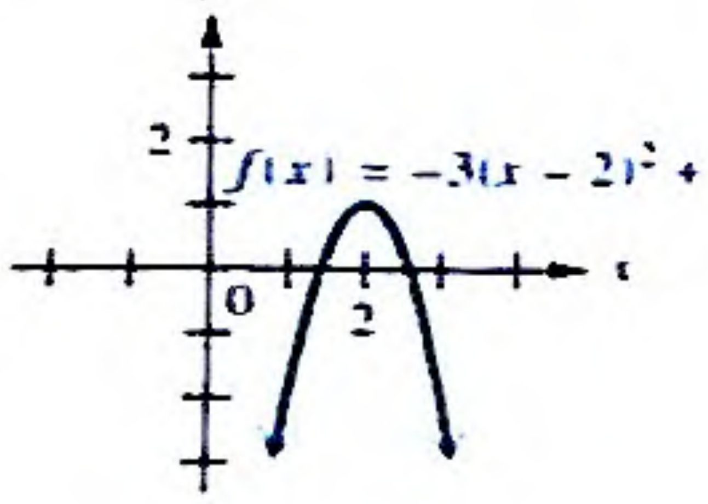


Concept Check *Calculator graphs of the functions in Exercises 5–8 are shown in Figures A–D. Match each function with its graph without actually entering it into your calculator. Then, after you have completed the exercises, check your answers with your calculator. Use the standard viewing window.*

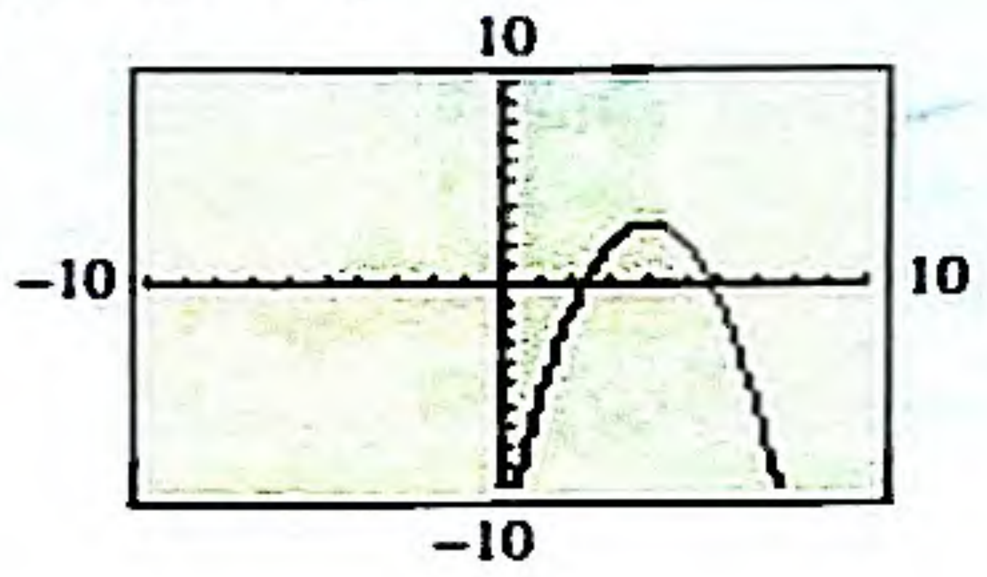
5. $f(x) = (x - 4)^2 - 3$

6. $f(x) = -(x - 4)^2 + 3$

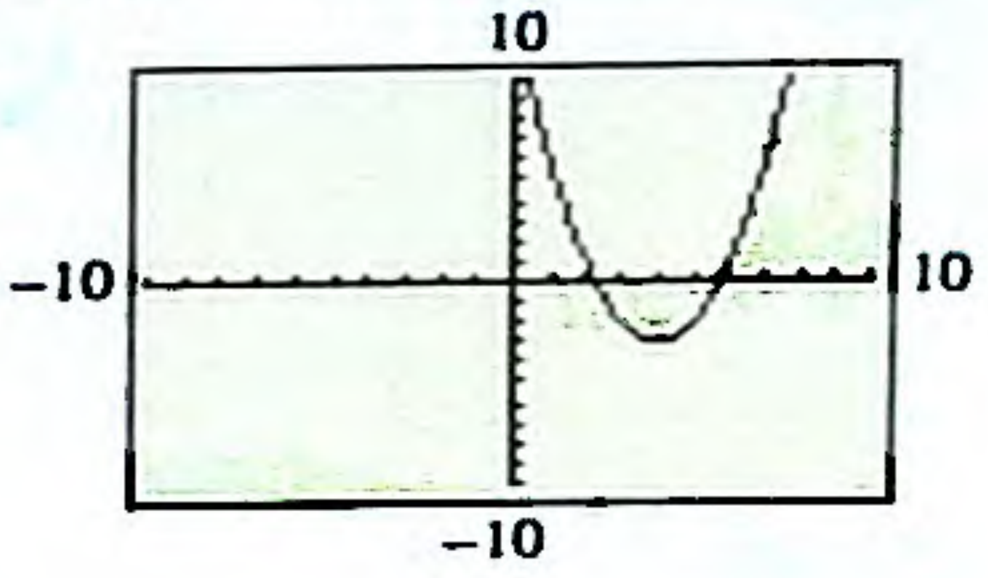
7. $f(x) = (x + 4)^2 - 3$

8. $f(x) = -(x + 4)^2 + 3$

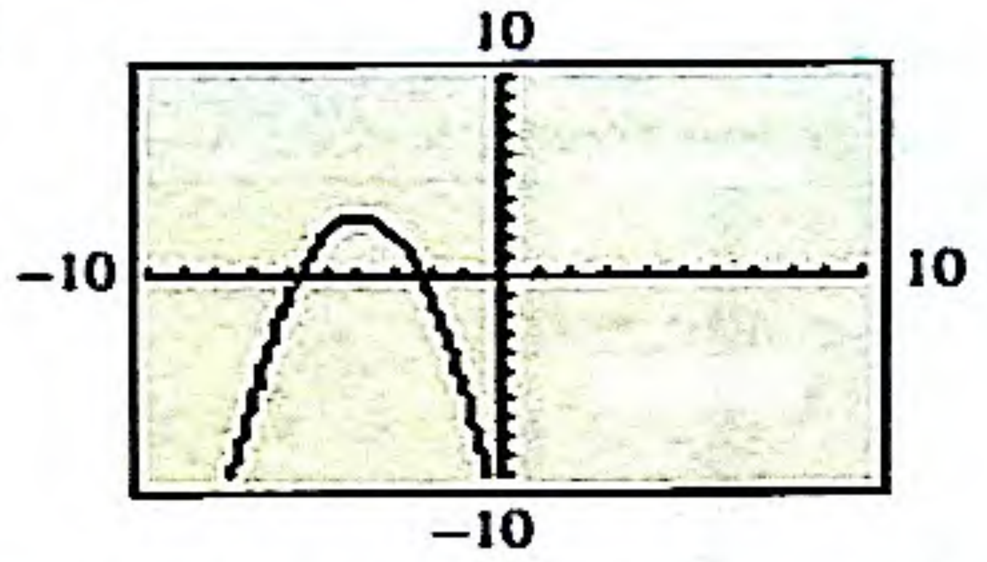
A.



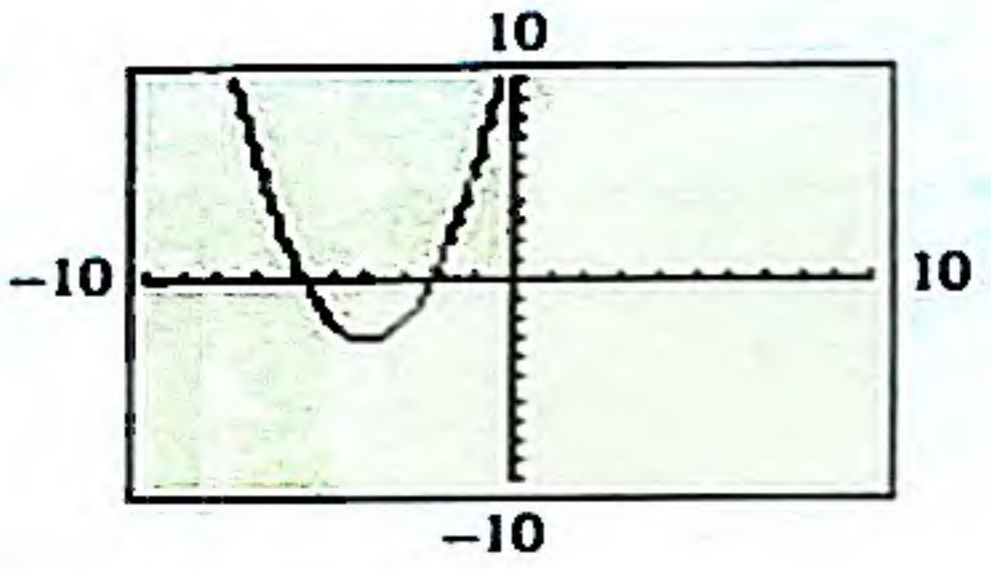
B.



C.



D.



9. Graph the following on the same coordinate system.

(a) $y = x^2$ (b) $y = 3x^2$ (c) $y = \frac{1}{3}x^2$

(d) How does the coefficient of x^2 affect the shape of the graph?

10. Graph the following on the same coordinate system.

(a) $y = x^2$ (b) $y = x^2 - 2$ (c) $y = x^2 + 2$

(d) How do the graphs in parts (b) and (c) differ from the graph of $y = x^2$?

11. Graph the following on the same coordinate system.

(a) $y = (x - 2)^2$ (b) $y = (x + 1)^2$ (c) $y = (x + 3)^2$

(d) How do these graphs differ from the graph of $y = x^2$?

12. ***Concept Check*** Match each equation with the description of the parabola that is its graph.

(a) $y = (x + 4)^2 + 2$	A. vertex $(-2, 4)$, opens up
(b) $y = (x + 2)^2 + 4$	B. vertex $(-2, 4)$, opens down
(c) $y = -(x + 4)^2 + 2$	C. vertex $(-4, 2)$, opens up
(d) $y = -(x + 2)^2 + 4$	D. vertex $(-4, 2)$, opens down

*Graph each quadratic function. Give the (a) vertex, (b) axis, (c) domain, and (d) range. Then determine (e) the interval of the domain for which the function is increasing and (f) the interval for which the function is decreasing. See **Examples 1–4.***

13. $f(x) = (x - 2)^2$

14. $f(x) = (x + 4)^2$

15. $f(x) = (x + 3)^2 - 4$

16. $f(x) = (x - 5)^2 - 4$

17. $f(x) = -\frac{1}{2}(x + 1)^2 - 3$

18. $f(x) = -3(x - 2)^2 + 1$

19. $f(x) = x^2 - 2x + 3$

20. $f(x) = x^2 + 6x + 5$

21. $f(x) = x^2 - 10x + 21$

22. $f(x) = 2x^2 - 4x + 5$

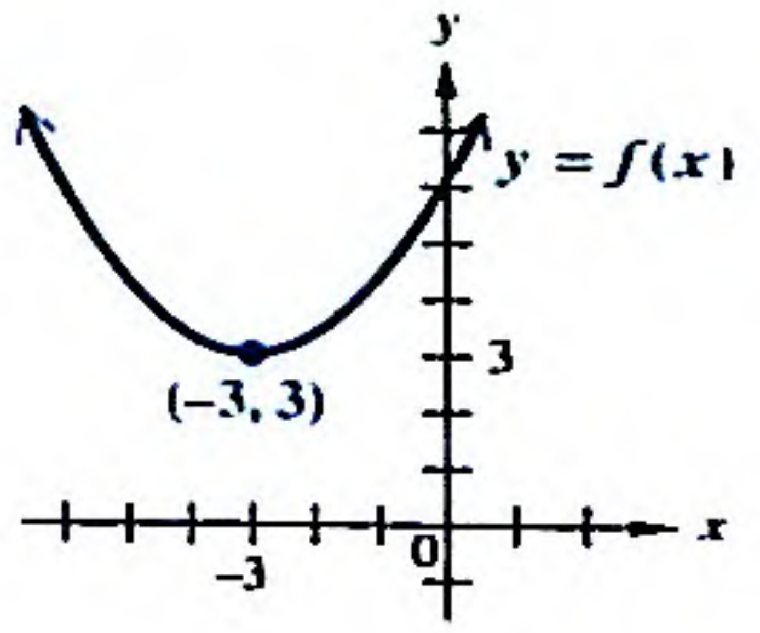
23. $f(x) = -2x^2 - 12x - 16$

24. $f(x) = -3x^2 + 24x - 46$

25. $f(x) = -\frac{1}{2}x^2 - 3x - \frac{1}{2}$

26. $f(x) = \frac{2}{3}x^2 - \frac{8}{3}x + \frac{5}{3}$

Concept Check *The figure shows the graph of a quadratic function $y = f(x)$. Use it to work Exercises 27–30.*



27. What is the minimum value of $f(x)$?

28. For what value of x is $f(x)$ as small as possible?

29. How many real solutions are there to the equation $f(x) = 1$?

30. How many real solutions are there to the equation $f(x) = 4$?

Concept Check *Several possible graphs of the quadratic function*

$$f(x) = ax^2 + bx + c$$

are shown below. For the restrictions on a, b, and c given in Exercises 31–36, select the corresponding graph from choices A–F. (Hint: Use the discriminant. See **Section 1.4** *if necessary.)*

31. $a < 0$; $b^2 - 4ac = 0$

32. $a > 0$; $b^2 - 4ac < 0$

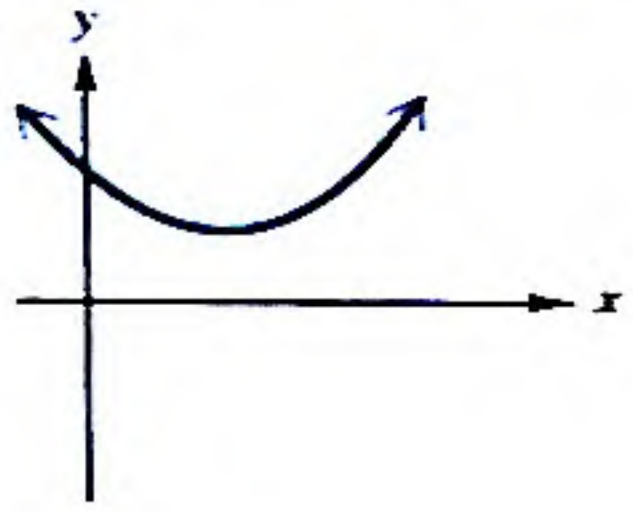
33. $a < 0$; $b^2 - 4ac < 0$

34. $a < 0$; $b^2 - 4ac > 0$

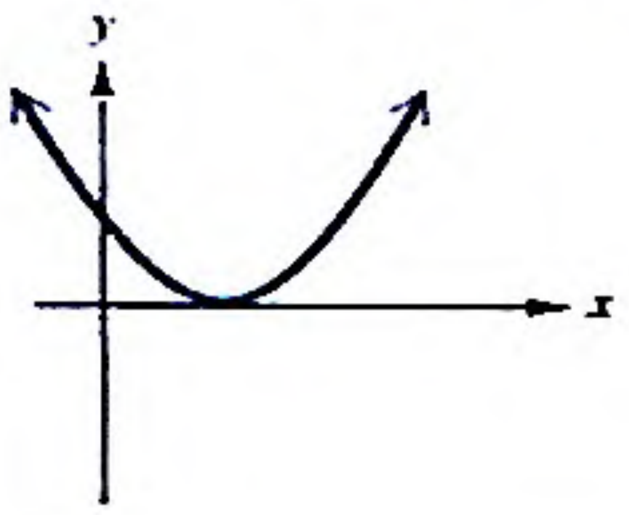
35. $a > 0$; $b^2 - 4ac > 0$

36. $a > 0$; $b^2 - 4ac = 0$

A.



B.



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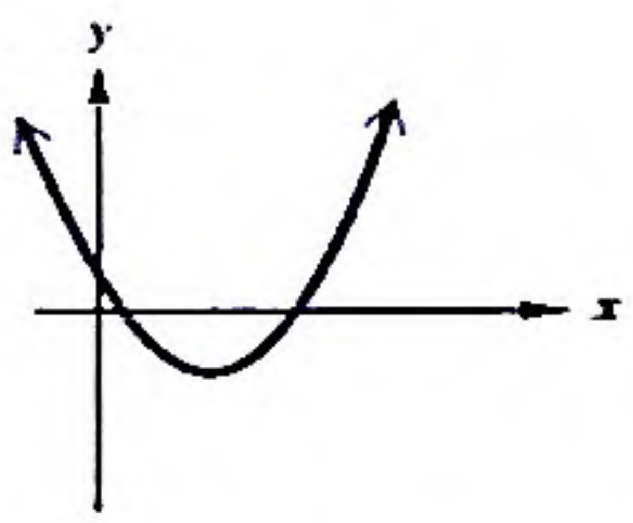

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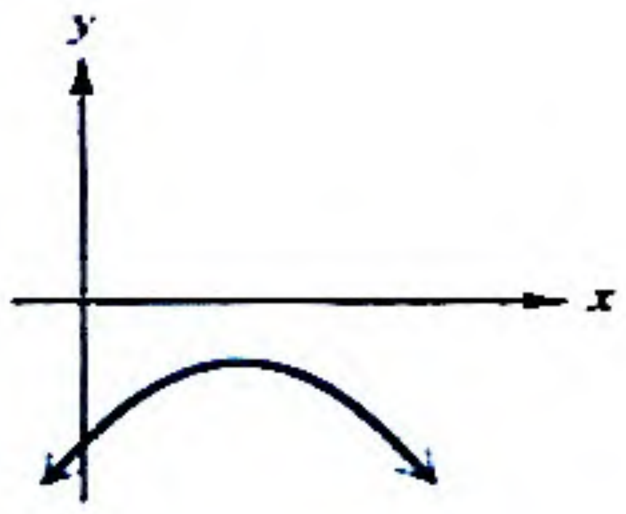
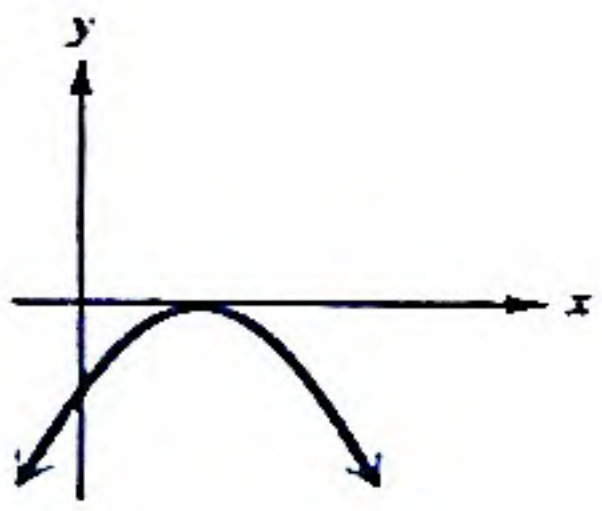
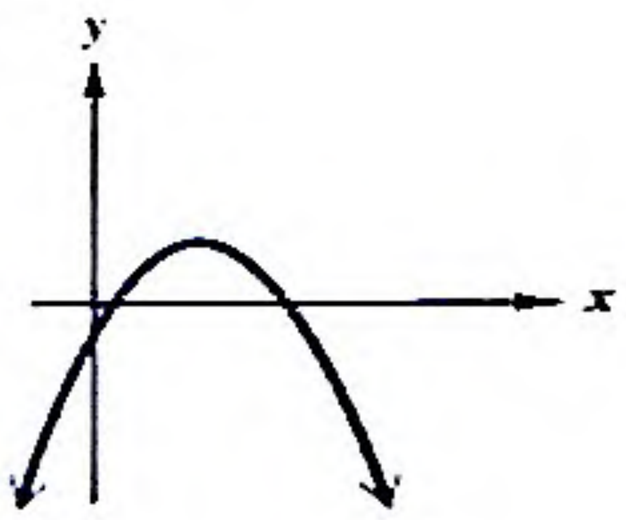


E.

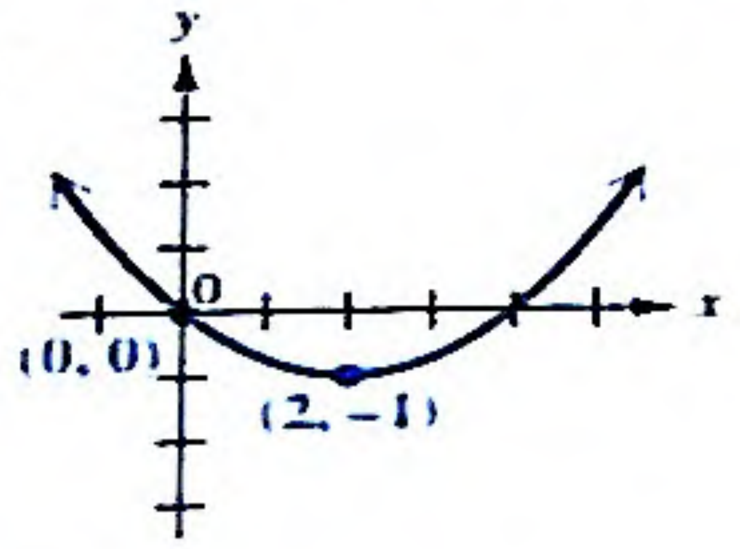


F.



Connecting Graphs with Equations *In Exercises 37–40, find a quadratic function f whose graph matches the one in the figure. (Hint: See the Note following* **Example 3.***)*

37.



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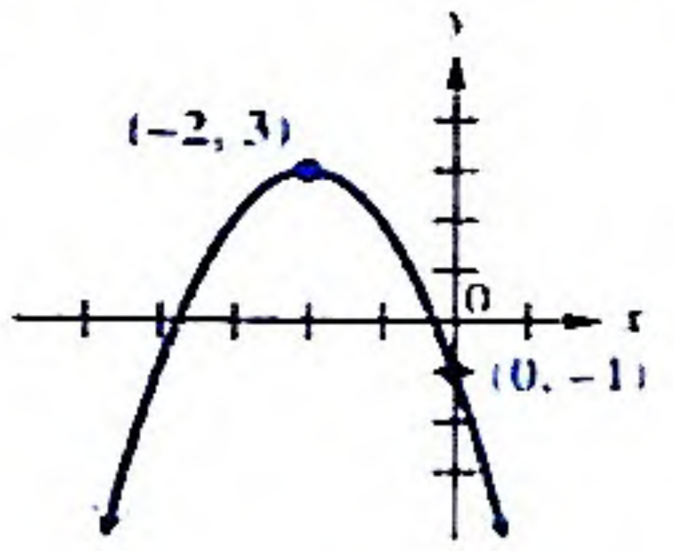

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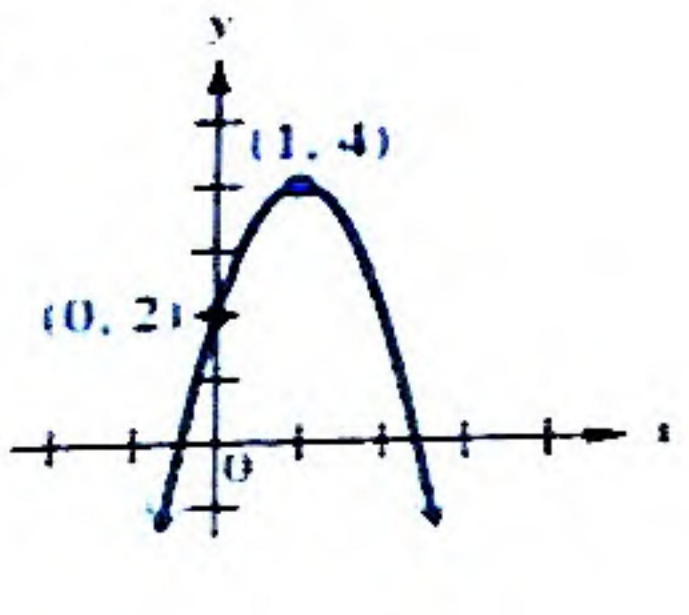
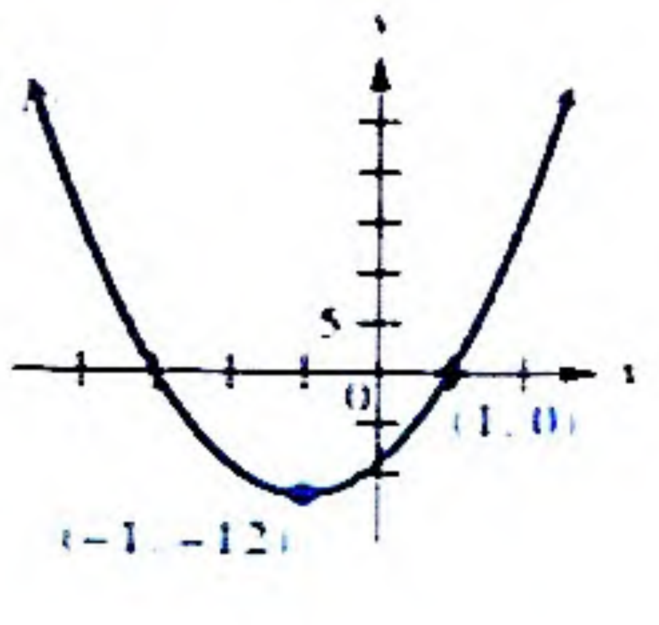
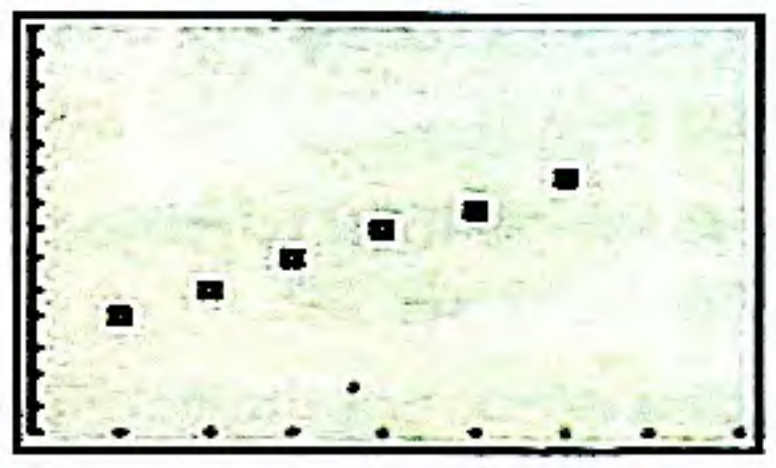


40.

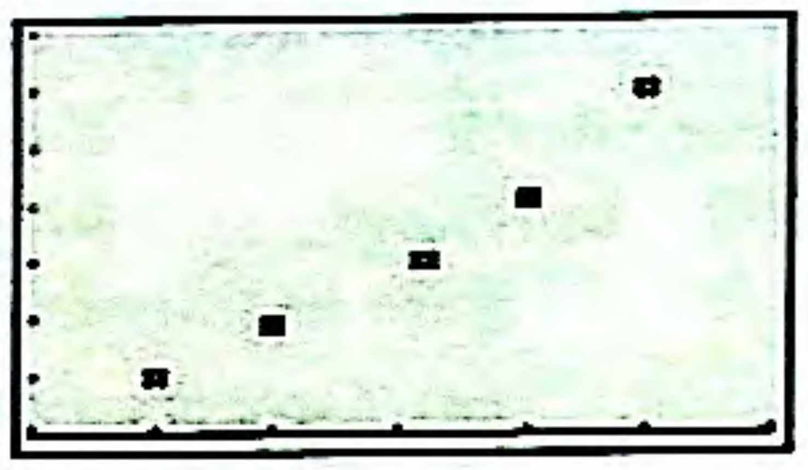


(Modeling) In each scatter diagram, tell whether a linear or a quadratic model is appropriate for the data. If linear, tell whether the slope should be positive or negative. If quadratic, tell whether the leading coefficient of x^2 should be positive or negative.

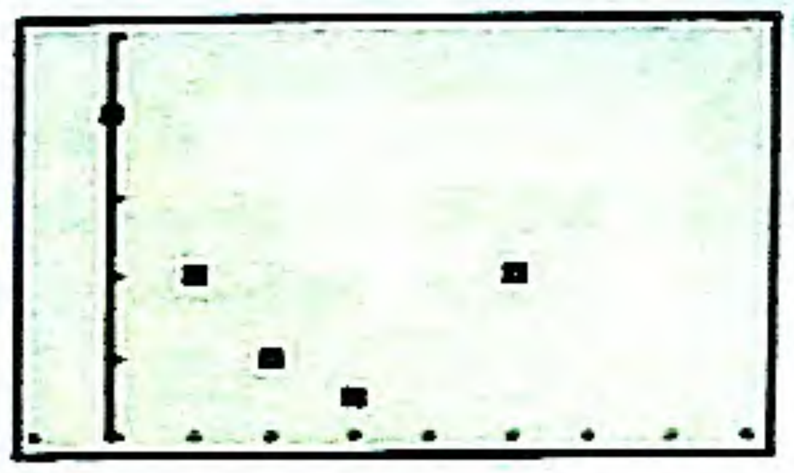
41. number of shopping centers as a function of time



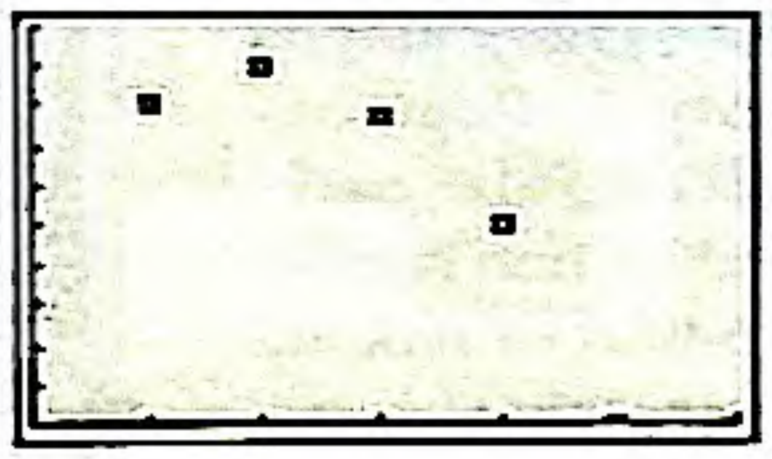
42. growth in science centers/museums as a function of time



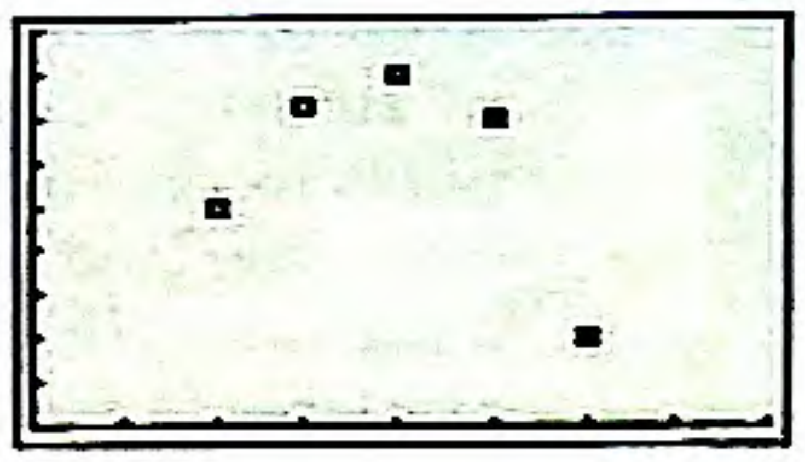
43. value of U.S. salmon catch as a function of time



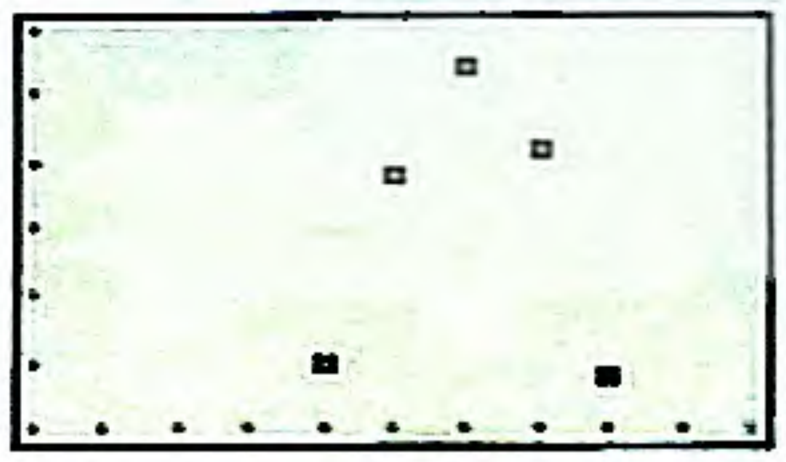
44. height of an object projected upward as a function of time



45. Social Security assets as a function of time



46. newborns with AIDS as a function of time



Solve each problem.

47. ***Sum and Product of Two Numbers*** Find two numbers whose sum is 20 and whose product is the maximum possible value. (*Hint: Let x be one number. Then $20 - x$ is the other number. Form a quadratic function by multiplying them, and then find the maximum value of the function.*)

48. ***Sum and Product of Two Numbers*** Find two numbers whose sum is 32 and whose product is the maximum possible value.

49. ***Minimum Cost*** Brigette Cole has a taco stand. She has found that her daily costs are approximated by

$$C(x) = x^2 - 40x + 610,$$

where $C(x)$ is the cost, in dollars, to sell x units of tacos. Find the number of units of tacos she should sell to minimize her costs. What is the minimum cost?

50. ***Maximum Revenue*** The revenue of a charter bus company depends on the number of unsold seats. If the revenue in dollars, $R(x)$, is given by

$$R(x) = -x^2 + 50x + 5000,$$

where x is the number of unsold seats, find the number of unsold seats that produce maximum revenue. What is the maximum revenue?

51. ***Maximum Number of Mosquitos*** The number of mosquitos, $M(x)$, in millions, in a certain area of Florida depends on the June rainfall, x, in inches. The function

$$M(x) = 10x - x^2$$

models this phenomenon. Find the amount of rainfall that will maximize the number of mosquitos. What is the maximum number of mosquitos?

52. ***Height of an Object*** If an object is projected upward from ground level with an initial velocity of 32 ft per sec, then its height in feet after t seconds is given by

$$s(t) = -16t^2 + 32t.$$

Find the number of seconds it will take to reach its maximum height. What is this maximum height?

(Modeling) Solve each problem. See Example 5.

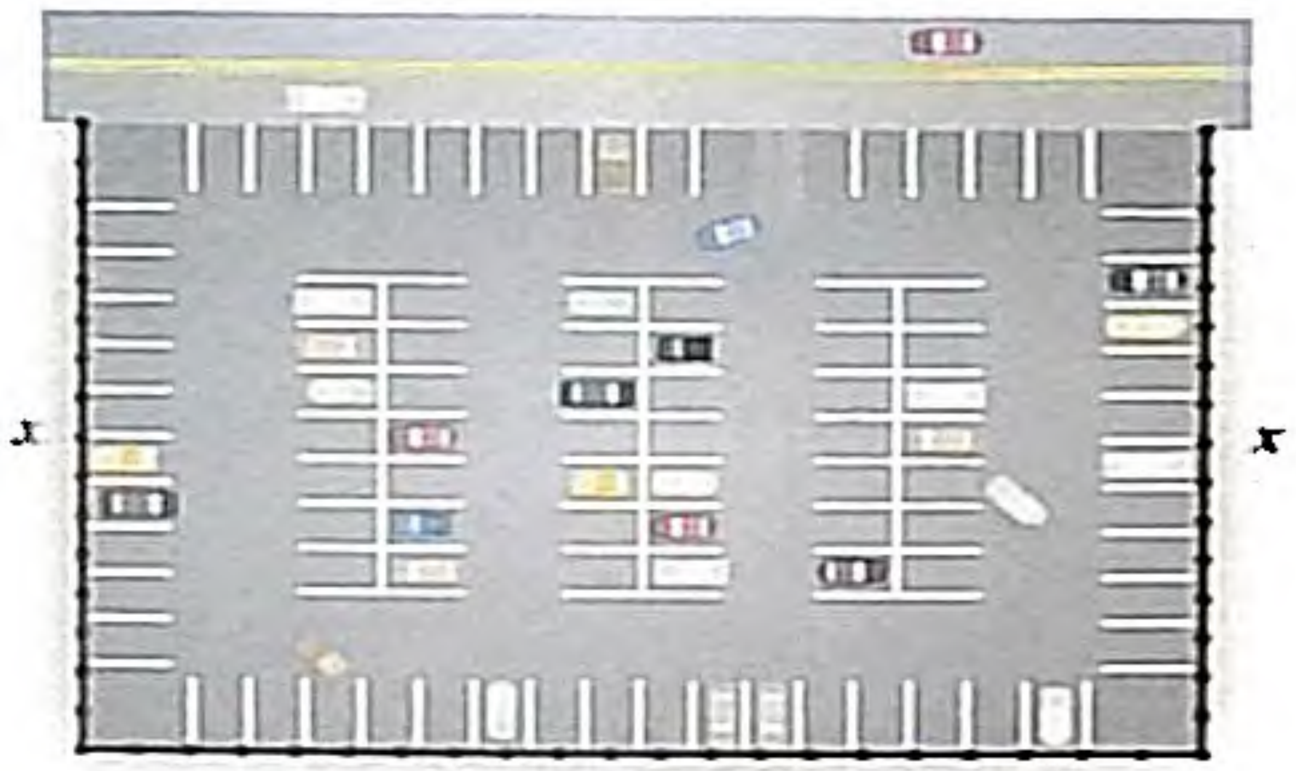
53. ***Height of a Toy Rocket*** A toy rocket (not internally powered) is launched straight up from the top of a building 50 ft tall at an initial velocity of 200 ft per sec.

(a) Give the function that describes the height of the rocket in terms of time t.
(b) Determine the time at which the rocket reaches its maximum height and the maximum height in feet.
(c) For what time interval will the rocket be more than 300 ft above ground level?
(d) After how many seconds will it hit the ground?

54. ***Height of a Projected Rock*** A rock is projected directly upward from ground level with an initial velocity of 90 ft per sec.

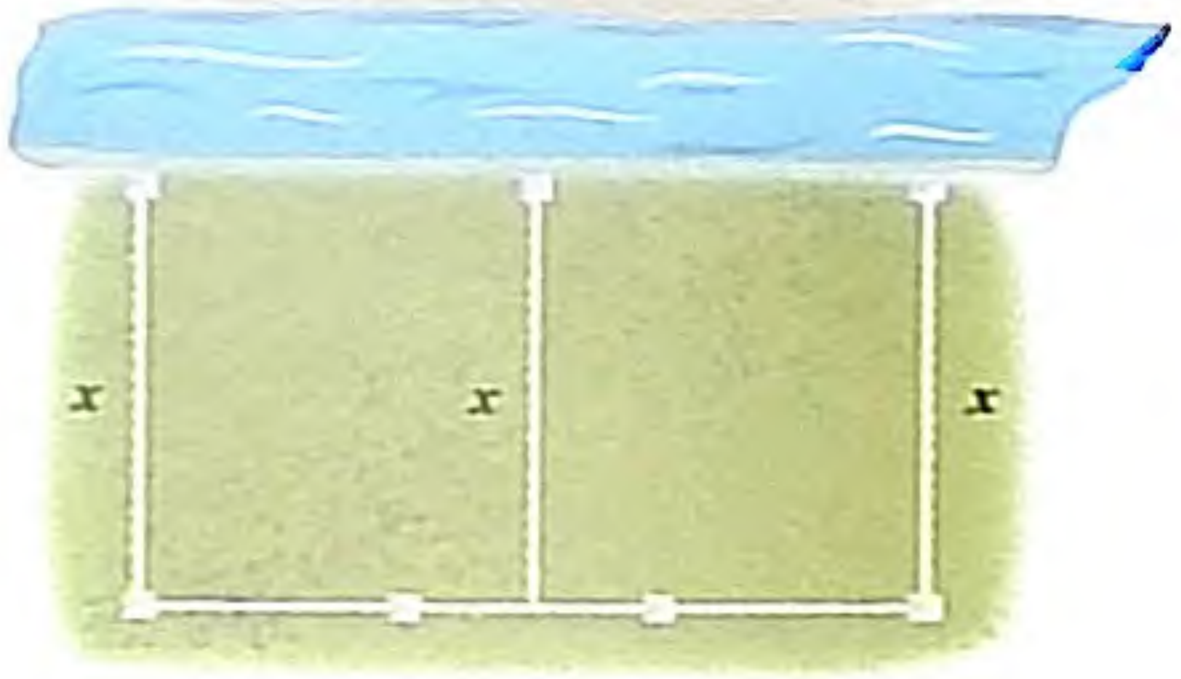
(a) Give the function that describes the height of the rock in terms of time t.
(b) Determine the time at which the rock reaches its maximum height and the maximum height in feet.
(c) For what time interval will the rock be more than 120 ft above ground level?
(d) After how many seconds will it return to the ground?

55. ***Area of a Parking Lot*** One campus of Houston Community College has plans to construct a rectangular parking lot on land bordered on one side by a highway. There are 640 ft of fencing available to fence the other three sides. Let x represent the length of each of the two parallel sides of fencing.



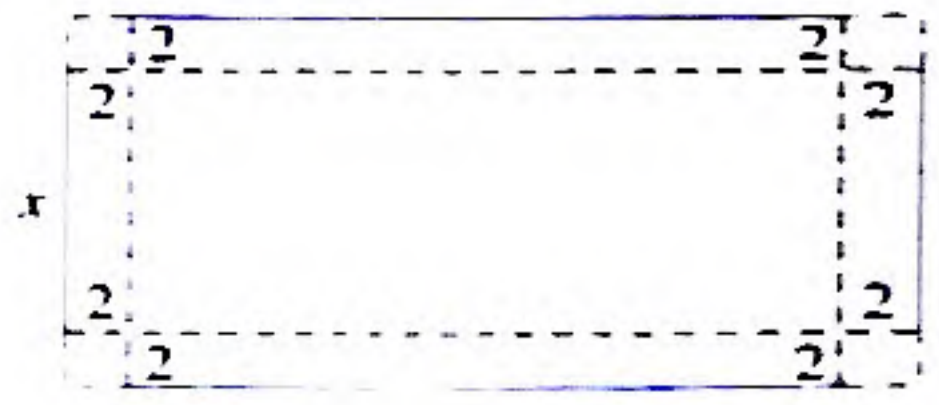
(a) Express the length of the remaining side to be fenced in terms of x.
(b) What are the restrictions on x?
(c) Determine a function $\mathscr{A}$ that represents the area of the parking lot in terms of x.
(d) Determine the values of x that will give an area between 30,000 and 40,000 ft^2.
(e) What dimensions will give a maximum area, and what will this area be?

56. ***Area of a Rectangular Region*** A farmer wishes to enclose a rectangular region bordering a river with fencing, as shown in the diagram. Suppose that x represents the length of each of the three parallel pieces of fencing. She has 600 ft of fencing available.

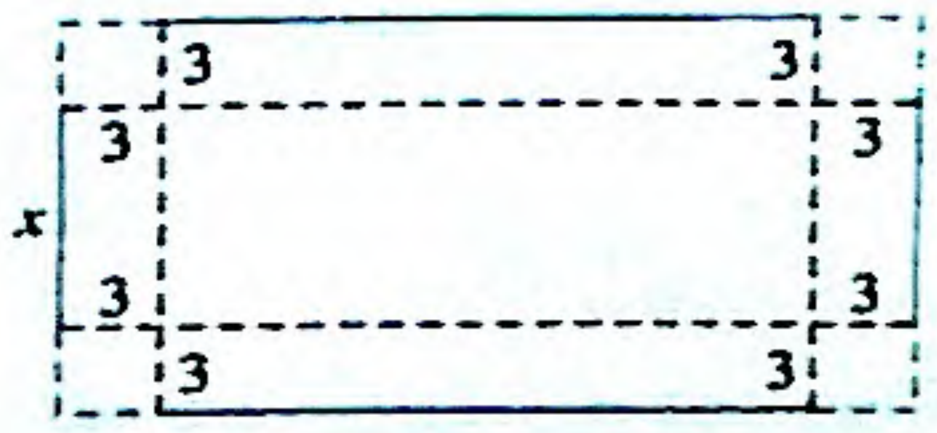


(a) What is the length of the remaining piece of fencing in terms of x?
(b) Determine a function $\mathscr{A}$ that represents the total area of the enclosed region. Give any restrictions on x.
(c) What dimensions for the total enclosed region would give an area of 22,500 ft^2?
(d) What is the maximum area that can be enclosed?

57. ***Volume of a Box*** A piece of cardboard is twice as long as it is wide. It is to be made into a box with an open top by cutting 2-in. squares from each corner and folding up the sides. Let x represent the width (in inches) of the original piece of cardboard.

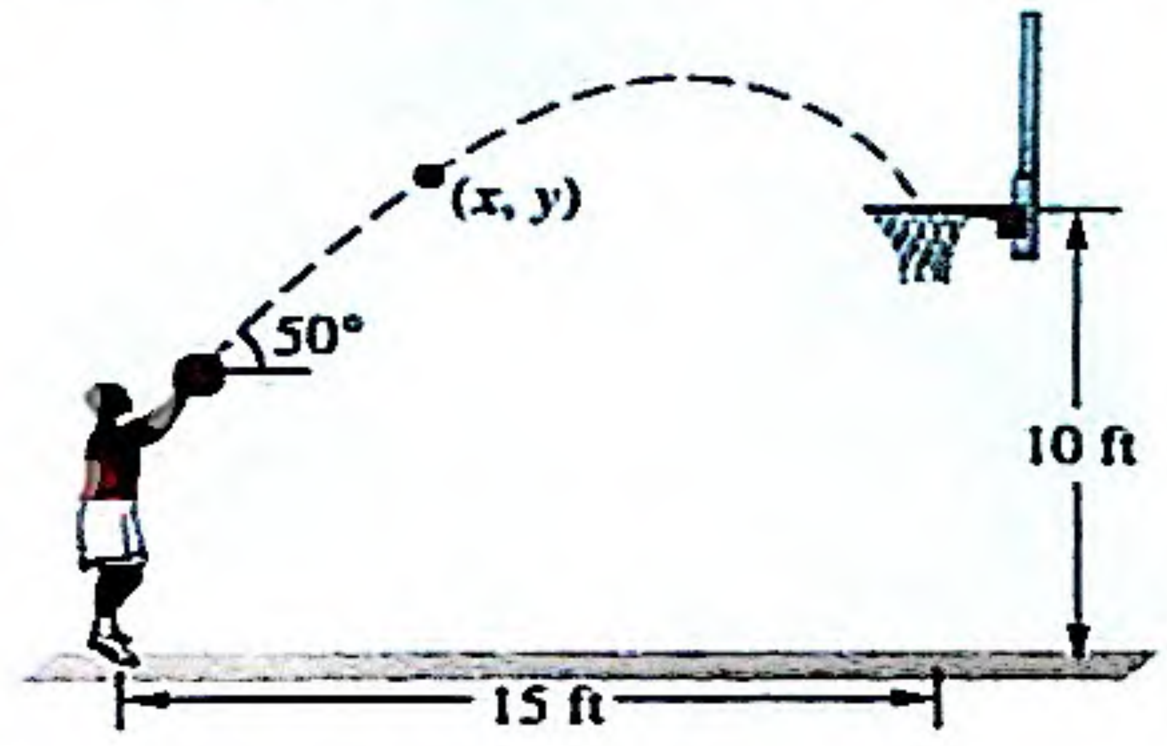


(a) Represent the length of the original piece of cardboard in terms of x.
(b) What will be the dimensions of the bottom rectangular base of the box? Give the restrictions on x.
(c) Determine a function V that represents the volume of the box in terms of x.
(d) For what dimensions of the bottom of the box will the volume be 320 in^3?
(e) Find the values of x (to the nearest tenth of an inch) if such a box is to have a volume between 400 and 500 in^3.



58. ***Volume of a Box*** A piece of sheet metal is 2.5 times as long as it is wide. It is to be made into a box with an open top by cutting 3-in. squares from each corner and folding up the sides. Let x represent the width (in inches) of the original piece.

(a) Represent the length of the original piece of sheet metal in terms of x.
(b) What are the restrictions on x?
(c) Determine a function V that represents the volume of the box in terms of x.
(d) For what values of x (that is, original widths) will the volume of the box be between 600 and 800 in.^{3}? Give values to the nearest tenth of an inch.

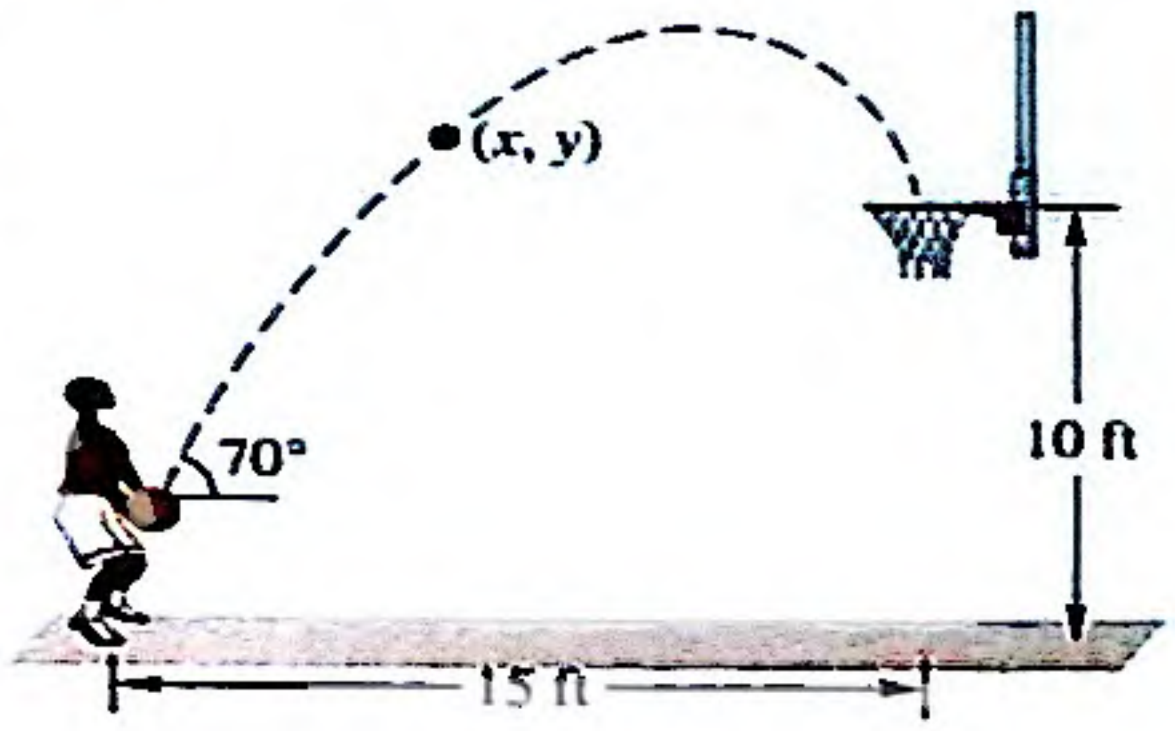


59. ***Shooting a Free Throw*** If a person shoots a free throw from a position 8 ft above the floor, then the path of the ball may be modeled by the parabola

$$y = \frac{-16x^2}{0.434v^2} + 1.15x + 8,$$

where v is the initial velocity of the ball in feet per second, as illustrated in the figure. (*Source:* Rist, C., "The Physics of Foul Shots," *Discover*.)

(a) If the basketball hoop is 10 ft high and located 15 ft away, what initial velocity v should the basketball have?
(b) What is the maximum height of the basketball?



60. ***Shooting a Free Throw*** See Exercise 59. If a person shoots a free throw from an underhand position 3 ft above the floor, the path of the ball may be modeled by

$$y = \frac{-16x^2}{0.117v^2} + 2.75x + 3.$$

Repeat parts (a) and (b) from Exercise 59. Then compare the paths for the overhand shot and the underhand shot.

(Modeling) Solve each problem. See Example 6.

61. ***Births to Unmarried Women*** The percent of births to unmarried women in the United States from 1996 to 2007 are shown in the table. The data are modeled by the quadratic function

$$f(x) = 0.0773x^2 - 0.2115x + 32.64,$$

where $x = 0$ corresponds to 1996 and $f(x)$ is the percent. If this model continues to apply, what would it predict for the percent of these births in 2012?

Year	Percent	Year	Percent
1996	32.4	2002	34.0
1997	32.4	2003	34.6
1998	32.8	2004	35.8
1999	33.0	2005	36.9
2000	33.2	2006	38.5
2001	33.5	2007	39.7

Source: National Center for Health Statistics.

62. *Concentration of Atmospheric CO_2* The quadratic function

$$f(x) = 0.0098x^2 + 0.9010x + 316.8$$

models the worldwide atmospheric concentration of carbon dioxide in parts per million (ppm) over the period 1960–2009, where $x = 0$ represents the year 1960. If this model continues to hold, what will be the atmospheric CO_2 concentration in 2020? (*Source*: U.S. Department of Energy.)

63. *Spending on Shoes and Clothing* The total amount spent by Americans on shoes and clothing from 2000 to 2009 can be modeled by

$$f(x) = -4.979x^2 + 71.73x + 97.29,$$

where $x = 0$ represents 2000 and $f(x)$ is in billions of dollars. Based on this model, in what year did spending on shoes and clothing reach a maximum? (*Source*: Bureau of Economic Analysis.)

64. *Accident Rate* According to data from the National Highway Traffic Safety Administration, the accident rate as a function of the age of the driver in years x can be approximated by the function

$$f(x) = 0.0232x^2 - 2.28x + 60.0,$$

for $16 \le x \le 85$. Find both the age at which the accident rate is a minimum and the minimum rate.

65. *AIDS Cases in the United States* The table* lists the total (cumulative) number of AIDS cases diagnosed in the United States through 2007. For example, a total of 361,509 AIDS cases were diagnosed through 1993.

Year	AIDS Cases	Year	AIDS Cases
1990	193,245	1999	718,676
1991	248,023	2000	759,434
1992	315,329	2001	801,302
1993	361,509	2002	844,047
1994	441,406	2003	888,279
1995	515,586	2004	932,387
1996	584,394	2005	978,056
1997	632,249	2006	982,498
1998	673,572	2007	1,018,428

(a) Plot the data. Let $x = 0$ correspond to the year 1990.

(b) Would a linear or a quadratic function model the data better? Explain.

(c) Find a quadratic function defined by $f(x) = ax^2 + bx + c$ that models the data.

(d) Plot the data together with f on the same coordinate plane. How well does f model the number of AIDS cases?

(e) Use f to estimate the total number of AIDS cases diagnosed in the years 2009 and 2010.

(f) According to the model, how many new cases were diagnosed in the year 2010?

66. *AIDS Deaths in the United States* The table* lists the total (cumulative) number of known deaths caused by AIDS in the United States up to 2007.

Year	AIDS Deaths	Year	AIDS Deaths
1990	119,821	1999	419,234
1991	154,567	2000	436,373
1992	191,508	2001	454,099
1993	220,592	2002	471,417
1994	269,992	2003	489,437
1995	320,692	2004	507,536
1996	359,892	2005	524,547
1997	381,738	2006	565,927
1998	400,743	2007	583,298

(a) Plot the data. Let $x = 0$ correspond to the year 1990.

(b) Would a linear or a quadratic function model the data better? Explain.

(c) Find a quadratic function defined by $g(x) = ax^2 + bx + c$ that models the data.

(d) Plot the data together with g on the same coordinate plane. How well does g model the number of AIDS cases?

*Sources: "Facts and Figures," Joint United Nations Programme on HIV/AIDS (UNAIDS), February, 1999; U.S. Census Bureau, 2007, and Centers for Disease Control and Prevention.

(e) Use g to estimate the total number of AIDS deaths in the year 2010.

(f) Consider the last two entries in the table for the years 2006 and 2007. Is it safe to assume that the quadratic model given for $g(x)$ will continue for years 2008 and beyond?

67. ***Foreign-Born Americans*** The table lists the percent of the U.S. population that was foreign-born for selected years.

Year	Percent
1930	11.6
1940	8.8
1950	6.9
1960	5.4
1970	4.7
1980	6.2
1990	8.0
2000	10.4
2008	12.2

Source: U.S. Census Bureau.

(a) Plot the data. Let $x = 0$ correspond to the year 1930, $x = 10$ correspond to 1940, and so on.

(b) Find a quadratic function $f(x) = a(x - h)^2 + k$ that models the data. Use $(40, 4.7)$ as the vertex and $(20, 6.9)$ as the other point to determine a.

(c) Plot the data together with f in the same window. How well does f model the percent of the U.S. population that is foreign-born?

(d) Use the quadratic regression feature of a graphing calculator to determine the quadratic function g that provides the best fit for the data.

(e) Use the functions f and g to predict the percent of the U.S. population in 2015 that will be foreign-born.

68. ***Automobile Stopping Distance*** Selected values of the stopping distance y, in feet, of a car traveling x miles per hour are given in the table.

(a) Plot the data.

(b) The quadratic function

$$f(x) = 0.056057x^2 + 1.06657x$$

is one model that has been used to approximate stopping distances. Find and interpret $f(45)$.

(c) How well does f model the car's stopping distance?

Speed (in mph)	Stopping Distance (in feet)
20	46
30	87
40	140
50	240
60	282
70	371

Source: National Safety Institute Student Workbook.

Concept Check *Work Exercises 69–74.*

69. Find a value of c so that $y = x^2 - 10x + c$ has exactly one x-intercept.

70. For what values of a does $y = ax^2 - 8x + 4$ have no x-intercepts?

71. Define the quadratic function f having x-intercepts 2 and 5, and y-intercept 5.

72. Define the quadratic function f having x-intercepts 1 and -2, and y-intercept 4.

73. The distance between the two points $P(x_1, y_1)$ and $R(x_2, y_2)$ is

$$d(P, R) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.\quad \text{Distance formula (Section 2.1)}$$

Find the closest point on the line $y = 2x$ to the point $(1, 7)$. (*Hint:* Every point on $y = 2x$ has the form $(x, 2x)$, and the closest point has the minimum distance.)

74. A quadratic equation $f(x) = 0$ has a solution $x = 2$. Its graph has vertex $(5, 3)$. What is the other solution of the equation?

Relating Concepts

For individual or collaborative investigation *(Exercises 75–78)*

In **Section 1.7** *we solved quadratic inequalities such as*

$$x^2 + 2x - 8 < 0,$$

by solving the related quadratic equation

$$x^2 + 2x - 8 = 0,$$

identifying intervals determined by the solutions of this equation, and then using a test value from each interval to determine which intervals form the solution set. **Work Exercises 75–78 in order** *to learn a graphical method of solving inequalities.*

75. Graph $f(x) = x^2 + 2x - 8$.

76. The real solutions of $x^2 + 2x - 8 = 0$ are the x-intercepts of the graph in **Exercise 75.** These are values of x for which $f(x) = 0$. What are the x-intercepts? What is the solution set of this equation?

77. The real solutions of $x^2 + 2x - 8 < 0$ are the x-values for which the graph in **Exercise 75** lies *below* the x-axis. These are values of x for which $f(x) < 0$. What interval of x-values represents the solution set of this inequality?

78. The real solutions of $x^2 + 2x - 8 > 0$ are the x-values for which the graph in **Exercise 75** lies *above* the x-axis. These are values of x for which $f(x) > 0$. What intervals of x-values represent the solution set of this inequality?

Use the technique described in Relating Concepts Exercises 75–78 to solve each inequality. Write the solution set in interval notation.

79. $x^2 - x - 6 < 0$

80. $x^2 - 9x + 20 < 0$

81. $2x^2 - 9x \geq 18$

82. $3x^2 + x \geq 4$

83. $-x^2 + 4x + 1 \geq 0$

84. $-x^2 + 2x + 6 > 0$

3.2 Synthetic Division

- Synthetic Division
- Evaluating Polynomial Functions Using the Remainder Theorem
- **Testing Potential Zeros**

A division problem can be written using multiplication, even when the division involves polynomials. The **division algorithm** illustrates this.

Division Algorithm

Let $f(x)$ and $g(x)$ be polynomials with $g(x)$ of lesser degree than $f(x)$ and $g(x)$ of degree 1 or more. There exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = g(x) \cdot q(x) + r(x),$$

where either $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $g(x)$.

For instance, we saw in **Example 11** of **Section R.3** that

$$\frac{3x^3 - 2x^2 - 150}{x^2 - 4} = 3x - 2 + \frac{12x - 158}{x^2 - 4}. \qquad \text{(Section R.3)}$$

We can express this result using the division algorithm.

$$\underbrace{3x^3 - 2x^2 - 150}_{\substack{f(x)\\ \text{Dividend}\\ \text{(original polynomial)}}} = \underbrace{(x^2 - 4)}_{\substack{g(x)\\ \text{Divisor}}} \underbrace{(3x - 2)}_{\substack{q(x)\\ \text{Quotient}}} + \underbrace{12x - 158}_{\substack{r(x)\\ \text{Remainder}}}$$

$$= \text{Divisor} \cdot \text{Quotient} + \text{Remainder}$$

Synthetic Division When a given polynomial in x is divided by a first-degree binomial of the form $x - k$, a shortcut method called **synthetic division** may be used. The example on the left below is simplified by omitting all variables and writing only coefficients, with 0 used to represent the coefficient of any missing terms. Since the coefficient of x in the divisor is always 1 in these divisions, it too can be omitted. These omissions simplify the problem, as shown on the right.

$$\begin{array}{rrrrr}
 & 3x^2 & +\,10x & +\,40 & \\ \cline{2-5}
x - 4) & 3x^3 & -\,2x^2 & +\,0x & -\,150 \\
 & 3x^3 & -\,12x^2 & & \\ \cline{2-3}
 & & 10x^2 & +\,0x & \\
 & & 10x^2 & -\,40x & \\ \cline{3-4}
 & & & 40x & -\,150 \\
 & & & 40x & -\,160 \\ \cline{4-5}
 & & & & 10
\end{array}
\qquad\qquad
\begin{array}{rrrrr}
 & 3 & 10 & 40 & \\ \cline{2-5}
-4) & 3 & -2 & 0 & -150 \\
 & 3 & -12 & & \\ \cline{2-3}
 & & 10 & 0 & \\
 & & 10 & -40 & \\ \cline{3-4}
 & & & 40 & -150 \\
 & & & 40 & -160 \\ \cline{4-5}
 & & & & 10
\end{array}$$

The numbers in color that are repetitions of the numbers directly above them can also be omitted, as shown on the left below.

$$\begin{array}{rrrrr}
 & 3 & 10 & 40 & \\ \cline{2-5}
-4) & 3 & -2 & 0 & -150 \\
 & & -12 & & \\ \cline{2-3}
 & & 10 & 0 & \\
 & & & -40 & \\ \cline{3-4}
 & & & 40 & -150 \\
 & & & & -160 \\ \cline{4-5}
 & & & & 10
\end{array}
\qquad\qquad
\begin{array}{rrrrr}
 & 3 & 10 & 40 & \\ \cline{2-5}
-4) & 3 & -2 & 0 & -150 \\
 & & -12 & & \\ \cline{2-3}
 & & 10 & & \\
 & & & -40 & \\ \cline{3-4}
 & & & 40 & \\
 & & & & -160 \\ \cline{4-5}
 & & & & 10
\end{array}$$

The numbers in color are again repetitions of those directly above them. They may be omitted, as shown on the right above.

The entire process can now be condensed vertically. The top row of numbers can be omitted since it duplicates the bottom row if the 3 is brought down.

$$\begin{array}{rrrrr}
-4) & 3 & -2 & 0 & -150 \\
 & & -12 & -40 & -160 \\ \cline{2-5}
 & 3 & 10 & 40 & 10
\end{array}$$

The rest of the bottom row is obtained by subtracting -12, -40, and -160 from the corresponding terms above them.

To simplify the arithmetic, we replace subtraction in the second row by addition and compensate by changing the -4 at the upper left to its additive inverse, 4.

$$\begin{array}{lrrrrrl}
\text{Additive inverse} \longrightarrow & 4) & 3 & -2 & 0 & -150 & \\
 & & & 12 & 40 & 160 & \longleftarrow \text{Signs changed} \\ \cline{3-6}
 & & 3 & 10 & 40 & 10 & \\
 & & \downarrow & \downarrow & \downarrow & \downarrow & \\
\text{Quotient} \longrightarrow & & 3x^2 & +\,10x & +\,40 & +\,\dfrac{10}{x-4} & \longleftarrow \text{Remainder}
\end{array}$$

Testing Potential Zeros A **zero** of a polynomial function $f(x)$ is a number k such that $f(k) = 0$. *The real number zeros are the x-intercepts of the graph of the function.*

The remainder theorem gives a quick way to decide whether a number k is a zero of a polynomial function defined by $f(x)$, as follows.

1. Use synthetic division to find $f(k)$.
2. If the remainder is 0, then $f(k) = 0$ and k is a zero of $f(x)$. If the remainder is not 0, then k is not a zero of $f(x)$.

A zero of $f(x)$ is a **root**, or **solution**, of the equation $f(x) = 0$.

EXAMPLE 3 Deciding Whether a Number Is a Zero

Decide whether the given number k is a zero of $f(x)$.

(a) $f(x) = x^3 - 4x^2 + 9x - 6$; $k = 1$

(b) $f(x) = x^4 + x^2 - 3x + 1$; $k = -1$

(c) $f(x) = x^4 - 2x^3 + 4x^2 + 2x - 5$; $k = 1 + 2i$

SOLUTION

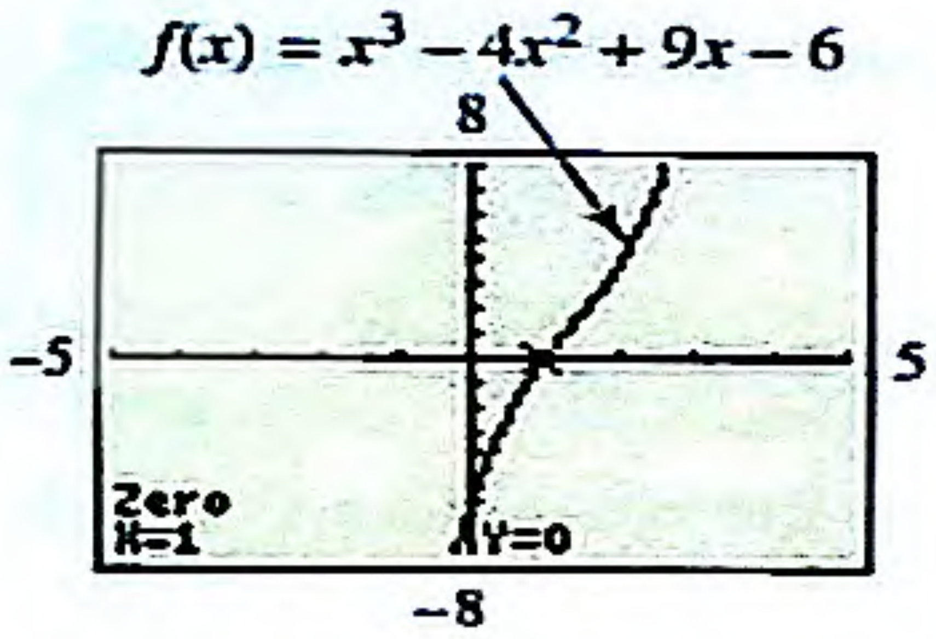


Figure 16

(a) Use synthetic division to decide whether 1 is a zero of $f(x) = x^3 - 4x^2 + 9x - 6$.

$$\text{Proposed zero} \longrightarrow 1\overline{)\begin{array}{rrrr} 1 & -4 & 9 & -6 \\ & 1 & -3 & 6 \\ \hline 1 & -3 & 6 & 0 \end{array}} \begin{array}{l} \longleftarrow f(x) = x^3 - 4x^2 + 9x - 6 \\ \\ \longleftarrow \text{Remainder} \end{array}$$

Since the remainder is 0, $f(1) = 0$, and 1 is a zero of the given polynomial function. An x-intercept of the graph of $f(x) = x^3 - 4x^2 + 9x - 6$ is 1, so the graph includes the point $(1, 0)$. The graph in **Figure 16** supports this.

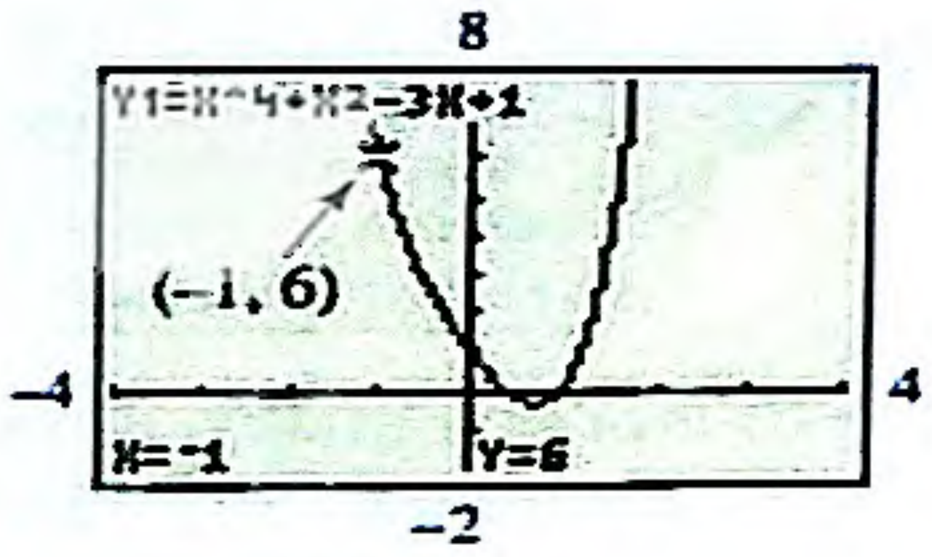


Figure 17

(b) For $f(x) = x^4 + x^2 - 3x + 1$, remember to use 0 as coefficient for the missing x^3-term in the synthetic division.

$$\text{Proposed zero} \longrightarrow -1\overline{)\begin{array}{rrrrr} 1 & 0 & 1 & -3 & 1 \\ & -1 & 1 & -2 & 5 \\ \hline 1 & -1 & 2 & -5 & 6 \end{array}} \begin{array}{l} \\ \\ \longleftarrow \text{Remainder} \end{array}$$

The remainder is not 0, so -1 is not a zero of $f(x) = x^4 + x^2 - 3x + 1$. In fact, $f(-1) = 6$, indicating that $(-1, 6)$ is on the graph of $f(x)$. The graph in **Figure 17** supports this.

(c) Use synthetic division and operations with complex numbers to determine whether $1 + 2i$ is a zero of $f(x) = x^4 - 2x^3 + 4x^2 + 2x - 5$.

$$1 + 2i\overline{)\begin{array}{rrrrr} 1 & -2 & 4 & 2 & -5 \\ & 1 + 2i & -5 & -1 - 2i & 5 \\ \hline 1 & -1 + 2i & -1 & 1 - 2i & 0 \end{array}} \begin{array}{l} \\ i^2 = -1 \text{ (Section 1.3)} \\ \longleftarrow \text{Remainder} \end{array}$$

$(1 + 2i)(-1 + 2i)$
$= -1 + 4i^2$
$= -5$

Since the remainder is 0, $1 + 2i$ is a zero of the given polynomial function. Notice that $1 + 2i$ is *not* a real number zero. Therefore, it cannot appear as an x-intercept on the graph of $f(x)$.

✔ *Now Try Exercises 43 and 55.*

3.2 Exercises

Use synthetic division to perform each division. See Example 1.

1. $\dfrac{x^3+3x^2+11x+9}{x+1}$

2. $\dfrac{x^3+7x^2+13x+6}{x+2}$

3. $\dfrac{5x^4+5x^3+2x^2-x-3}{x+1}$

4. $\dfrac{2x^4-x^3-7x^2+7x-10}{x-2}$

5. $\dfrac{x^4+4x^3+2x^2+9x+4}{x+4}$

6. $\dfrac{x^4+5x^3+4x^2-3x+9}{x+3}$

7. $\dfrac{x^5+3x^4+2x^3+2x^2+3x+1}{x+2}$

8. $\dfrac{x^6-3x^4+2x^3-6x^2-5x+3}{x+2}$

9. $\dfrac{-9x^3+8x^2-7x+2}{x-2}$

10. $\dfrac{-11x^4+2x^3-8x^2-4}{x+1}$

11. $\dfrac{\frac{1}{3}x^3-\frac{2}{9}x^2+\frac{2}{27}x-\frac{1}{81}}{x-\frac{1}{3}}$

12. $\dfrac{x^3+x^2+\frac{1}{2}x+\frac{1}{8}}{x+\frac{1}{2}}$

13. $\dfrac{x^4-3x^3-4x^2+12x}{x-2}$

14. $\dfrac{x^4-x^3-5x^2-3x}{x+1}$

15. $\dfrac{x^3-1}{x-1}$

16. $\dfrac{x^4-1}{x-1}$

17. $\dfrac{x^5+1}{x+1}$

18. $\dfrac{x^7+1}{x+1}$

Express $f(x)$ in the form $f(x)=(x-k)q(x)+r$ for the given value of k.

19. $f(x)=2x^3+x^2+x-8;\ k=-1$

20. $f(x)=2x^3+3x^2-16x+10;\ k=-4$

21. $f(x)=x^3+4x^2+5x+2;\ k=-2$

22. $f(x)=-x^3+x^2+3x-2;\ k=2$

23. $f(x)=4x^4-3x^3-20x^2-x;\ k=3$

24. $f(x)=2x^4+x^3-15x^2+3x;\ k=-3$

25. $f(x)=3x^4+4x^3-10x^2+15;\ k=-1$

26. $f(x)=-5x^4+x^3+2x^2+3x+1;\ k=1$

For each polynomial function, use the remainder theorem and synthetic division to find $f(k)$. See Example 2.

27. $f(x)=x^2+5x+6;\ k=-2$

28. $f(x)=x^2-4x-5;\ k=5$

29. $f(x)=2x^2-3x-3;\ k=2$

30. $f(x)=-x^3+8x^2+63;\ k=4$

31. $f(x)=x^3-4x^2+2x+1;\ k=-1$

32. $f(x)=2x^3-3x^2-5x+4;\ k=2$

33. $f(x)=2x^5-10x^3-19x^2-50;\ k=3$

34. $f(x)=x^4+6x^3+9x^2+3x-3;\ k=4$

35. $f(x)=6x^4+x^3-8x^2+5x+6;\ k=\frac{1}{2}$

36. $f(x)=6x^3-31x^2-15x;\ k=-\frac{1}{2}$

37. $f(x)=x^2-5x+1;\ k=2+i$

38. $f(x)=x^2-x+3;\ k=3-2i$

39. $f(x)=x^2+4;\ k=2i$

40. $f(x)=2x^2+10;\ k=i\sqrt{5}$

Use synthetic division to decide whether the given number k is a zero of the given polynomial function. If it is not, give the value of $f(k)$. See Examples 2 and 3.

41. $f(x) = x^2 + 2x - 8;\ k = 2$

42. $f(x) = x^2 + 4x - 5;\ k = -5$

43. $f(x) = x^3 - 3x^2 + 4x - 4;\ k = 2$

44. $f(x) = x^3 + 2x^2 - x + 6;\ k = -3$

45. $f(x) = 2x^3 - 6x^2 - 9x + 4;\ k = 1$

46. $f(x) = 2x^3 + 9x^2 - 16x + 12;\ k = 1$

47. $f(x) = x^3 + 7x^2 + 10x;\ k = 0$

48. $f(x) = 2x^3 - 3x^2 - 5x;\ k = 0$

49. $f(x) = 4x^4 + x^2 + 17x + 3;\ k = -\frac{3}{2}$

50. $f(x) = 3x^4 + 13x^3 - 10x + 8;\ k = -\frac{4}{3}$

51. $f(x) = 5x^4 + 2x^3 - x + 3;\ k = \frac{2}{5}$

52. $f(x) = 16x^4 + 3x^2 - 2;\ k = \frac{1}{2}$

53. $f(x) = x^2 - 2x + 2;\ k = 1 - i$

54. $f(x) = x^2 - 4x + 5;\ k = 2 - i$

55. $f(x) = x^2 + 3x + 4;\ k = 2 + i$

56. $f(x) = x^2 - 3x + 5;\ k = 1 - 2i$

57. $f(x) = x^3 + 3x^2 - x + 1;\ k = 1 + i$

58. $f(x) = 2x^3 - x^2 + 3x - 5;\ k = 2 - i$

Relating Concepts

For individual or collaborative investigation *(Exercises 59–68)*

The remainder theorem indicates that when a polynomial $f(x)$ is divided by $x - k$, the remainder is equal to $f(k)$. For

$$f(x) = x^3 - 2x^2 - x + 2,$$

use the remainder theorem to find each of the following. Then determine the coordinates of the corresponding point on the graph of $f(x)$.

59. $f(-2)$

60. $f(-1)$

61. $f\left(-\frac{1}{2}\right)$

62. $f(0)$

63. $f(1)$

64. $f\left(\frac{3}{2}\right)$

65. $f(2)$

66. $f(3)$

67. Use the results from **Exercises 59–66** to plot eight points on the graph of $f(x)$. Connect these points with a smooth curve. Describe a method for graphing polynomial functions using the remainder theorem.

68. Use the method described in **Exercises 59–67** to sketch the graph of $f(x) = -x^3 - x^2 + 2x$.

3.3 Zeros of Polynomial Functions

- Factor Theorem
- Rational Zeros Theorem
- Number of Zeros
- Conjugate Zeros Theorem
- Finding Zeros of a Polynomial Function
- Descartes' Rule of Signs

Factor Theorem Consider the polynomial function

$$f(x) = x^2 + x - 2,$$

which is written in factored form as

$$f(x) = (x - 1)(x + 2). \quad \text{(Section R.4)}$$

For this function, $f(1) = 0$ and $f(-2) = 0$, and thus 1 and -2 are zeros of $f(x)$. Notice the special relationship between each linear factor and its corresponding zero. The **factor theorem** summarizes this relationship.

Factor Theorem

For any polynomial function $f(x)$, $x - k$ is a factor of the polynomial if and only if $f(k) = 0$.

EXAMPLE 1 **Deciding Whether $x - k$ Is a Factor**

Determine whether $x - 1$ is a factor of each polynomial.

(a) $f(x) = 2x^4 + 3x^2 - 5x + 7$

(b) $f(x) = 3x^5 - 2x^4 + x^3 - 8x^2 + 5x + 1$

SOLUTION

(a) By the factor theorem, $x - 1$ will be a factor if $f(1) = 0$. Use synthetic division and the remainder theorem to decide.

Use a zero coefficient for the missing term.

$$\begin{array}{r|rrrrr} 1 & 2 & 0 & 3 & -5 & 7 \\ & & 2 & 2 & 5 & 0 \\ \hline & 2 & 2 & 5 & 0 & 7 \leftarrow f(1) = 7 \end{array} \quad \text{(Section 3.2)}$$

The remainder is 7 and not 0, so $x - 1$ is not a factor of $2x^4 + 3x^2 - 5x + 7$.

(b)

$$\begin{array}{r|rrrrrr} 1 & 3 & -2 & 1 & -8 & 5 & 1 \\ & & 3 & 1 & 2 & -6 & -1 \\ \hline & 3 & 1 & 2 & -6 & -1 & 0 \leftarrow f(1) = 0 \end{array}$$

Because the remainder is 0, $x - 1$ is a factor. Additionally, we can determine from the coefficients in the bottom row that the other factor is

$$3x^4 + x^3 + 2x^2 - 6x - 1.$$

Thus, we can express the polynomial in factored form.

$$f(x) = (x - 1)(3x^4 + x^3 + 2x^2 - 6x - 1)$$

✓ *Now Try Exercises 5 and 7.*

We can use the factor theorem to factor a polynomial of greater degree into linear factors of the form $ax - b$.

EXAMPLE 2 **Factoring a Polynomial Given a Zero**

Factor $f(x) = 6x^3 + 19x^2 + 2x - 3$ into linear factors if -3 is a zero of f.

SOLUTION Since -3 is a zero of f, $x - (-3) = x + 3$ is a factor.

$$\begin{array}{r|rrrr} -3 & 6 & 19 & 2 & -3 \\ & & -18 & -3 & 3 \\ \hline & 6 & 1 & -1 & 0 \end{array} \quad \text{Use synthetic division to divide } f(x) \text{ by } x + 3.$$

The quotient is $6x^2 + x - 1$, which is the factor that accompanies $x + 3$.

$$f(x) = (x + 3)(6x^2 + x - 1)$$

$$f(x) = (x + 3)(2x + 1)(3x - 1) \quad \text{Factor } 6x^2 + x - 1$$

These factors are all linear.

✓ *Now Try Exercise 17.*

EXAMPLE 7 Applying Descartes' Rule of Signs

Determine the different possibilities for the numbers of positive, negative, and nonreal complex zeros of

$$f(x) = x^4 - 6x^3 + 8x^2 + 2x - 1.$$

SOLUTION We first consider the possible number of positive zeros by observing that $f(x)$ has three variations in signs.

$$f(x) = \underbrace{+x^4 \;-}_{1}\underbrace{\;6x^3 \;+}_{2}\; 8x^2 \underbrace{+\;2x \;-}_{3}\; 1$$

Thus, by Descartes' rule of signs, $f(x)$ has either three or one (since $3 - 2 = 1$) positive real zeros.

For negative zeros, consider the variations in signs for $f(-x)$.

$$f(-x) = (-x)^4 - 6(-x)^3 + 8(-x)^2 + 2(-x) - 1$$
$$= x^4 + 6x^3 \underbrace{+\;8x^2 \;-}_{1}\; 2x - 1$$

Since there is only one variation in sign, $f(x)$ has exactly one negative real zero.

Because $f(x)$ is a fourth-degree polynomial function, it must have four complex zeros, some of which may be repeated. Descartes' rule of signs has indicated that exactly one of these zeros is a negative real number.

- One possible combination of the zeros is one negative real zero, three positive real zeros, and no nonreal complex zeros.
- Another possible combination of the zeros is one negative real zero, one positive real zero, and two nonreal complex zeros.

By the conjugate zeros theorem, any possible nonreal complex zeros must occur in conjugate pairs since $f(x)$ has real coefficients. The table below summarizes these possibilities.

Possible Number of Zeros		
Positive	Negative	Nonreal Complex
3	1	0
1	1	2

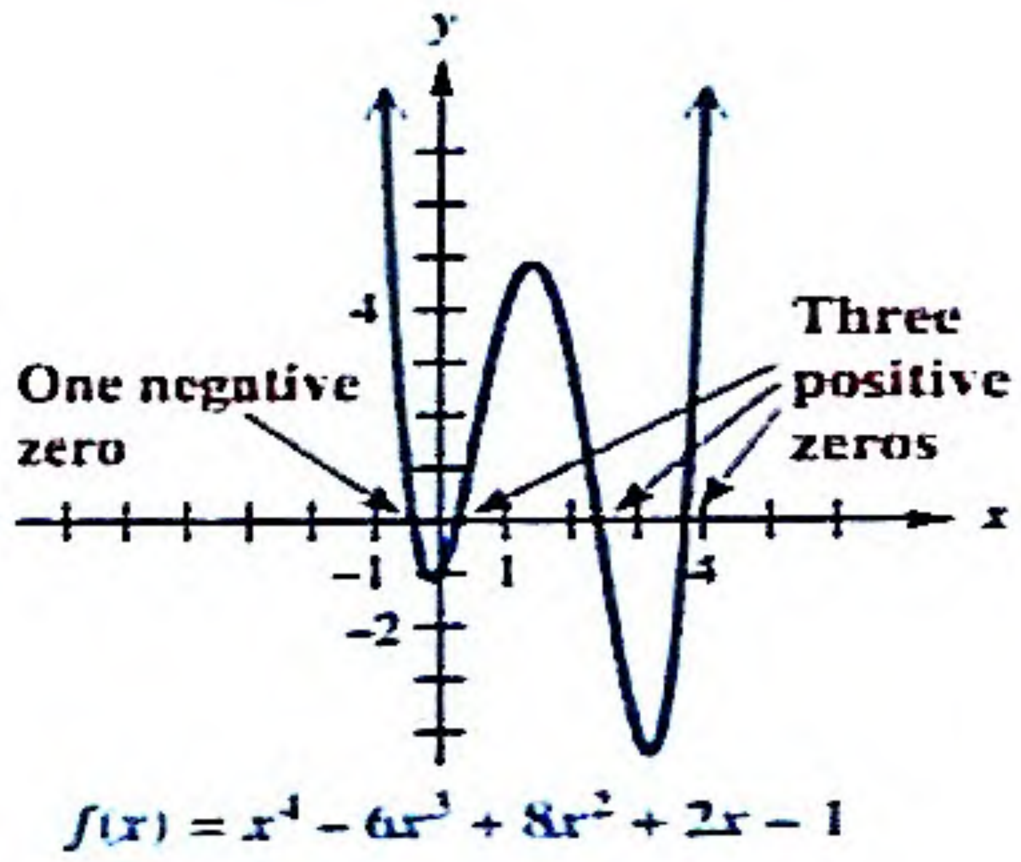


Figure 19

The graph of $f(x)$ in **Figure 19** verifies the correct combination of three positive real zeros with one negative real zero, as seen in the first row of the table.*

✓ *Now Try Exercise 75.*

NOTE ***Descartes' rule of signs does not identify the multiplicity of the zeros of a function.*** For example, if it indicates that a function $f(x)$ has exactly two positive real zeros, then $f(x)$ may have two distinct positive real zeros or one positive real zero of multiplicity 2.

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* The authors would like to thank Mary Hill of College of DuPage for her input into **Example 7.**

3.3 Exercises

Concept Check *Decide whether each statement is true or false. If false, tell why.*

1. Since $x - 1$ is a factor of $f(x) = x^6 - x^4 + 2x^2 - 2$, we can conclude that $f(1) = 0$.
2. Since $f(1) = 0$ for $f(x) = x^6 - x^4 + 2x^2 - 2$, we can conclude that $x - 1$ is a factor of $f(x)$.
3. For $f(x) = (x + 2)^4(x - 3)$, 2 is a zero of multiplicity 4.
4. Since $2 + 3i$ is a zero of $f(x) = x^2 - 4x + 13$, we can conclude that $2 - 3i$ is also a zero.

Use the factor theorem and synthetic division to decide whether the second polynomial is a factor of the first. See Example 1.

5. $x^3 - 5x^2 + 3x + 1;\ x - 1$
6. $x^3 + 6x^2 - 2x - 7;\ x + 1$
7. $2x^4 + 5x^3 - 8x^2 + 3x + 13;\ x + 1$
8. $-3x^4 + x^3 - 5x^2 + 2x + 4;\ x - 1$
9. $-x^3 + 3x - 2;\ x + 2$
10. $-2x^3 + x^2 - 63;\ x + 3$
11. $4x^2 + 2x + 54;\ x - 4$
12. $5x^2 - 14x + 10;\ x + 2$
13. $x^3 + 2x^2 + 3;\ x - 1$
14. $2x^3 + x + 2;\ x + 1$
15. $2x^4 + 5x^3 - 2x^2 + 5x + 6;\ x + 3$
16. $5x^4 + 16x^3 - 15x^2 + 8x + 16;\ x + 4$

Factor $f(x)$ into linear factors given that k is a zero of $f(x)$. See Example 2.

17. $f(x) = 2x^3 - 3x^2 - 17x + 30;\ k = 2$
18. $f(x) = 2x^3 - 3x^2 - 5x + 6;\ k = 1$
19. $f(x) = 6x^3 + 13x^2 - 14x + 3;\ k = -3$
20. $f(x) = 6x^3 + 17x^2 - 63x + 10;\ k = -5$
21. $f(x) = 6x^3 + 25x^2 + 3x - 4;\ k = -4$
22. $f(x) = 8x^3 + 50x^2 + 47x - 15;\ k = -5$
23. $f(x) = x^3 + (7 - 3i)x^2 + (12 - 21i)x - 36i;\ k = 3i$
24. $f(x) = 2x^3 + (3 + 2i)x^2 + (1 + 3i)x + i;\ k = -i$
25. $f(x) = 2x^3 + (3 - 2i)x^2 + (-8 - 5i)x + (3 + 3i);\ k = 1 + i$
26. $f(x) = 6x^3 + (19 - 6i)x^2 + (16 - 7i)x + (4 - 2i);\ k = -2 + i$
27. $f(x) = x^4 + 2x^3 - 7x^2 - 20x - 12;\ k = -2$ (multiplicity 2)
28. $f(x) = 2x^4 + x^3 - 9x^2 - 13x - 5;\ k = -1$ (multiplicity 3)

For each polynomial function, one zero is given. Find all others. See Examples 2 and 6.

29. $f(x) = x^3 - x^2 - 4x - 6;\ 3$
30. $f(x) = x^3 + 4x^2 - 5;\ 1$
31. $f(x) = x^3 - 7x^2 + 17x - 15;\ 2 - i$
32. $f(x) = 4x^3 + 6x^2 - 2x - 1;\ \frac{1}{2}$
33. $f(x) = x^4 + 5x^2 + 4;\ -i$
34. $f(x) = x^4 + 26x^2 + 25;\ i$

For each polynomial function, (a) list all possible rational zeros, (b) find all rational zeros, and (c) factor $f(x)$. See Example 3.

35. $f(x) = x^3 - 2x^2 - 13x - 10$
36. $f(x) = x^3 + 5x^2 + 2x - 8$
37. $f(x) = x^3 + 6x^2 - x - 30$
38. $f(x) = x^3 - x^2 - 10x - 8$

39. $f(x) = 6x^3 + 17x^2 - 31x - 12$

40. $f(x) = 15x^3 + 61x^2 + 2x - 8$

41. $f(x) = 24x^3 + 40x^2 - 2x - 12$

42. $f(x) = 24x^3 + 80x^2 + 82x + 24$

For each polynomial function, find all zeros and their multiplicities.

43. $f(x) = (x - 2)^3(x^2 - 7)$

44. $f(x) = (x + 1)^2(x - 1)^3(x^2 - 10)$

45. $f(x) = 3x(x - 2)(x + 3)(x^2 - 1)$

46. $f(x) = 5x^2(x^2 - 16)(x + 5)$

47. $f(x) = (x^2 + x - 2)^5\left(x - 1 + \sqrt{3}\right)^2$

48. $f(x) = (2x^2 - 7x + 3)^3\left(x - 2 - \sqrt{5}\right)$

Find a polynomial function $f(x)$ of degree 3 with real coefficients that satisfies the given conditions. See Example 4.

49. Zeros of -3, 1, and 4; $f(2) = 30$

50. Zeros of 1, -1, and 0; $f(2) = 3$

51. Zeros of -2, 1, and 0; $f(-1) = -1$

52. Zeros of 2, -3, and 5; $f(3) = 6$

53. Zero of -3 having multiplicity 3; $f(3) = 36$

54. Zero of 2 and zero of 4 having multiplicity 2; $f(1) = -18$

55. Zero of 0 and zero of 1 having multiplicity 2; $f(2) = 10$

56. Zero of -4 and zero of 0 having multiplicity 2; $f(-1) = -6$

Find a polynomial function $f(x)$ of least degree having only real coefficients with zeros as given. See Examples 4–6.

57. $5 + i$ and $5 - i$

58. $7 - 2i$ and $7 + 2i$

59. 0, i, and $1 + i$

60. 0, $-i$, and $2 + i$

61. $1 + \sqrt{2}$, $1 - \sqrt{2}$, and 1

62. $1 - \sqrt{3}$, $1 + \sqrt{3}$, and 1

63. $2 - i$, 3, and -1

64. $3 + 2i$, -1, and 2

65. 2 and $3 + i$

66. -1 and $4 - 2i$

67. $1 - \sqrt{2}$, $1 + \sqrt{2}$, and $1 - i$

68. $2 + \sqrt{3}$, $2 - \sqrt{3}$, and $2 + 3i$

69. $2 - i$ and $6 - 3i$

70. $5 + i$ and $4 - i$

71. 4, $1 - 2i$, and $3 + 4i$

72. -1, $5 - i$, and $1 + 4i$

73. $1 + 2i$ and 2 (multiplicity 2)

74. $2 + i$ and -3 (multiplicity 2)

Use Descartes' rule of signs to determine the different possibilities for the numbers of positive, negative, and nonreal complex zeros for each function. See Example 7.

75. $f(x) = 2x^3 - 4x^2 + 2x + 7$

76. $f(x) = x^3 + 2x^2 + x - 10$

77. $f(x) = 4x^3 - x^2 + 2x - 7$

78. $f(x) = 3x^3 + 6x^2 + x + 7$

79. $f(x) = 5x^4 + 3x^2 + 2x - 9$

80. $f(x) = 3x^4 + 2x^3 - 8x^2 - 10x - 1$

81. $f(x) = -8x^4 + 3x^3 - 6x^2 + 5x - 7$

82. $f(x) = 6x^4 + 2x^3 + 9x^2 + x + 5$

83. $f(x) = x^5 + 3x^4 - x^3 + 2x + 3$

84. $f(x) = 2x^5 - x^4 + x^3 - x^2 + x + 5$

85. $f(x) = 7x^5 + 6x^4 + 2x^3 + 9x^2 + x + 5$

86. $f(x) = -2x^5 + 10x^4 - 6x^3 + 8x^2 - x + 1$

87. $f(x) = 2x^5 - 7x^3 + 6x + 8$

88. $f(x) = 11x^5 - x^3 + 7x - 5$

89. $f(x) = 5x^6 - 6x^5 + 7x^3 - 4x^2 + x + 2$

90. $f(x) = 9x^6 - 7x^4 + 8x^2 + x + 6$

*Find all complex zeros of each polynomial function. Give exact values. List multiple zeros as necessary.**

91. $f(x) = x^4 + 2x^3 - 3x^2 + 24x - 180$

92. $f(x) = x^3 - x^2 - 8x + 12$

93. $f(x) = x^4 + x^3 - 9x^2 + 11x - 4$

94. $f(x) = x^3 - 14x + 8$

95. $f(x) = 2x^5 + 11x^4 + 16x^3 + 15x^2 + 36x$

96. $f(x) = 3x^3 - 9x^2 - 31x + 5$

97. $f(x) = x^5 - 6x^4 + 14x^3 - 20x^2 + 24x - 16$

98. $f(x) = 9x^4 + 30x^3 + 241x^2 + 720x + 600$

99. $f(x) = 2x^4 - x^3 + 7x^2 - 4x - 4$

100. $f(x) = 32x^4 - 188x^3 + 261x^2 + 54x - 27$

101. $f(x) = 5x^3 - 9x^2 + 28x + 6$

102. $f(x) = 4x^3 + 3x^2 + 8x + 6$

103. $f(x) = x^4 + 29x^2 + 100$

104. $f(x) = x^4 + 4x^3 + 6x^2 + 4x + 1$

105. $f(x) = x^4 + 2x^2 + 1$

106. $f(x) = x^4 - 8x^3 + 24x^2 - 32x + 16$

107. $f(x) = x^4 - 6x^3 + 7x^2$

108. $f(x) = 4x^4 - 65x^2 + 16$

109. $f(x) = x^4 - 8x^3 + 29x^2 - 66x + 72$

110. $f(x) = 12x^4 - 43x^3 + 50x^2 + 38x - 12$

111. $f(x) = x^6 - 9x^4 - 16x^2 + 144$

112. $f(x) = x^6 - x^5 - 26x^4 + 44x^3 + 91x^2 - 139x + 30$

If c and d are complex numbers, prove each statement. (Hint: Let $c = a + bi$ *and* $d = m + ni$ *and form all the conjugates, the sums, and the products.)*

113. $\overline{c + d} = \overline{c} + \overline{d}$

114. $\overline{c \cdot d} = \overline{c} \cdot \overline{d}$

115. $\overline{a} = a$ for any real number a

116. $\overline{c^2} = (\overline{c})^2$

In 1545, a method of solving a cubic equation of the form

$$x^3 + mx = n,$$

developed by Niccolo Tartaglia, was published in the Ars Magna, *a work by Girolamo Cardano. The formula for finding the one real solution of the equation is*

$$x = \sqrt[3]{\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}} - \sqrt[3]{\frac{-n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}}.$$

(Source: Gullberg, J., *Mathematics from the Birth of Numbers*, W.W. Norton & Company.) *Use the formula to solve each equation for the one real solution.*

117. $x^3 + 9x = 26$

118. $x^3 + 15x = 124$

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* The authors would like to thank Aileen Solomon of Trident Technical College for preparing and suggesting the inclusion of **Exercises 91–104.**

3.4 Exercises

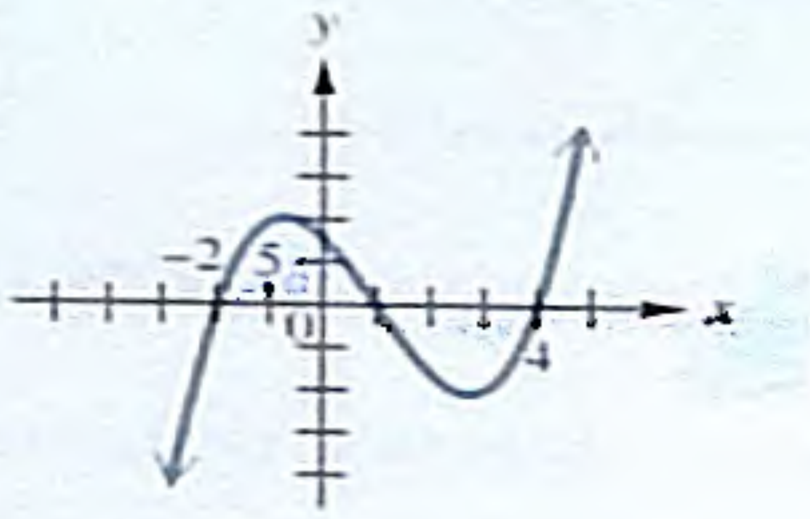
Concept Check Comprehensive graphs of four polynomial functions are shown in **A–D.** *They represent the graphs of functions defined by these four equations, but not necessarily in the order listed.*

$$y = x^3 - 3x^2 - 6x + 8 \qquad y = x^4 + 7x^3 - 5x^2 - 75x$$

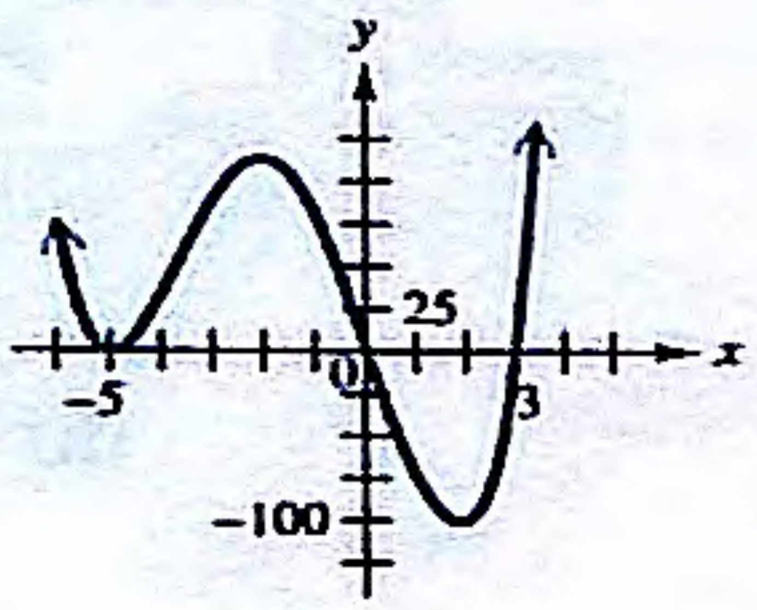
$$y = -x^3 + 9x^2 - 27x + 17 \qquad y = -x^5 + 36x^3 - 22x^2 - 147x - 90$$

Apply the concepts of this section to work Exercises 1–8.

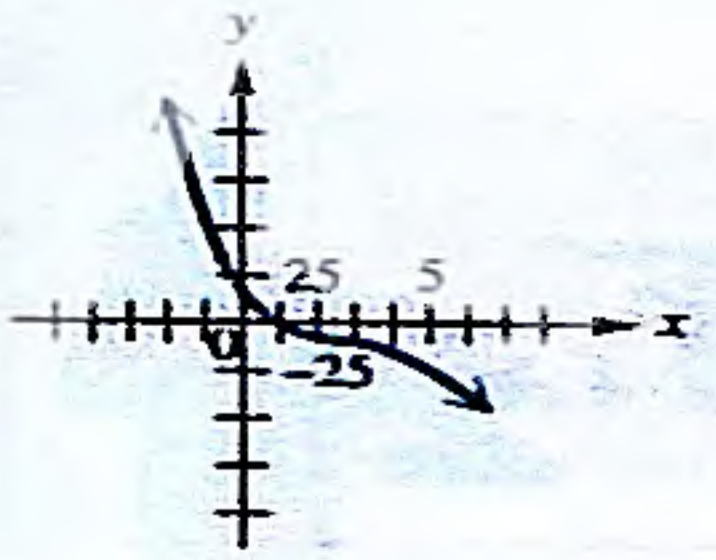
A.



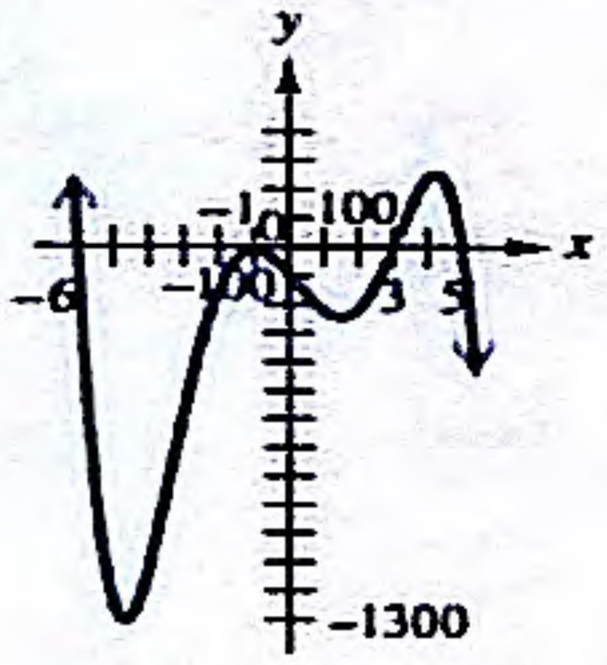
B.



C.



D.



1. Which one of the graphs is that of $y = x^3 - 3x^2 - 6x + 8$?
2. Which one of the graphs is that of $y = x^4 + 7x^3 - 5x^2 - 75x$?
3. How many real zeros does the function graphed in C have?
4. Which one of C and D is the graph of $y = -x^3 + 9x^2 - 27x + 17$?
5. Which of the graphs cannot be that of a cubic polynomial function?
6. Which one of the graphs is that of a function whose range is *not* $(-\infty, \infty)$?
7. The function $f(x) = x^4 + 7x^3 - 5x^2 - 75x$ has the graph shown in B. Use the graph to factor the polynomial.
8. The function $f(x) = -x^5 + 36x^3 - 22x^2 - 147x - 90$ has the graph shown in D. Use the graph to factor the polynomial.

Sketch the graph of each polynomial function. Determine the intervals of the domain for which each function is (a) increasing or (b) decreasing. See Example 1.

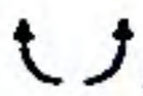
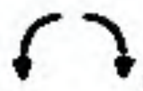
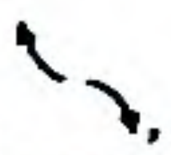
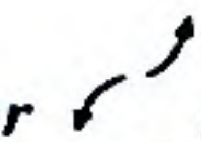
9. $f(x) = 2x^4$
10. $f(x) = \frac{1}{4}x^6$
11. $f(x) = -\frac{2}{3}x^5$
12. $f(x) = -\frac{5}{4}x^5$
13. $f(x) = \frac{1}{2}x^3 + 1$
14. $f(x) = -x^4 + 2$
15. $f(x) = -(x + 1)^3 + 1$
16. $f(x) = (x + 2)^3 - 1$

17. $f(x) = (x - 1)^4 + 2$

18. $f(x) = \frac{1}{3}(x + 3)^4 - 3$

19. $f(x) = \frac{1}{2}(x - 2)^2 + 4$

20. $f(x) = \frac{1}{3}(x + 1)^3 - 3$

Use an end behavior diagram, , , , *or* , *to describe the end behavior of the graph of each polynomial function.* ***See Example 2.***

21. $f(x) = 5x^5 + 2x^3 - 3x + 4$

22. $f(x) = -x^3 - 4x^2 + 2x - 1$

23. $f(x) = -4x^3 + 3x^2 - 1$

24. $f(x) = 4x^7 - x^5 + x^3 - 1$

25. $f(x) = 9x^6 - 3x^4 + x^2 - 2$

26. $f(x) = 10x^6 - x^5 + 2x - 2$

27. $f(x) = 3 + 2x - 4x^2 - 5x^{10}$

28. $f(x) = 7 + 2x - 5x^2 - 10x^4$

Graph each polynomial function. Factor first if the expression is not in factored form. ***See Examples 3 and 4.***

29. $f(x) = x^3 + 5x^2 + 2x - 8$

30. $f(x) = x^3 + 3x^2 - 13x - 15$

31. $f(x) = 2x(x - 3)(x + 2)$

32. $f(x) = x(x + 1)(x - 1)$

33. $f(x) = x^2(x - 2)(x + 3)^2$

34. $f(x) = x^2(x - 5)(x + 3)(x - 1)$

35. $f(x) = (3x - 1)(x + 2)^2$

36. $f(x) = (4x + 3)(x + 2)^2$

37. $f(x) = x^3 + 5x^2 - x - 5$

38. $f(x) = x^3 + x^2 - 36x - 36$

39. $f(x) = x^3 - x^2 - 2x$

40. $f(x) = 3x^4 + 5x^3 - 2x^2$

41. $f(x) = 2x^3(x^2 - 4)(x - 1)$

42. $f(x) = x^2(x - 3)^3(x + 1)$

43. $f(x) = 2x^3 - 5x^2 - x + 6$

44. $f(x) = 2x^4 + x^3 - 6x^2 - 7x - 2$

45. $f(x) = 3x^4 - 7x^3 - 6x^2 + 12x + 8$

46. $f(x) = x^4 + 3x^3 - 3x^2 - 11x - 6$

Use the intermediate value theorem for polynomials to show that each polynomial function has a real zero between the numbers given. ***See Example 5.***

47. $f(x) = 2x^2 - 7x + 4$; 2 and 3

48. $f(x) = 3x^2 - x - 4$; 1 and 2

49. $f(x) = 2x^3 - 5x^2 - 5x + 7$; 0 and 1

50. $f(x) = 2x^3 - 9x^2 + x + 20$; 2 and 2.5

51. $f(x) = 2x^4 - 4x^2 + 4x - 8$; 1 and 2

52. $f(x) = x^4 - 4x^3 - x + 3$; 0.5 and 1

53. $f(x) = x^4 + x^3 - 6x^2 - 20x - 16$; 3.2 and 3.3

54. $f(x) = x^4 - 2x^3 - 2x^2 - 18x + 5$; 3.7 and 3.8

55. $f(x) = x^4 - 4x^3 - 20x^2 + 32x + 12$; -1 and 0

56. $f(x) = x^5 + 2x^4 + x^3 + 3$; -1.8 and -1.7

Show that the real zeros of each polynomial function satisfy the given conditions. ***See Example 6.***

57. $f(x) = x^4 - x^3 + 3x^2 - 8x + 8$; no real zero greater than 2

58. $f(x) = 2x^5 - x^4 + 2x^3 - 2x^2 + 4x - 4$; no real zero greater than 1

59. $f(x) = x^4 + x^3 - x^2 + 3$; no real zero less than -2

60. $f(x) = x^5 + 2x^3 - 2x^2 + 5x + 5$; no real zero less than -1

61. $f(x) = 3x^4 + 2x^3 - 4x^2 + x - 1$; no real zero greater than 1

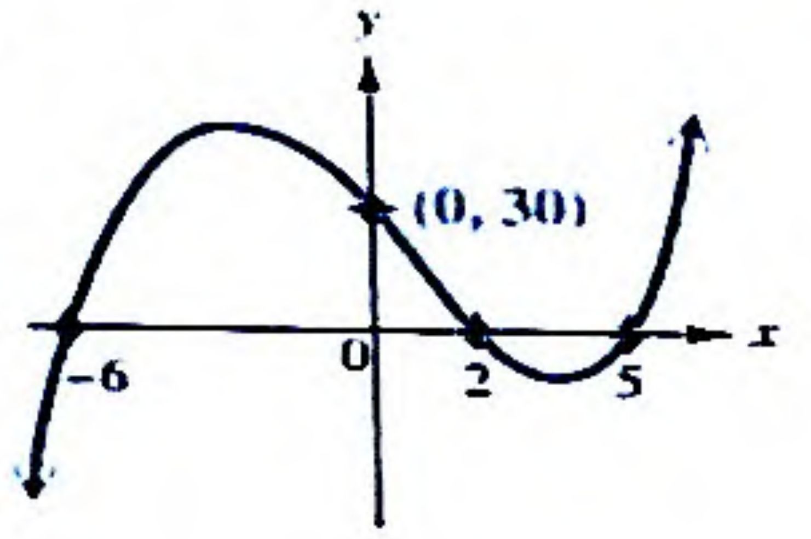
62. $f(x) = 3x^4 + 2x^3 - 4x^2 + x - 1$; no real zero less than -2

63. $f(x) = x^5 - 3x^3 + x + 2$; no real zero greater than 2

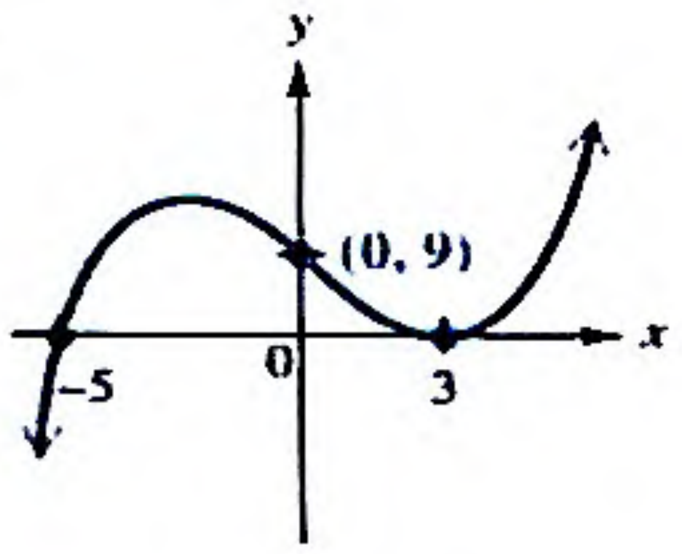
64. $f(x) = x^5 - 3x^3 + x + 2$; no real zero less than -3

Connecting Graphs with Equations *In Exercises 65–70, find a polynomial function $f(x)$ of least possible degree having the graph shown. (Hint: See the Note following Example 4.)*

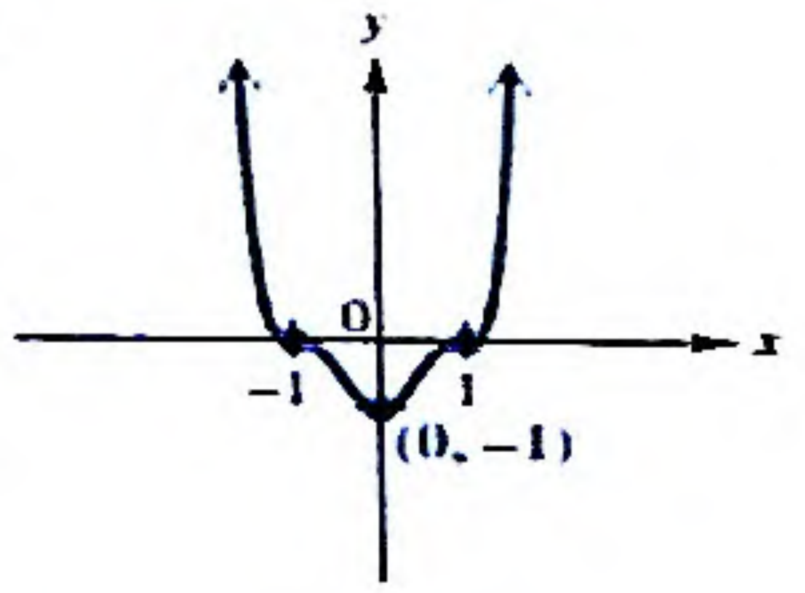
65.



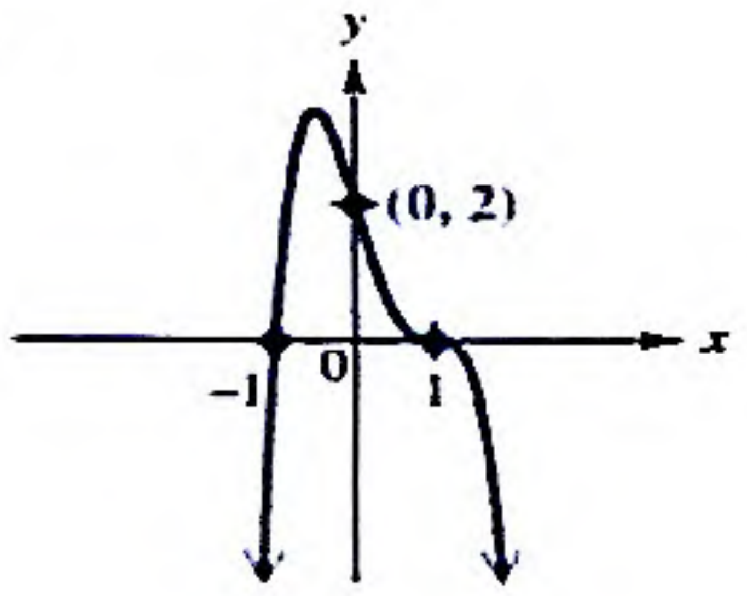
66.



67.



68.



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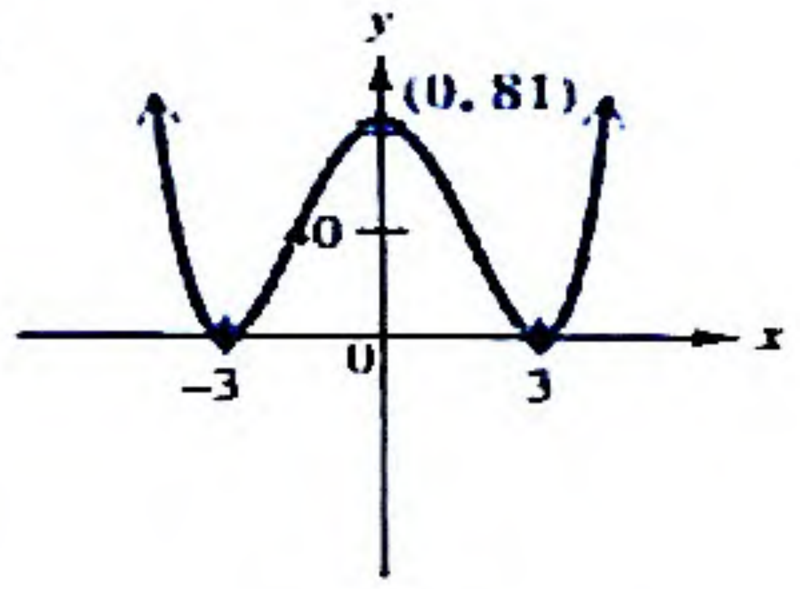
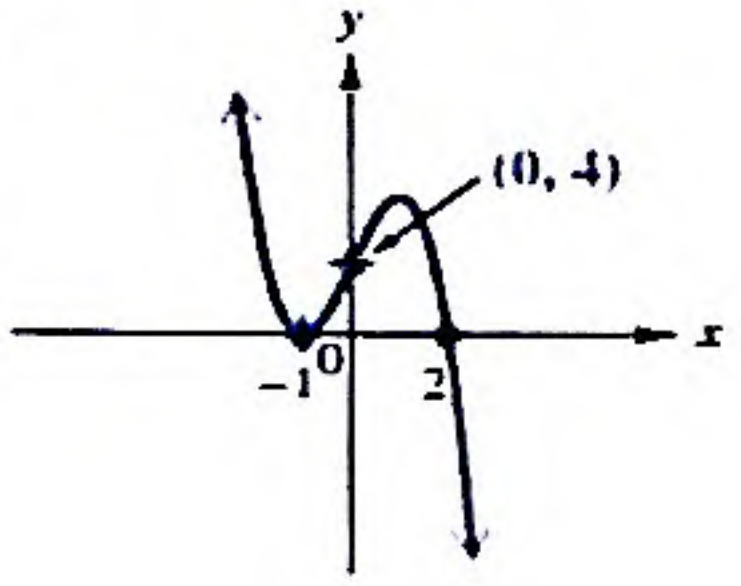


70.



Use a graphing calculator to graph the function in the viewing window specified. Compare the graph to the one shown in the answer section of this text. Then use the graph to find $f(1.25)$.

71. $f(x) = 2x(x - 3)(x + 2)$; window: $[-3, 4]$ by $[-20, 12]$
Compare to **Exercise 31.**

72. $f(x) = x^2(x - 2)(x + 3)^2$; window: $[-4, 3]$ by $[-24, 4]$
Compare to **Exercise 33.**

73. $f(x) = (3x - 1)(x + 2)^2$; window: $[-4, 2]$ by $[-15, 15]$
Compare to **Exercise 35.**

74. $f(x) = x^3 + 5x^2 - x - 5$; window: $[-6, 2]$ by $[-30, 30]$
Compare to **Exercise 37.**

Use a graphing calculator to approximate the real zero discussed in each specified exercise. See Example 7.

75. Exercise 47

76. Exercise 49

77. Exercise 51

78. Exercise 50

For the given polynomial function, approximate each zero as a decimal to the nearest tenth. See Example 7.

79. $f(x) = x^3 + 3x^2 - 2x - 6$

80. $f(x) = x^3 - 3x + 3$

81. $f(x) = -2x^4 - x^2 + x + 5$

82. $f(x) = -x^4 + 2x^3 + 3x^2 + 6$

Use a graphing calculator to find the coordinates of the turning points of the graph of each polynomial function in the given domain interval. Give answers to the nearest hundredth.

83. $f(x) = x^3 + 4x^2 - 8x - 8;\ [-3.8, -3]$

84. $f(x) = x^3 + 4x^2 - 8x - 8;\ [0.3, 1]$

85. $f(x) = 2x^3 - 5x^2 - x + 1;\ [-1, 0]$

86. $f(x) = 2x^3 - 5x^2 - x + 1;\ [1.4, 2]$

87. $f(x) = x^4 - 7x^3 + 13x^2 + 6x - 28;\ [-1, 0]$

88. $f(x) = x^3 - x + 3;\ [-1, 0]$

Solve each problem.

89. *(Modeling) Social Security Numbers* Your Social Security number (SSN) is unique, and with it you can construct your own personal Social Security polynomial. Let the polynomial function be defined as follows, where a_i represents the ith digit in your SSN:

$$\text{SSN}(x) = (x - a_1)(x + a_2)(x - a_3)(x + a_4)(x - a_5) \cdot (x + a_6)(x - a_7)(x + a_8)(x - a_9).$$

For example, if the SSN is 539-58-0954, the polynomial function is

$$\text{SSN}(x) = (x - 5)(x + 3)(x - 9)(x + 5)(x - 8)(x + 0)(x - 9)(x + 5)(x - 4).$$

A comprehensive graph of this function is shown in **Figure A.** In **Figure B,** we show a screen obtained by zooming in on the positive zeros, as the comprehensive graph does not show the local behavior well in this region. Use a graphing calculator to graph your own "personal polynomial."

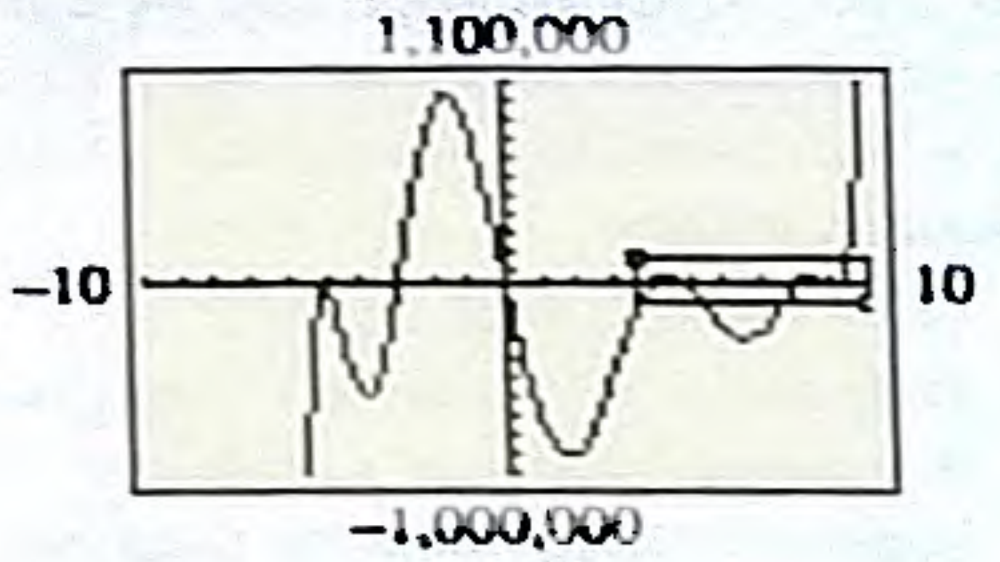


Figure A

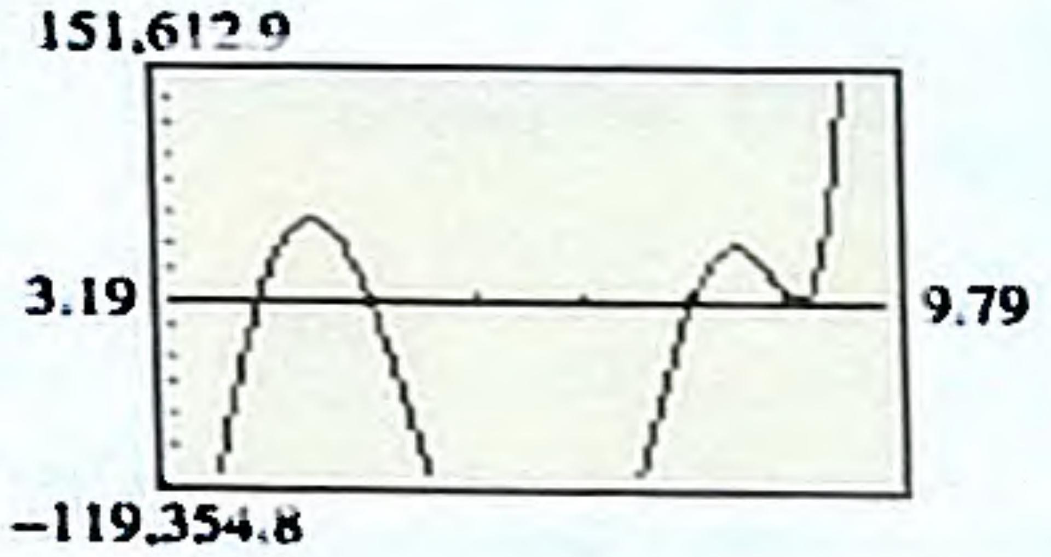
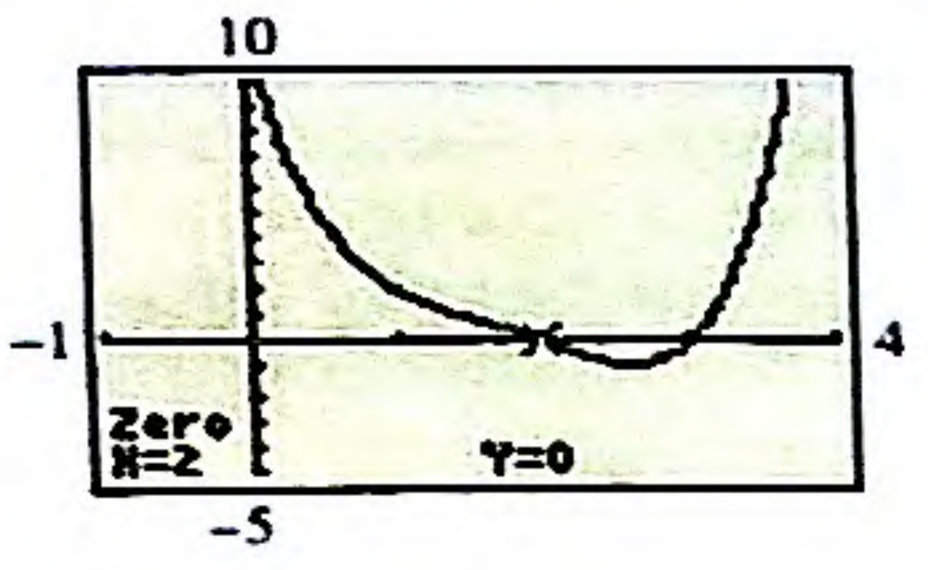
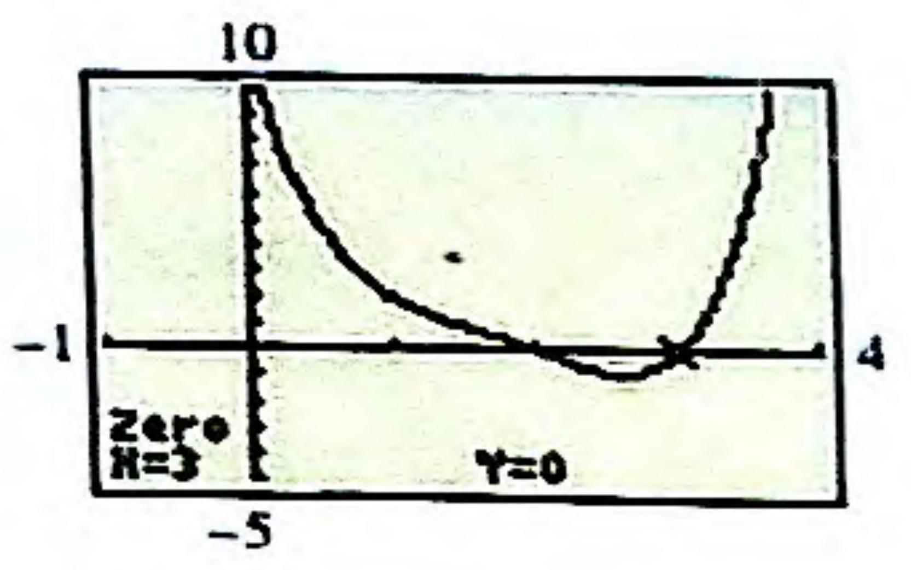


Figure B

90. A comprehensive graph of $f(x) = x^4 - 7x^3 + 18x^2 - 22x + 12$ is shown in the two screens, along with displays of the two real zeros. Find the two remaining nonreal complex zeros.





Relating Concepts

For individual or collaborative investigation (Exercises 91–96)

For any function $y = f(x)$, the following hold true.

(a) The real solutions of $f(x) = 0$ are the x-intercepts of the graph.

(b) The real solutions of $f(x) < 0$ are the x-values for which the graph lies *below* the x-axis.

(c) The real solutions of $f(x) > 0$ are the x-values for which the graph lies *above* the x-axis.

In Exercises 91–96, a polynomial function $f(x)$ is given in both expanded and factored forms. Graph the function, and solve the equations and inequalities. ***Give multiplicities of solutions when applicable.***

91. $f(x) = x^3 - 3x^2 - 6x + 8$
$= (x - 4)(x - 1)(x + 2)$

(a) $f(x) = 0$ **(b)** $f(x) < 0$
(c) $f(x) > 0$

92. $f(x) = x^3 + 4x^2 - 11x - 30$
$= (x - 3)(x + 2)(x + 5)$

(a) $f(x) = 0$ **(b)** $f(x) < 0$
(c) $f(x) > 0$

93. $f(x) = 2x^4 - 9x^3 - 5x^2 + 57x - 45$
$= (x - 3)^2(2x + 5)(x - 1)$

(a) $f(x) = 0$ **(b)** $f(x) < 0$
(c) $f(x) > 0$

94. $f(x) = 4x^4 + 27x^3 - 42x^2 - 445x - 300$
$= (x + 5)^2(4x + 3)(x - 4)$

(a) $f(x) = 0$ **(b)** $f(x) < 0$
(c) $f(x) > 0$

95. $f(x) = -x^4 - 4x^3 + 3x^2 + 18x$
$= x(2 - x)(x + 3)^2$

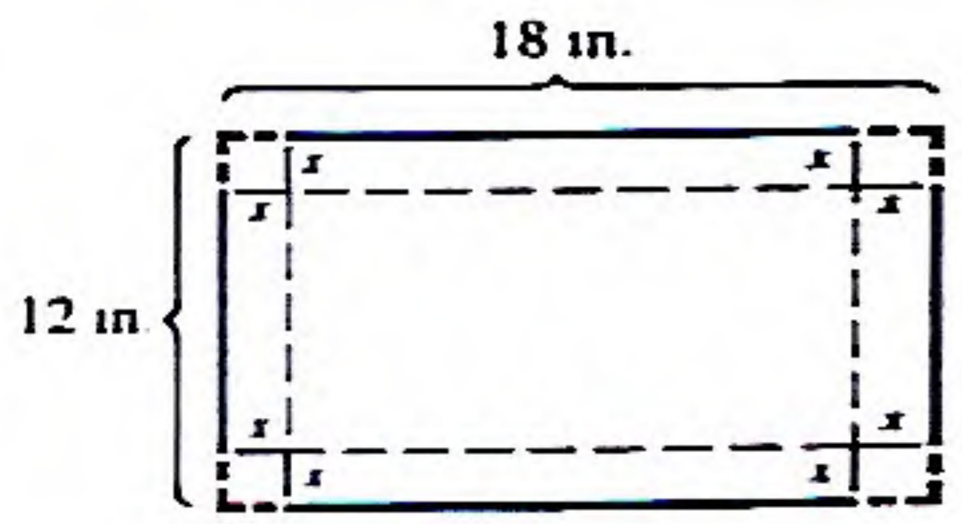
(a) $f(x) = 0$ **(b)** $f(x) \geq 0$
(c) $f(x) \leq 0$

96. $f(x) = -x^4 + 2x^3 + 8x^2$
$= x^2(4 - x)(x + 2)$

(a) $f(x) = 0$ **(b)** $f(x) \geq 0$
(c) $f(x) \leq 0$

(Modeling) ***Exercises 97–104 are geometric in nature and lead to polynomial models. Solve each problem.***

97. ***Volume of a Box*** A rectangular piece of cardboard measuring 12 in. by 18 in. is to be made into a box with an open top by cutting equal-size squares from each corner and folding up the sides. Let x represent the length of a side of each such square in inches.



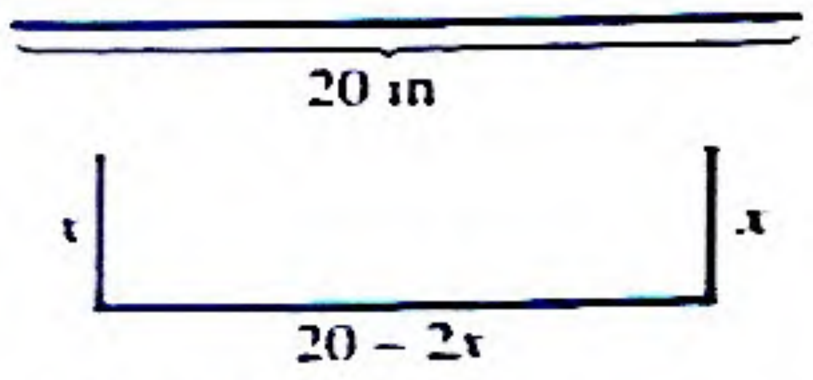
(a) Give the restrictions on x.

(b) Determine a function V that gives the volume of the box as a function of x.

(c) For what value of x will the volume be a maximum? What is this maximum volume? (*Hint:* Use the function of a graphing calculator that allows you to determine a maximum point within a given interval.)

(d) For what values of x will the volume be greater than 80 in.^{3}?

98. ***Construction of a Rain Gutter*** A piece of rectangular sheet metal is 20 in. wide. It is to be made into a rain gutter by turning up the edges to form parallel sides. Let x represent the length of each of the parallel sides.



(a) Give the restrictions on x.

(b) Determine a function $\mathcal{A}$ that gives the area of a cross section of the gutter.

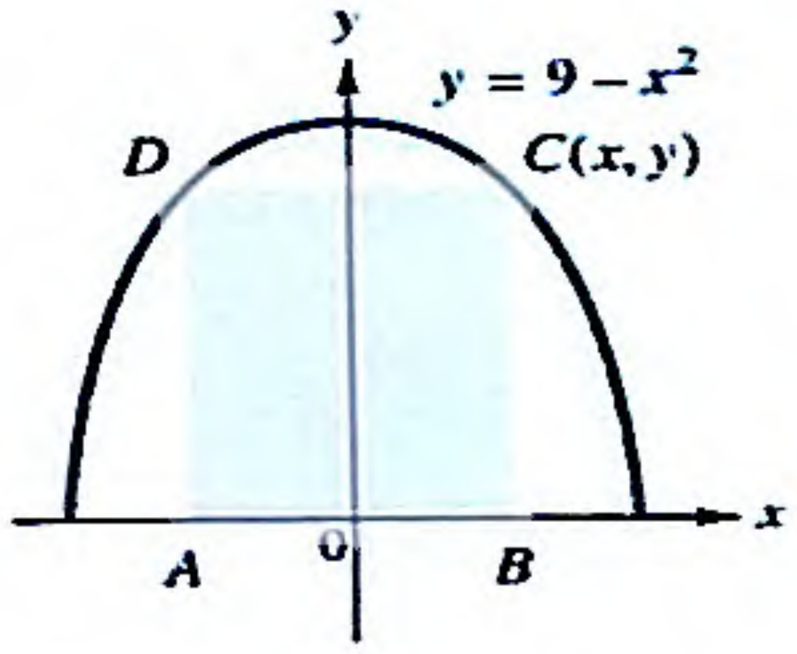
(c) For what value of x will $\mathcal{A}$ be a maximum (and thus maximize the amount of water that the gutter will hold)? What is this maximum area?

(d) For what values of x will the area of a cross section be less than 40 in.^{2}?

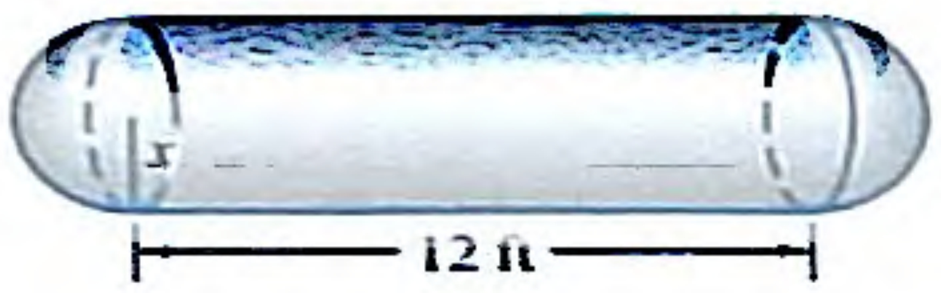
99. ***Sides of a Right Triangle*** A certain right triangle has area 84 in.^{2}. One leg of the triangle measures 1 in. less than the hypotenuse. Let x represent the length of the hypotenuse.

(a) Express the length of the leg mentioned above in terms of x. Give the domain of x.
(b) Express the length of the other leg in terms of x.
(c) Write an equation based on the information determined thus far. Square both sides and then write the equation with one side as a polynomial with integer coefficients, in descending powers, and the other side equal to 0.
(d) Solve the equation in part (c) graphically. Find the lengths of the three sides of the triangle.

100. ***Area of a Rectangle*** Find the value of x in the figure that will maximize the area of rectangle $ABCD$.



101. ***Butane Gas Storage*** A storage tank for butane gas is to be built in the shape of a right circular cylinder of altitude 12 ft, with a half sphere attached to each end. If x represents the radius of each half sphere, what radius should be used to cause the volume of the tank to be 144π ft^3?



102. ***Volume of a Box*** A standard piece of notebook paper measuring 8.5 in. by 11 in. is to be made into a box with an open top by cutting equal-size squares from each corner and folding up the sides. Let x represent the length of a side of each such square in inches. Use the table feature of your graphing calculator to do the following.

(a) Find the maximum volume of the box.
(b) Determine when the volume of the box will be greater than 40 in.^{3}.

103. ***Floating Ball*** The polynomial function

$$f(x) = \frac{\pi}{3}x^3 - 5\pi x^2 + \frac{500\pi d}{3}$$

can be used to find the depth that a ball 10 cm in diameter sinks in water. The constant d is the density of the ball, where the density of water is 1. The smallest *positive* zero of $f(x)$ equals the depth that the ball sinks. Approximate this depth for each material and interpret the results.

(a) A wooden ball with $d = 0.8$
(b) A solid aluminum ball with $d = 2.7$
(c) A spherical water balloon with $d = 1$

104. ***Floating Ball*** Refer to **Exercise 103.** If a ball has a 20-cm diameter, then the function becomes

$$f(x) = \frac{\pi}{3}x^3 - 10\pi x^2 + \frac{4000\pi d}{3}.$$

This function can be used to determine the depth that the ball sinks in water. Find the depth that this size ball sinks when $d = 0.6$.

(Modeling) Solve each problem involving a polynomial function model. See ***Example 8.***

105. ***Highway Design*** To allow enough distance for cars to pass on two-lane highways, engineers calculate minimum sight distances between curves and hills. The table shows the minimum sight distance y in feet for a car traveling at x miles per hour.

x (in mph)	20	30	40	50	60	65	70
y (in feet)	810	1090	1480	1840	2140	2310	2490

Source: Haefner, L., *Introduction to Transportation Systems,* Holt, Rinehart and Winston.

(a) Make a scatter diagram of the data.
(b) Use the regression feature of a calculator to find the best-fitting linear function for the data. Graph the function with the data.
(c) Repeat part (b) for a cubic function.
(d) Estimate the minimum sight distance for a car traveling 43 mph using the functions from parts (b) and (c).
(e) By comparing the graphs of the functions in parts (b) and (c) with the data, decide which function best fits the given data.

106. ***Water Pollution*** Copper in high doses can be lethal to aquatic life. The table lists copper concentrations in freshwater mussels after 45 days at various distances downstream from an electroplating plant. The concentration C is measured in micrograms of copper per gram of mussel x kilometers downstream.

x	5	21	37	53	59
C	20	13	9	6	5

Source: Foster, R., and J. Bates, "Use of mussels to monitor point source industrial discharges," *Environ. Sci. Technol.*, Mason, C., *Biology of Freshwater Pollution,* John Wiley & Sons.

(a) Make a scatter diagram of the data.
(b) Use the regression feature of a calculator to find the best-fitting quadratic function for the data. Graph the function with the data.
(c) Repeat part (b) for a cubic function.
(d) By comparing graphs of the functions in parts (b) and (c) with the data, decide which function best fits the given data.
(e) Concentrations above 10 are lethal to mussels. Find the values of x (using the cubic function) for which this is the case.

107. ***Government Spending on Health Research and Training*** The table lists the annual amount (in billions of dollars) spent by the federal government on health research and training programs over an 8-yr period.

Year	Amount	Year	Amount
2002	21.4	2006	28.8
2003	24.0	2007	29.3
2004	27.1	2008	29.9
2005	28.1	2009	30.6

Source: U.S. Office of Management and Budget.

Which one of the following provides the best model for these data, where x represents the year?

A. $f(x) = 0.5(x - 2002)^2 + 21.4$
B. $g(x) = (x - 2002) + 21.4$
C. $h(x) = 3.5\sqrt{x - 2002} + 21.4$
D. $k(x) = (x - 2002)^3 + 21.4$

108. ***Swing of a Pendulum*** Grandfather clocks use pendulums to keep accurate time. The relationship between the length of a pendulum L and the time T for one complete oscillation can be expressed by the equation $L = kT^n$, where k is a constant and n is a positive integer to be determined. The data in the table were taken for different lengths of pendulums.

L (ft)	T (sec)	L (ft)	T (sec)
1.0	1.11	3.0	1.92
1.5	1.36	3.5	2.08
2.0	1.57	4.0	2.22
2.5	1.76		

(a) As the length of the pendulum increases, what happens to T?

(b) Discuss how n and k could be found.

(c) Use the data to approximate k and determine the best value for n.

(d) Using the values of k and n from part (c), predict T for a pendulum having length 5 ft.

(e) If the length L of a pendulum doubles, what happens to the period T?

Summary Exercises on Polynomial Functions, Zeros, and Graphs

We use all of the theorems for finding complex zeros of polynomial functions in the next example.

EXAMPLE **Finding All Zeros of a Polynomial Function**

Find all zeros of $f(x) = x^4 - 3x^3 + 6x^2 - 12x + 8$.

SOLUTION We consider the number of positive zeros by observing the variations in signs for $f(x)$.

$$f(x) = +x^4 \underset{1}{-} 3x^3 \underset{2}{+} 6x^2 \underset{3}{-} 12x \underset{4}{+} 8 \qquad \text{(Section 3.3)}$$

Since $f(x)$ has four sign changes, we can use Descartes' rule of signs to determine that there are four, two, or zero positive real zeros. For negative zeros, we consider the variations in signs for $f(-x)$.

$$f(-x) = (-x)^4 - 3(-x)^3 + 6(-x)^2 - 12(-x) + 8$$
$$f(-x) = x^4 + 3x^3 + 6x^2 + 12x + 8$$

Since $f(-x)$ has no sign changes, there are no negative real zeros. Because the function is of degree 4, it has a maximum of four zeros with possibilities summarized as follows.

Positive	Negative	Nonreal Complex
4	0	0
2	0	2
0	0	4

We can now use the rational zeros theorem to determine that the possible rational zeros are ± 1, ± 2, ± 4, and ± 8. Based on Descartes' rule of signs, we discard the negative rational zeros from this list and try to find a positive rational zero. Suppose we start by using synthetic division to check 4.

$$\text{Proposed zero} \longrightarrow \begin{array}{r|rrrrr} 4 & 1 & -3 & 6 & -12 & 8 \\ & & 4 & 4 & 40 & 112 \\ \hline & 1 & 1 & 10 & 28 & 120 \end{array} \longleftarrow f(4) = 120 \qquad \text{(Section 3.2)}$$

We find that 4 is not a zero. However, $4 > 0$, and the numbers in the bottom row of the synthetic division are nonnegative. Thus, the boundedness theorem indicates that there are no zeros greater than 4. We can discard 8 as a possible rational zero and use synthetic division to show that 1 and 2 are zeros.

$$\begin{array}{r|rrrrr} 1 & 1 & -3 & 6 & -12 & 8 \\ & & 1 & -2 & 4 & -8 \\ \hline 2 & 1 & -2 & 4 & -8 & 0 \\ & & 2 & 0 & 8 & \\ \hline & 1 & 0 & 4 & 0 & \end{array} \quad \begin{array}{l} \\ \\ \longleftarrow f(1) = 0 \\ \\ \longleftarrow f(2) = 0 \end{array}$$

The polynomial now factors as

$$f(x) = (x - 1)(x - 2)(x^2 + 4).$$

We find the remaining two zeros using algebra to solve for x in the quadratic factor of the following equation.

$$(x - 1)(x - 2)(x^2 + 4) = 0$$

$$x - 1 = 0 \quad \text{or} \quad x - 2 = 0 \quad \text{or} \quad x^2 + 4 = 0 \qquad \text{Zero-factor property (Section 1.4)}$$

$$x = 1 \quad \text{or} \quad x = 2 \quad \text{or} \quad x^2 = -4$$

$$x = \pm 2i \qquad \text{Square root property (Section 1.4)}$$

The linear factored form of the polynomial is

$$f(x) = (x - 1)(x - 2)(x - 2i)(x + 2i),$$

and the corresponding zeros are 1, 2, $2i$, and $-2i$.

✓ ***Now Try Exercise 3.***

For each polynomial function, do the following in order.

- **(a)** Use Descartes' rule of signs to determine the different possibilities for the numbers of positive, negative, and nonreal complex zeros.
- **(b)** Use the rational zeros theorem to determine the possible rational zeros.
- **(c)** Use synthetic division with the boundedness theorem where appropriate and/or factoring to find the rational zeros, if any.
- **(d)** Find all other complex zeros (both real and nonreal), if any.

1. $f(x) = 6x^3 - 41x^2 + 26x + 24$

2. $f(x) = 2x^3 - 5x^2 - 4x + 3$

3. $f(x) = 3x^4 - 5x^3 + 14x^2 - 20x + 8$

4. $f(x) = 2x^4 - 3x^3 + 16x^2 - 27x - 18$

5. $f(x) = 6x^4 - 5x^3 - 11x^2 + 10x - 2$

6. $f(x) = 5x^4 + 8x^3 - 19x^2 - 24x + 12$

7. $f(x) = x^5 - 6x^4 + 16x^3 - 24x^2 + 16x$ (*Hint:* Factor out x first.)

8. $f(x) = 2x^4 + 8x^3 - 7x^2 - 42x - 9$

9. $f(x) = 8x^4 + 8x^3 - x - 1$ (*Hint:* Factor the polynomial.)

10. $f(x) = 2x^5 + 5x^4 - 9x^3 - 11x^2 + 19x - 6$

For each polynomial function, do the following in order.

(a) Use Descartes' rule of signs to determine the different possibilities for the numbers of positive, negative, and nonreal complex zeros.
(b) Use the rational zeros theorem to determine the possible rational zeros.
(c) Find the rational zeros, if any.
(d) Find all other real zeros, if any.
(e) Find any other complex zeros (that is, zeros that are not real), if any.
(f) Find the x-intercepts of the graph, if any.
(g) Find the y-intercept of the graph.
(h) Use synthetic division to find $f(4)$, and give the coordinates of the corresponding point on the graph.
(i) Determine the end behavior of the graph.
(j) Sketch the graph.

11. $f(x) = x^4 + 3x^3 - 3x^2 - 11x - 6$

12. $f(x) = -2x^5 + 5x^4 + 34x^3 - 30x^2 - 84x + 45$

13. $f(x) = 2x^5 - 10x^4 + x^3 - 5x^2 - x + 5$

14. $f(x) = 3x^4 - 4x^3 - 22x^2 + 15x + 18$

15. $f(x) = -2x^4 - x^3 + x + 2$

16. $f(x) = 4x^5 + 8x^4 + 9x^3 + 27x^2 + 27x$ (*Hint:* Factor out x first.)

17. $f(x) = 3x^4 - 14x^2 - 5$ (*Hint:* Factor the polynomial.)

18. $f(x) = -x^5 - x^4 + 10x^3 + 10x^2 - 9x - 9$

19. $f(x) = -3x^4 + 22x^3 - 55x^2 + 52x - 12$

20. For the polynomial functions in **Exercises 11–19** that have irrational zeros, find approximations to the nearest thousandth.

3.5 Rational Functions: Graphs, Applications, and Models

- **The Reciprocal Function $f(x) = \frac{1}{x}$**
- **The Function $f(x) = \frac{1}{x^2}$**
- **Asymptotes**
- **Steps for Graphing Rational Functions**
- **Rational Function Models**

A rational expression is a fraction that is the quotient of two polynomials. A function defined by a rational expression is a *rational function.*

Rational Function

A function f of the form

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials, with $q(x) \neq 0$, is a **rational function.**

Some examples of rational functions are

$$f(x) = \frac{1}{x}, \quad f(x) = \frac{x+1}{2x^2 + 5x - 3}, \quad \text{and} \quad f(x) = \frac{3x^2 - 3x - 6}{x^2 + 8x + 16}.$$

Rational functions

Since any values of x such that $q(x) = 0$ are excluded from the domain of a rational function, this type of function often has a **discontinuous graph**—that is, a graph that has one or more breaks in it. (See **Chapter 2.**)

60. $f(x) = x^5 + 2x^3 - 2x^2 + 5x + 5$; no real zero less than -1

61. $f(x) = 3x^4 + 2x^3 - 4x^2 + x - 1$; no real zero greater than 1

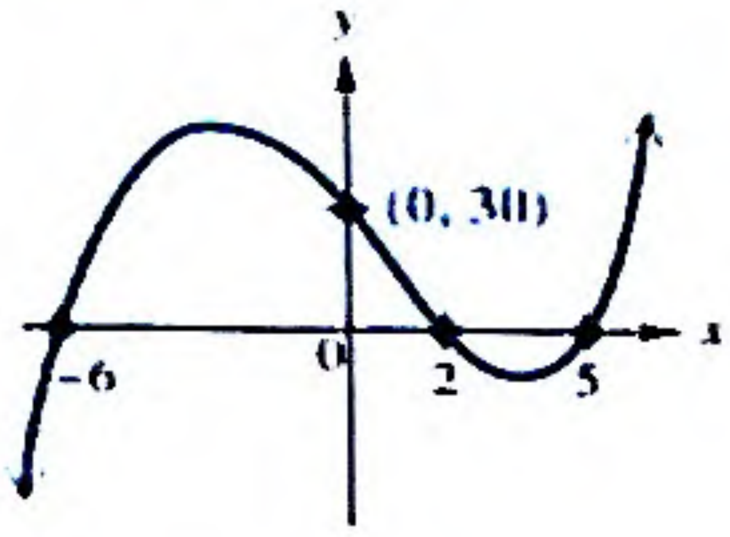
62. $f(x) = 3x^4 + 2x^3 - 4x^2 + x - 1$; no real zero less than -2

63. $f(x) = x^5 - 3x^3 + x + 2$; no real zero greater than 2

64. $f(x) = x^5 - 3x^3 + x + 2$; no real zero less than -3

Connecting Graphs with Equations In Exercises 65–70, find a polynomial function $f(x)$ of least possible degree having the graph shown. (Hint: See the Note following Example 4.)

65.



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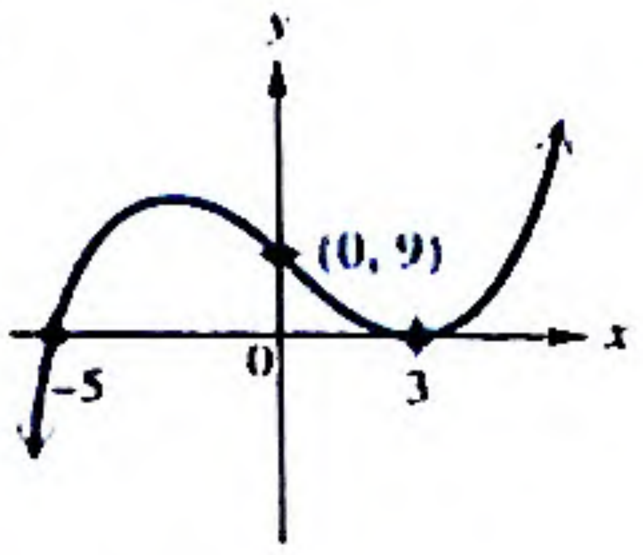

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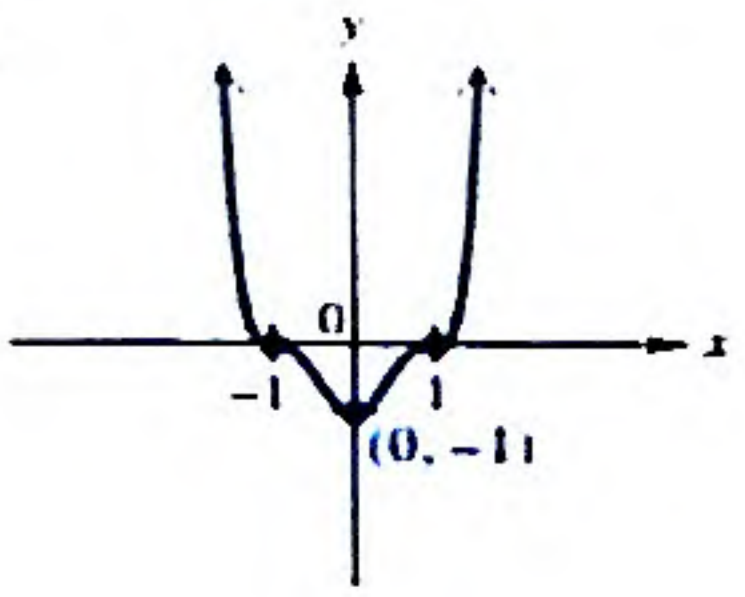

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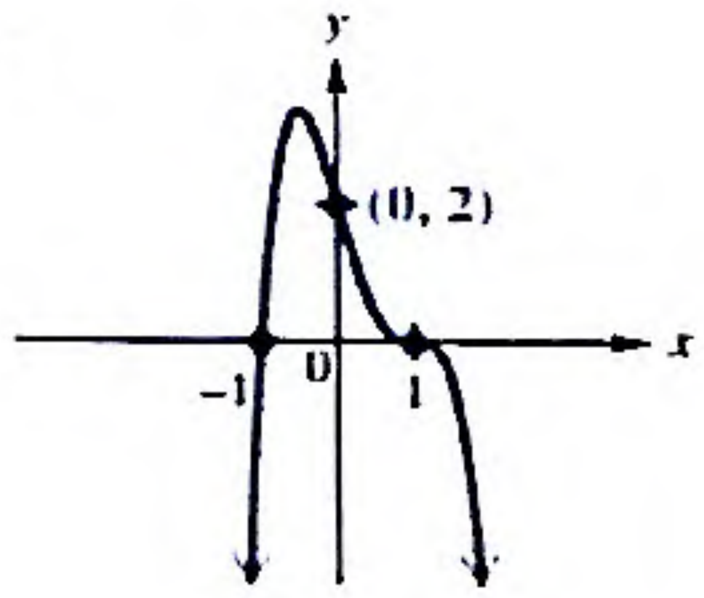

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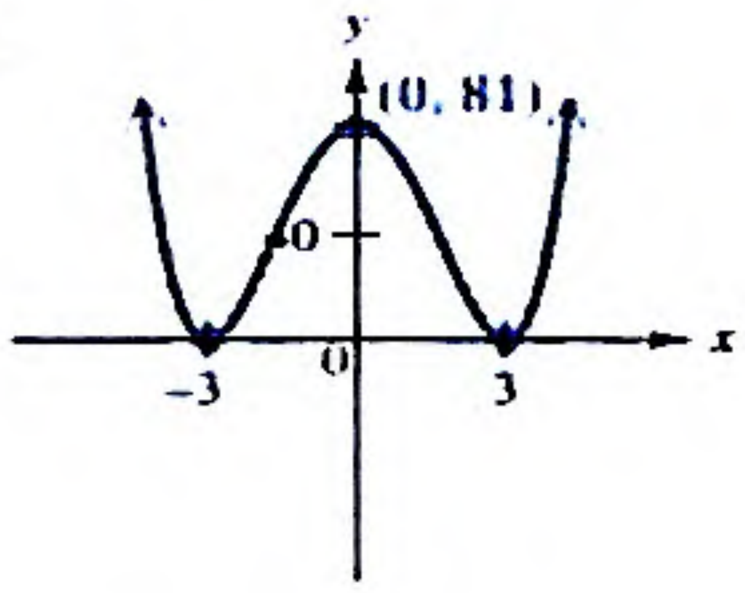
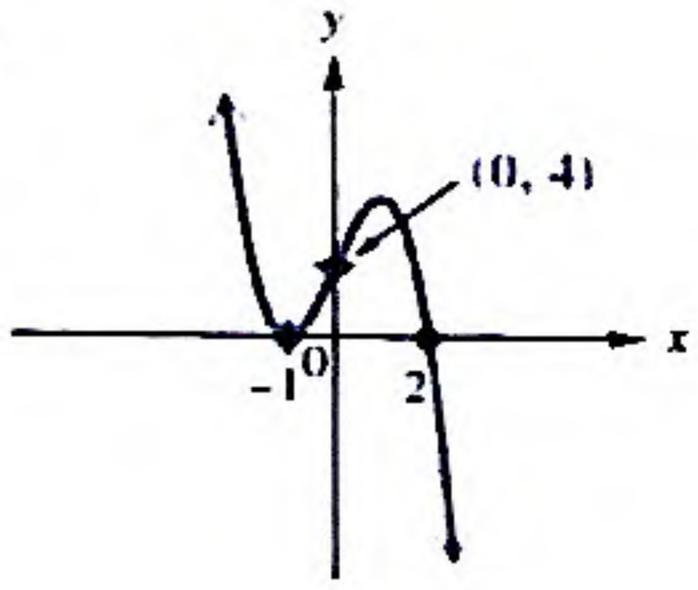


70.



Use a graphing calculator to graph the function in the viewing window specified. Compare the graph to the one shown in the answer section of this text. Then use the graph to find $f(1.25)$.

71. $f(x) = 2x(x - 3)(x + 2)$; window: $[-3, 4]$ by $[-20, 12]$
Compare to Exercise 31.

72. $f(x) = x^2(x - 2)(x + 3)^2$, window: $[-4, 3]$ by $[-24, 4]$
Compare to Exercise 33.

73. $f(x) = (3x - 1)(x + 2)^2$, window: $[-4, 2]$ by $[-15, 15]$
Compare to Exercise 35.

74. $f(x) = x^3 + 5x^2 - x - 5$, window: $[-6, 2]$ by $[-30, 30]$
Compare to Exercise 37.

Use a graphing calculator to approximate the real zero discussed in each specified exercise. See Example 7.

75. Exercise 47

76. Exercise 49

77. Exercise 51

78. Exercise 50

For the given polynomial function, approximate each zero as a decimal to the nearest tenth. See Example 7.

79. $f(x) = x^3 + 3x^2 - 2x - 6$

80. $f(x) = x^3 - 3x + 3$

81. $f(x) = -2x^4 - x^2 + x + 5$

82. $f(x) = -x^4 + 2x^3 + 3x^2 + 6$

Use a graphing calculator to find the coordinates of the turning points of the graph of each polynomial function in the given domain interval. Give answers to the nearest hundredth.

83. $f(x) = x^3 + 4x^2 - 8x - 8;\ [-3.8, -3]$

84. $f(x) = x^3 + 4x^2 - 8x - 8,\ [0.3, 1]$

85. $f(x) = 2x^3 - 5x^2 - x + 1;\ [-1, 0]$

86. $f(x) = 2x^3 - 5x^2 - x + 1,\ [1.4, 2]$

87. $f(x) = x^4 - 7x^3 + 13x^2 + 6x - 28;\ [-1, 0]$

88. $f(x) = x^3 - x + 3;\ [-1, 0]$

Solve each problem.

89. ***(Modeling) Social Security Numbers*** Your Social Security number (SSN) is unique, and with it you can construct your own personal Social Security polynomial. Let the polynomial function be defined as follows, where a_i represents the ith digit in your SSN:

$$\text{SSN}(x) = (x - a_1)(x + a_2)(x - a_3)(x + a_4)(x - a_5) \cdot$$
$$(x + a_6)(x - a_7)(x + a_8)(x - a_9).$$

For example, if the SSN is 539-58-0954, the polynomial function is

$$\text{SSN}(x) = (x - 5)(x + 3)(x - 9)(x + 5)(x - 8)(x + 0)(x - 9)(x + 5)(x - 4).$$

A comprehensive graph of this function is shown in **Figure A.** In **Figure B,** we show a screen obtained by zooming in on the positive zeros, as the comprehensive graph does not show the local behavior well in this region. Use a graphing calculator to graph your own "personal polynomial."

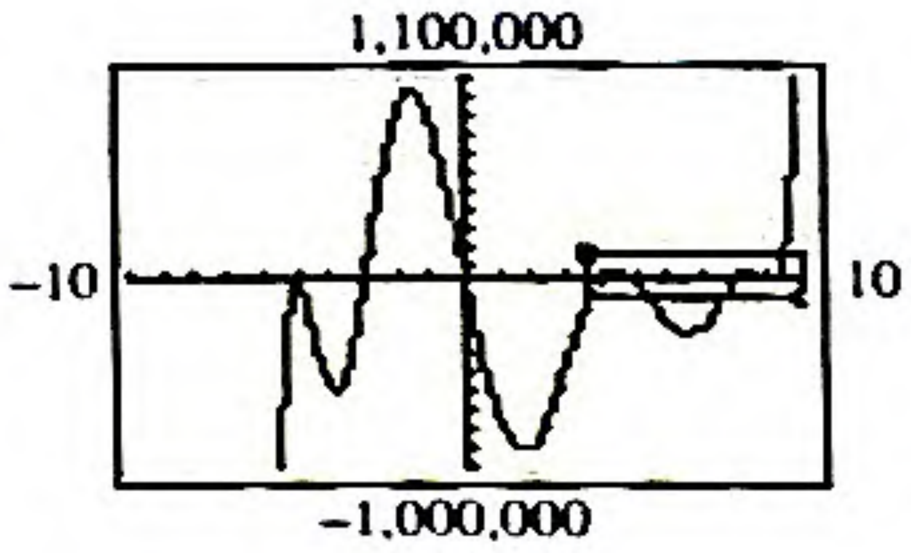


Figure A

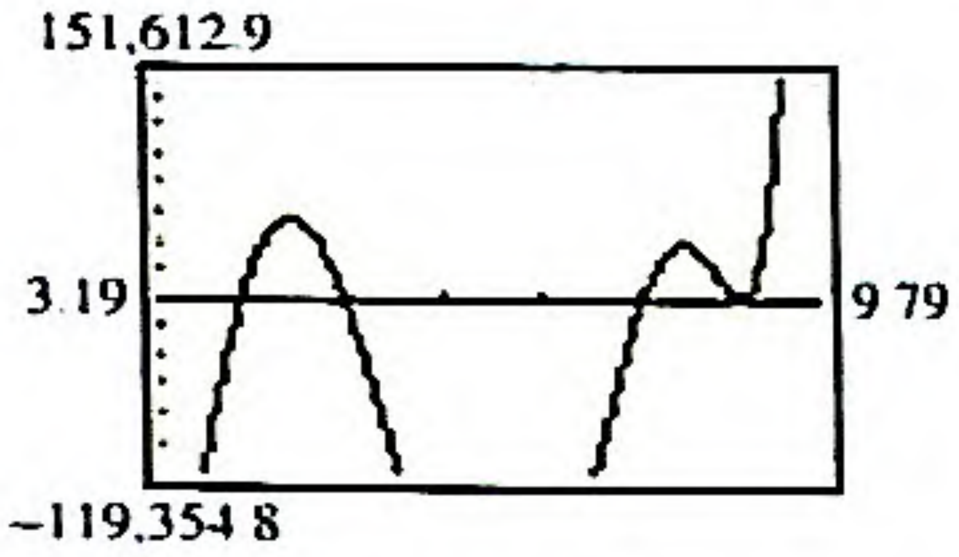
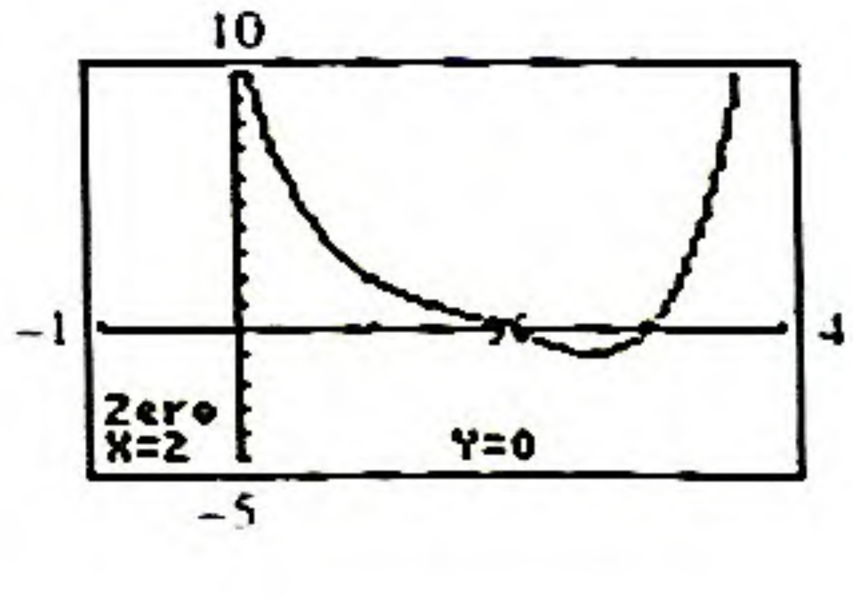
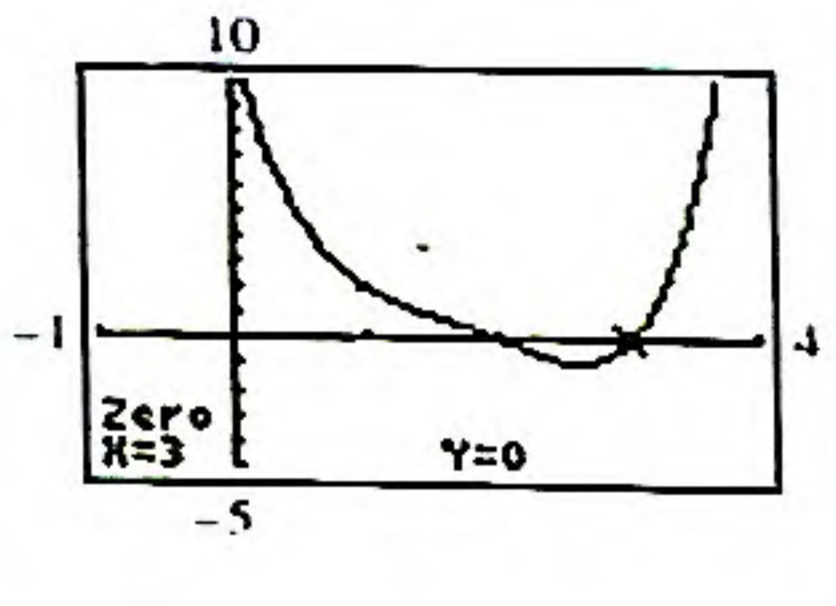


Figure B

90. A comprehensive graph of $f(x) = x^4 - 7x^3 + 18x^2 - 22x + 12$ is shown in the two screens, along with displays of the two real zeros. Find the two remaining nonreal complex zeros.





Relating Concepts

For individual or collaborative investigation *(Exercises 91–96)*

For any function $y = f(x)$, the following hold true.

(a) The real solutions of $f(x) = 0$ are the x-intercepts of the graph.

(b) The real solutions of $f(x) < 0$ are the x-values for which the graph lies *below* the x-axis.

(c) The real solutions of $f(x) > 0$ are the x-values for which the graph lies *above* the x-axis.

In Exercises 91–96, a polynomial function $f(x)$ is given in both expanded and factored forms. Graph the function, and solve the equations and inequalities. Give multiplicities of solutions when applicable.

91. $f(x) = x^3 - 3x^2 - 6x + 8$
$= (x - 4)(x - 1)(x + 2)$

(a) $f(x) = 0$ **(b)** $f(x) < 0$
(c) $f(x) > 0$

92. $f(x) = x^3 + 4x^2 - 11x - 30$
$= (x - 3)(x + 2)(x + 5)$

(a) $f(x) = 0$ **(b)** $f(x) < 0$
(c) $f(x) > 0$

93. $f(x) = 2x^4 - 9x^3 - 5x^2 + 57x - 45$
$= (x - 3)^2(2x + 5)(x - 1)$

(a) $f(x) = 0$ **(b)** $f(x) < 0$
(c) $f(x) > 0$

94. $f(x) = 4x^4 + 27x^3 - 42x^2$
$- 445x - 300$
$= (x + 5)^2(4x + 3)(x - 4)$

(a) $f(x) = 0$ **(b)** $f(x) < 0$
(c) $f(x) > 0$

95. $f(x) = -x^4 - 4x^3 + 3x^2 + 18x$
$= x(2 - x)(x + 3)^2$

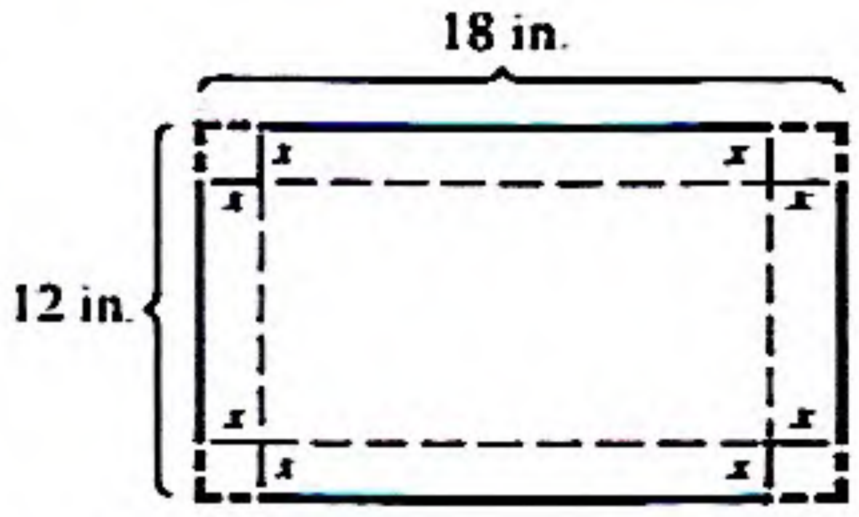
(a) $f(x) = 0$ **(b)** $f(x) \geq 0$
(c) $f(x) \leq 0$

96. $f(x) = -x^4 + 2x^3 + 8x^2$
$= x^2(4 - x)(x + 2)$

(a) $f(x) = 0$ **(b)** $f(x) \geq 0$
(c) $f(x) \leq 0$

***(Modeling)* Exercises 97–104 are geometric in nature and lead to polynomial models. Solve each problem.**

97. *Volume of a Box* A rectangular piece of cardboard measuring 12 in. by 18 in. is to be made into a box with an open top by cutting equal-size squares from each corner and folding up the sides. Let x represent the length of a side of each such square in inches.



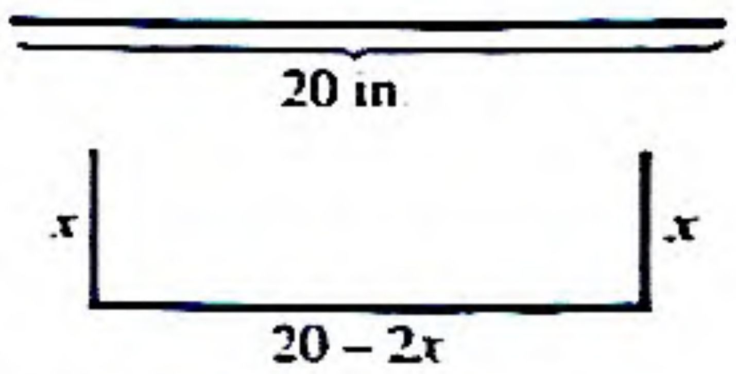
(a) Give the restrictions on x.

(b) Determine a function V that gives the volume of the box as a function of x.

(c) For what value of x will the volume be a maximum? What is this maximum volume? (*Hint:* Use the function of a graphing calculator that allows you to determine a maximum point within a given interval.)

(d) For what values of x will the volume be greater than 80 in.^{3}?

98. *Construction of a Rain Gutter* A piece of rectangular sheet metal is 20 in. wide. It is to be made into a rain gutter by turning up the edges to form parallel sides. Let x represent the length of each of the parallel sides.



(a) Give the restrictions on x.

(b) Determine a function $\mathcal{A}$ that gives the area of a cross section of the gutter.

(c) For what value of x will $\mathcal{A}$ be a maximum (and thus maximize the amount of water that the gutter will hold)? What is this maximum area?

(d) For what values of x will the area of a cross section be less than 40 in.^{2}?

99. ***Sides of a Right Triangle*** A certain right triangle has area 84 in.^{2}. One leg of the triangle measures 1 in. less than the hypotenuse. Let x represent the length of the hypotenuse.

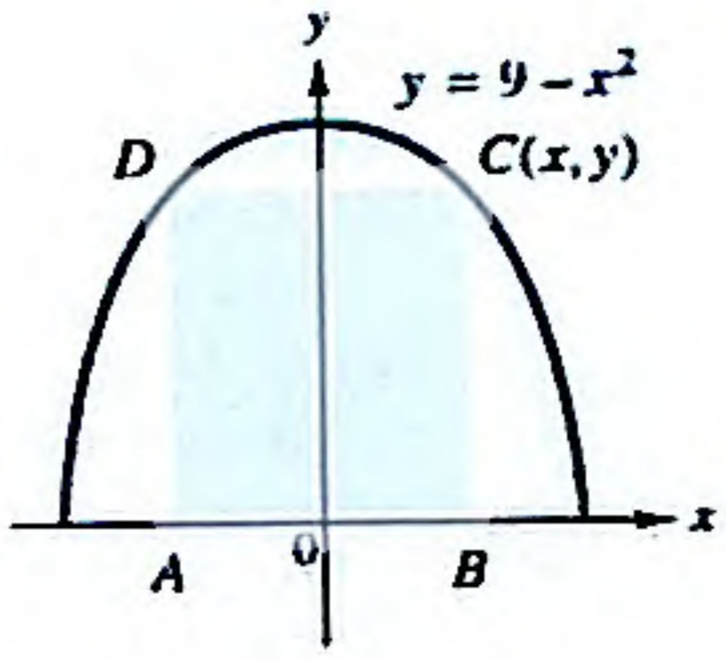
(a) Express the length of the leg mentioned above in terms of x. Give the domain of x.

(b) Express the length of the other leg in terms of x.

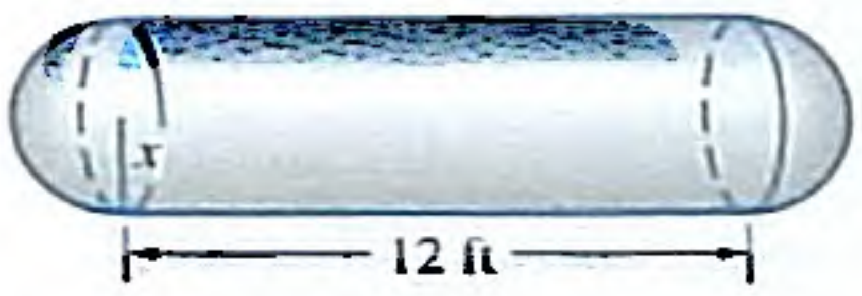
(c) Write an equation based on the information determined thus far. Square both sides and then write the equation with one side as a polynomial with integer coefficients, in descending powers, and the other side equal to 0.

(d) Solve the equation in part (c) graphically. Find the lengths of the three sides of the triangle.

100. ***Area of a Rectangle*** Find the value of x in the figure that will maximize the area of rectangle $ABCD$.



101. ***Butane Gas Storage*** A storage tank for butane gas is to be built in the shape of a right circular cylinder of altitude 12 ft, with a half sphere attached to each end. If x represents the radius of each half sphere, what radius should be used to cause the volume of the tank to be 144π ft^3?



102. ***Volume of a Box*** A standard piece of notebook paper measuring 8.5 in. by 11 in. is to be made into a box with an open top by cutting equal-size squares from each corner and folding up the sides. Let x represent the length of a side of each such square in inches. Use the table feature of your graphing calculator to do the following.

(a) Find the maximum volume of the box.

(b) Determine when the volume of the box will be greater than 40 in.^{3}.

103. ***Floating Ball*** The polynomial function

$$f(x) = \frac{\pi}{3}x^3 - 5\pi x^2 + \frac{500\pi d}{3}$$

can be used to find the depth that a ball 10 cm in diameter sinks in water. The constant d is the density of the ball, where the density of water is 1. The smallest *positive* zero of $f(x)$ equals the depth that the ball sinks. Approximate this depth for each material and interpret the results.

(a) A wooden ball with $d = 0.8$

(b) A solid aluminum ball with $d = 2.7$

(c) A spherical water balloon with $d = 1$

104. ***Floating Ball*** Refer to Exercise 103. If a ball has a 20-cm diameter, then the function becomes

$$f(x) = \frac{\pi}{3}x^3 - 10\pi x^2 + \frac{4000\pi d}{3}.$$

This function can be used to determine the depth that the ball sinks in water. Find the depth that this size ball sinks when $d = 0.6$.

(Modeling) Solve each problem involving a polynomial function model. See Example 8.

105. *Highway Design* To allow enough distance for cars to pass on two-lane highways, engineers calculate minimum sight distances between curves and hills. The table shows the minimum sight distance y in feet for a car traveling at x miles per hour.

x (in mph)	20	30	40	50	60	65	70
y (in feet)	810	1090	1480	1840	2140	2310	2490

Source: Haefner, L., *Introduction to Transportation Systems*, Holt, Rinehart and Winston.

(a) Make a scatter diagram of the data.
(b) Use the regression feature of a calculator to find the best-fitting linear function for the data. Graph the function with the data.
(c) Repeat part (b) for a cubic function.
(d) Estimate the minimum sight distance for a car traveling 43 mph using the functions from parts (b) and (c).
(e) By comparing the graphs of the functions in parts (b) and (c) with the data, decide which function best fits the given data.

106. *Water Pollution* Copper in high doses can be lethal to aquatic life. The table lists copper concentrations in freshwater mussels after 45 days at various distances downstream from an electroplating plant. The concentration C is measured in micrograms of copper per gram of mussel x kilometers downstream.

x	5	21	37	53	59
C	20	13	9	6	5

Source: Foster, R., and J. Bates, "Use of mussels to monitor point source industrial discharges," *Environ. Sci. Technol.*, Mason, C., *Biology of Freshwater Pollution*, John Wiley & Sons.

(a) Make a scatter diagram of the data.
(b) Use the regression feature of a calculator to find the best-fitting quadratic function for the data. Graph the function with the data.
(c) Repeat part (b) for a cubic function.
(d) By comparing graphs of the functions in parts (b) and (c) with the data, decide which function best fits the given data.
(e) Concentrations above 10 are lethal to mussels. Find the values of x (using the cubic function) for which this is the case.

107. *Government Spending on Health Research and Training* The table lists the annual amount (in billions of dollars) spent by the federal government on health research and training programs over an 8-yr period.

Year	Amount	Year	Amount
2002	21.4	2006	28.8
2003	24.0	2007	29.3
2004	27.1	2008	29.9
2005	28.1	2009	30.6

Source: U.S. Office of Management and Budget.

Which one of the following provides the best model for these data, where x represents the year?

A. $f(x) = 0.5(x - 2002)^2 + 21.4$
B. $g(x) = (x - 2002) + 21.4$
C. $h(x) = 3.5\sqrt{x - 2002} + 21.4$
D. $k(x) = (x - 2002)^3 + 21.4$

L (ft)	T (sec)	L (ft)	T (sec)
1.0	1.11	3.0	1.92
1.5	1.36	3.5	2.08
2.0	1.57	4.0	2.22
2.5	1.76		

108. ***Swing of a Pendulum*** Grandfather clocks use pendulums to keep accurate time. The relationship between the length of a pendulum L and the time T for one complete oscillation can be expressed by the equation $L = kT^n$, where k is a constant and n is a positive integer to be determined. The data in the table were taken for different lengths of pendulums.

(a) As the length of the pendulum increases, what happens to T?

(b) Discuss how n and k could be found.

(c) Use the data to approximate k and determine the best value for n.

(d) Using the values of k and n from part (c), predict T for a pendulum having length 5 ft.

(e) If the length L of a pendulum doubles, what happens to the period T?

Summary Exercises on Polynomial Functions, Zeros, and Graphs

We use all of the theorems for finding complex zeros of polynomial functions in the next example.

EXAMPLE **Finding All Zeros of a Polynomial Function**

Find all zeros of $f(x) = x^4 - 3x^3 + 6x^2 - 12x + 8$.

SOLUTION We consider the number of positive zeros by observing the variations in signs for $f(x)$.

$$f(x) = +x^4 \underset{1}{\smile} -3x^3 \underset{2}{\smile} +6x^2 \underset{3}{\smile} -12x \underset{4}{\smile} +8 \qquad \text{(Section 3.3)}$$

Since $f(x)$ has four sign changes, we can use Descartes' rule of signs to determine that there are four, two, or zero positive real zeros. For negative zeros, we consider the variations in signs for $f(-x)$.

$$f(-x) = (-x)^4 - 3(-x)^3 + 6(-x)^2 - 12(-x) + 8$$
$$f(-x) = x^4 + 3x^3 + 6x^2 + 12x + 8$$

Since $f(-x)$ has no sign changes, there are no negative real zeros. Because the function is of degree 4, it has a maximum of four zeros with possibilities summarized as follows.

Positive	Negative	Nonreal Complex
4	0	0
2	0	2
0	0	4

We can now use the rational zeros theorem to determine that the possible rational zeros are ± 1, ± 2, ± 4, and ± 8. Based on Descartes' rule of signs, we discard the negative rational zeros from this list and try to find a positive rational zero. Suppose we start by using synthetic division to check 4.

$$\text{Proposed zero} \longrightarrow \begin{array}{r|rrrrr} 4 & 1 & -3 & 6 & -12 & 8 \\ & & 4 & 4 & 40 & 112 \\ \hline & 1 & 1 & 10 & 28 & 120 \end{array} \longleftarrow f(4) = 120$$

(Section 3.2)

We find that 4 is not a zero. However, $4 > 0$, and the numbers in the bottom row of the synthetic division are nonnegative. Thus, the boundedness theorem indicates that there are no zeros greater than 4. We can discard 8 as a possible rational zero and use synthetic division to show that 1 and 2 are zeros.

$$\begin{array}{r|rrrrr} 1 & 1 & -3 & 6 & -12 & 8 \\ & & 1 & -2 & 4 & -8 \\ \hline 2 & 1 & -2 & 4 & -8 & 0 \\ & & 2 & 0 & 8 & \\ \hline & 1 & 0 & 4 & 0 & \end{array} \quad \begin{array}{l} \\ \\ \longleftarrow f(1) = 0 \\ \\ \longleftarrow f(2) = 0 \end{array}$$

The polynomial now factors as

$$f(x) = (x - 1)(x - 2)(x^2 + 4).$$

We find the remaining two zeros using algebra to solve for x in the quadratic factor of the following equation.

$$(x - 1)(x - 2)(x^2 + 4) = 0$$

$$x - 1 = 0 \quad \text{or} \quad x - 2 = 0 \quad \text{or} \quad x^2 + 4 = 0$$

Zero-factor property (Section 1.4)

$$x = 1 \quad \text{or} \quad x = 2 \quad \text{or} \quad x^2 = -4$$

$$x = \pm 2i$$

Square root property (Section 1.4)

The linear factored form of the polynomial is

$$f(x) = (x - 1)(x - 2)(x - 2i)(x + 2i),$$

and the corresponding zeros are 1, 2, $2i$, and $-2i$.

✓ ***Now Try Exercise 3.***

For each polynomial function, do the following in order.

- **(a)** Use Descartes' rule of signs to determine the different possibilities for the numbers of positive, negative, and nonreal complex zeros.
- **(b)** Use the rational zeros theorem to determine the possible rational zeros.
- **(c)** Use synthetic division with the boundedness theorem where appropriate and/or factoring to find the rational zeros, if any.
- **(d)** Find all other complex zeros (both real and nonreal), if any.

1. $f(x) = 6x^3 - 41x^2 + 26x + 24$

2. $f(x) = 2x^3 - 5x^2 - 4x + 3$

3. $f(x) = 3x^4 - 5x^3 + 14x^2 - 20x + 8$

4. $f(x) = 2x^4 - 3x^3 + 16x^2 - 27x - 18$

5. $f(x) = 6x^4 - 5x^3 - 11x^2 + 10x - 2$

6. $f(x) = 5x^4 + 8x^3 - 19x^2 - 24x + 12$

7. $f(x) = x^5 - 6x^4 + 16x^3 - 24x^2 + 16x$ (*Hint:* Factor out x first.)

8. $f(x) = 2x^4 + 8x^3 - 7x^2 - 42x - 9$

9. $f(x) = 8x^4 + 8x^3 - x - 1$ (*Hint:* Factor the polynomial.)

10. $f(x) = 2x^5 + 5x^4 - 9x^3 - 11x^2 + 19x - 6$

For each polynomial function, do the following in order.

- (a) Use Descartes' rule of signs to determine the different possibilities for the numbers of positive, negative, and nonreal complex zeros.
- (b) Use the rational zeros theorem to determine the possible rational zeros.
- (c) Find the rational zeros, if any.
- (d) Find all other real zeros, if any.
- (e) Find any other complex zeros (that is, zeros that are not real), if any.
- (f) Find the x-intercepts of the graph, if any.
- (g) Find the y-intercept of the graph.
- (h) Use synthetic division to find $f(4)$, and give the coordinates of the corresponding point on the graph.
- (i) Determine the end behavior of the graph.
- (j) Sketch the graph.

11. $f(x) = x^4 + 3x^3 - 3x^2 - 11x - 6$

12. $f(x) = -2x^5 + 5x^4 + 34x^3 - 30x^2 - 84x + 45$

13. $f(x) = 2x^5 - 10x^4 + x^3 - 5x^2 - x + 5$

14. $f(x) = 3x^4 - 4x^3 - 22x^2 + 15x + 18$

15. $f(x) = -2x^4 - x^3 + x + 2$

16. $f(x) = 4x^5 + 8x^4 + 9x^3 + 27x^2 + 27x$ (*Hint:* Factor out x first.)

17. $f(x) = 3x^4 - 14x^2 - 5$ (*Hint:* Factor the polynomial.)

18. $f(x) = -x^5 - x^4 + 10x^3 + 10x^2 - 9x - 9$

19. $f(x) = -3x^4 + 22x^3 - 55x^2 + 52x - 12$

20. For the polynomial functions in **Exercises 11–19** that have irrational zeros, find approximations to the nearest thousandth.

3.5 Rational Functions: Graphs, Applications, and Models

- **The Reciprocal Function $f(x) = \frac{1}{x}$**
- **The Function $f(x) = \frac{1}{x^2}$**
- **Asymptotes**
- **Steps for Graphing Rational Functions**
- **Rational Function Models**

A rational expression is a fraction that is the quotient of two polynomials. A function defined by a rational expression is a *rational function.*

Rational Function

A function f of the form

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials, with $q(x) \neq 0$, is a **rational function.**

Some examples of rational functions are

$$f(x) = \frac{1}{x}, \quad f(x) = \frac{x+1}{2x^2 + 5x - 3}, \quad \text{and} \quad f(x) = \frac{3x^2 - 3x - 6}{x^2 + 8x + 16}.$$

Rational functions

Since any values of x such that $q(x) = 0$ are excluded from the domain of a rational function, this type of function often has a **discontinuous graph**—that is, a graph that has one or more breaks in it. (See **Chapter 2.**)

EXAMPLE 8 Graphing a Rational Function with an Oblique Asymptote

Graph $f(x) = \dfrac{x^2 + 1}{x - 2}$.

SOLUTION As shown in **Example 4,** the vertical asymptote has equation $x = 2$, and the graph has an oblique asymptote with equation $y = x + 2$. The y-intercept is $-\frac{1}{2}$, and the graph has no x-intercepts since the numerator, $x^2 + 1$, has no real zeros. The graph does not intersect its oblique asymptote because the following has no solution.

$$\frac{x^2 + 1}{x - 2} = x + 2$$

$$x^2 + 1 = x^2 - 4 \qquad \text{Multiply each side by } x - 2.$$

$$1 = -4 \qquad \text{False}$$

Using the y-intercept, asymptotes, the points $\left(4, \frac{17}{2}\right)$ and $\left(-1, -\frac{2}{3}\right)$, and the general behavior of the graph near its asymptotes leads to the graph in **Figure 55.**

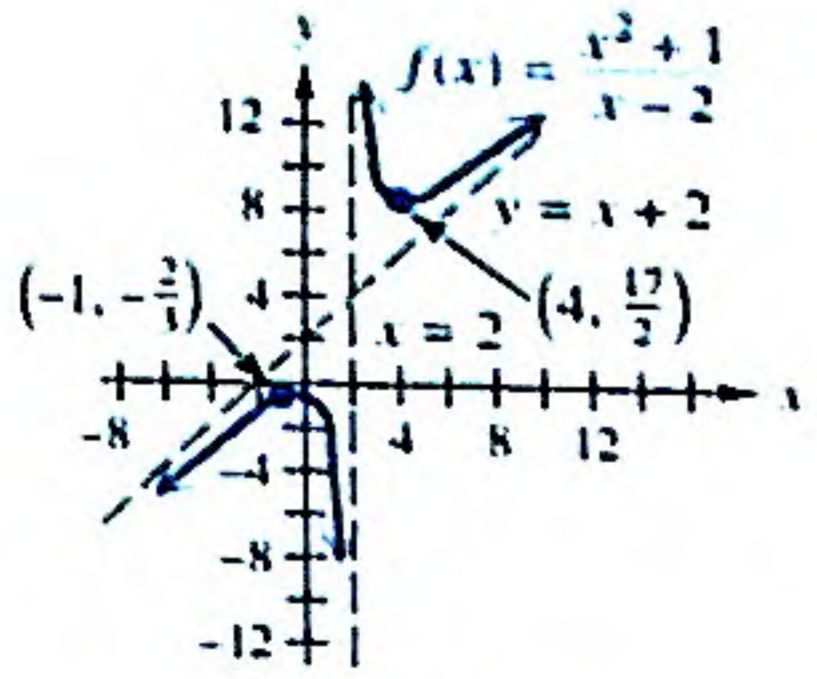


Figure 55

✔ *Now Try Exercise 87.*

A rational function that is not in lowest terms often has a "hole," or **point of discontinuity,** in its graph.

EXAMPLE 9 Graphing a Rational Function Defined by an Expression That Is Not in Lowest Terms

Graph $f(x) = \dfrac{x^2 - 4}{x - 2}$.

ALGEBRAIC SOLUTION

The domain of this function cannot include 2. The expression $\frac{x^2 - 4}{x - 2}$ should be written in lowest terms.

$$f(x) = \frac{x^2 - 4}{x - 2} \qquad \text{Factor and then divide.}$$

$$= \frac{(x + 2)(x - 2)}{x - 2} \qquad \text{Factor}$$

$$= x + 2, \quad x \neq 2$$

Therefore, the graph of this function will be the same as the graph of $y = x + 2$ (a straight line), with the exception of the point with x-value 2. A "hole" appears in the graph at $(2, 4)$. See **Figure 56.**

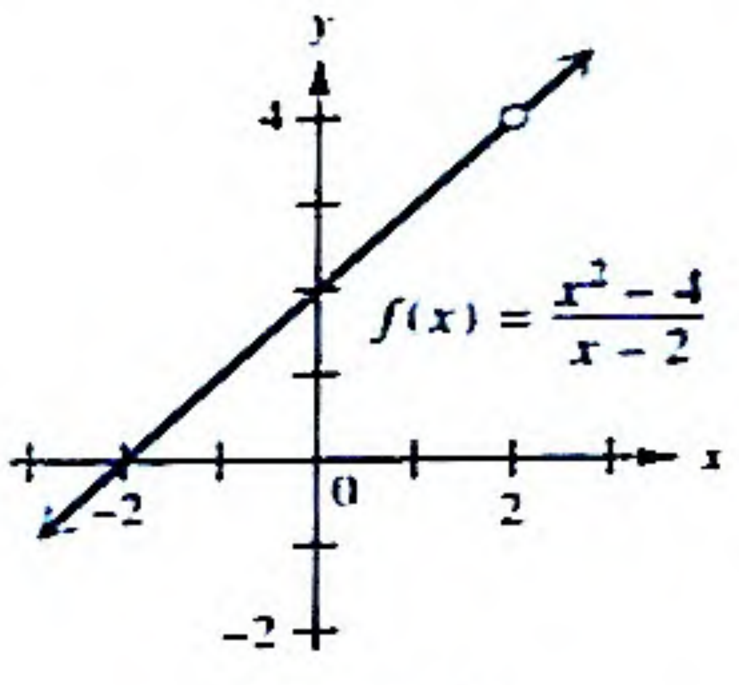


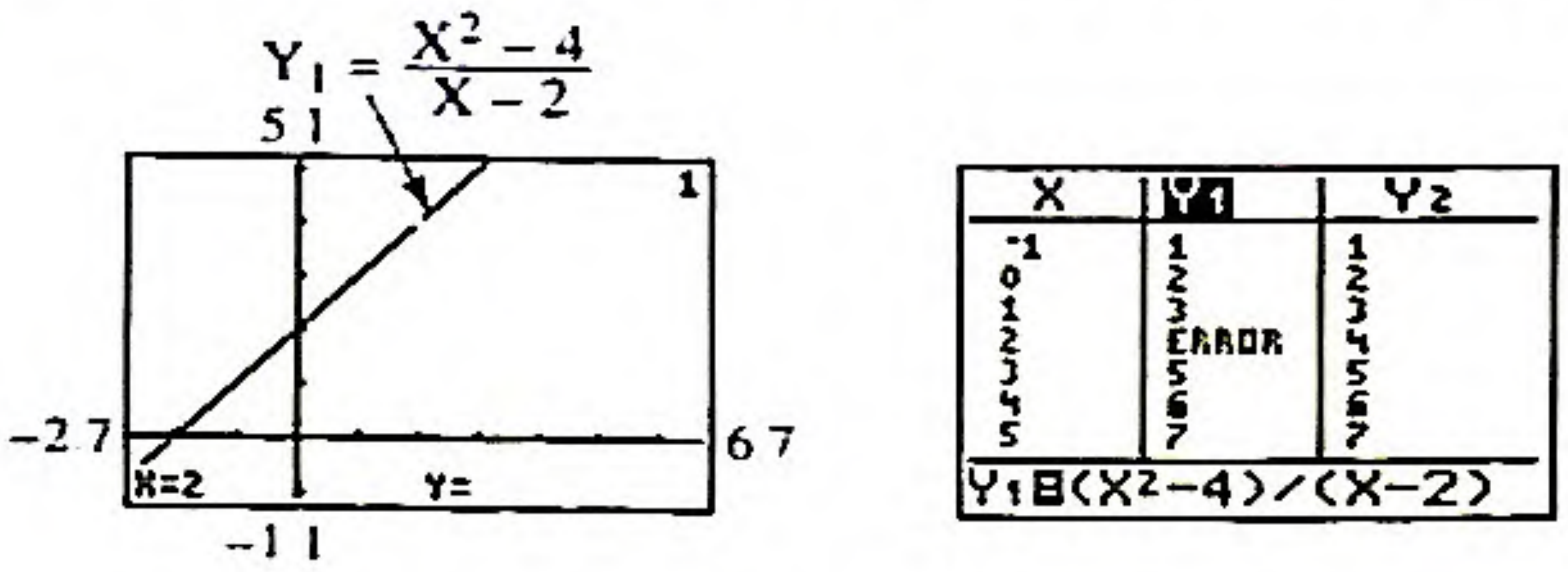
Figure 56

GRAPHING CALCULATOR SOLUTION

If we set the window of a graphing calculator so that an x-value of 2 is displayed, then we can see that the calculator cannot determine a value for y. We define

$$Y_1 = \frac{X^2 - 4}{X - 2}$$

and graph it in such a window, as in **Figure 57.** The error message in the table further supports the existence of a discontinuity at $X = 2$. (For the table, $Y_2 = X + 2$.)



X	Y1	Y2
-1	1	1
0	2	2
1	3	3
2	ERROR	4
3	5	5
4	6	6
5	7	7

Y1 B (X²−4)/(X−2)

Figure 57

Notice the visible discontinuity at $X = 2$ in the graph. The window was chosen so the "hole" would be visible. This requires a decimal viewing window or a window with x-values centered at 2. Other window choices may not show this discontinuity.

✔ *Now Try Exercise 91*

Rational Function Models

Rational functions have a variety of applications.

EXAMPLE 10 Modeling Traffic Intensity with a Rational Function

Vehicles arrive randomly at a parking ramp at an average rate of 2.6 vehicles per minute. The parking attendant can admit 3.2 vehicles per minute. However, since arrivals are random, lines form at various times. (*Source:* Mannering, F. and W. Kilareski, *Principles of Highway Engineering and Traffic Analysis*, 2nd ed., John Wiley & Sons.)

(a) The **traffic intensity** x is defined as the ratio of the average arrival rate to the average admittance rate. Determine x for this parking ramp.

(b) The average number of vehicles waiting in line to enter the ramp is given by

$$f(x) = \frac{x^2}{2(1-x)},$$

where $0 \le x < 1$ is the traffic intensity. Graph $f(x)$ and compute $f(0.8125)$ for this parking ramp.

(c) What happens to the number of vehicles waiting as the traffic intensity approaches 1?

SOLUTION

(a) The average arrival rate is 2.6 vehicles and the average admittance rate is 3.2 vehicles, so

$$x = \frac{2.6}{3.2} = 0.8125.$$

(b) A calculator graph of f is shown in **Figure 58.**

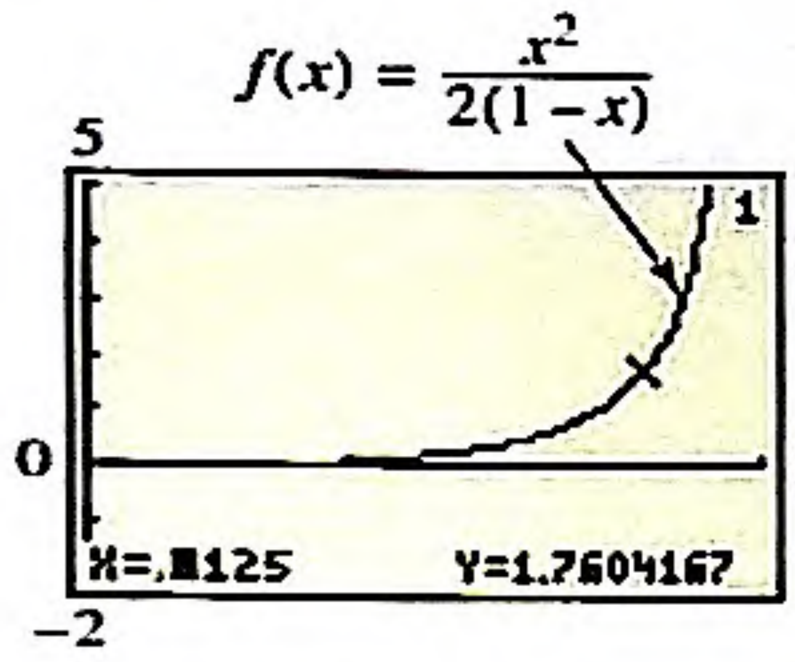


Figure 58

$$f(0.8125) = \frac{0.8125^2}{2(1-0.8125)} \approx 1.76 \text{ vehicles}$$

(c) From the graph we see that as x approaches 1, $y = f(x)$ gets very large. Thus, the average number of waiting vehicles gets very large. This is what we would expect.

✔ *Now Try Exercise 113.*

3.5 Exercises

Concept Check Provide a short answer to each question.

1. What is the domain of $f(x) = \frac{1}{x}$? What is its range?

2. What is the domain of $f(x) = \frac{1}{x^2}$? What is its range?

3. What is the interval over which $f(x) = \frac{1}{x}$ increases? decreases? is constant?

4. What is the interval over which $f(x) = \frac{1}{x^2}$ increases? decreases? is constant?

5. What is the equation of the vertical asymptote of the graph of $y = \frac{1}{x-3} + 2$? of the horizontal asymptote?

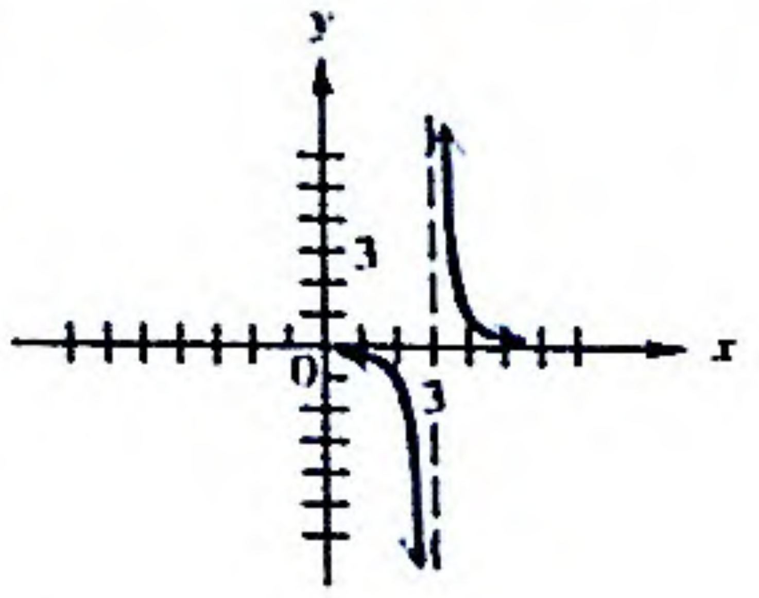
6. What is the equation of the vertical asymptote of the graph of $y = \frac{1}{(x+2)^2} - 4$? of the horizontal asymptote?

7. Is $f(x) = \frac{1}{x^2}$ an even or an odd function? What symmetry does its graph exhibit?

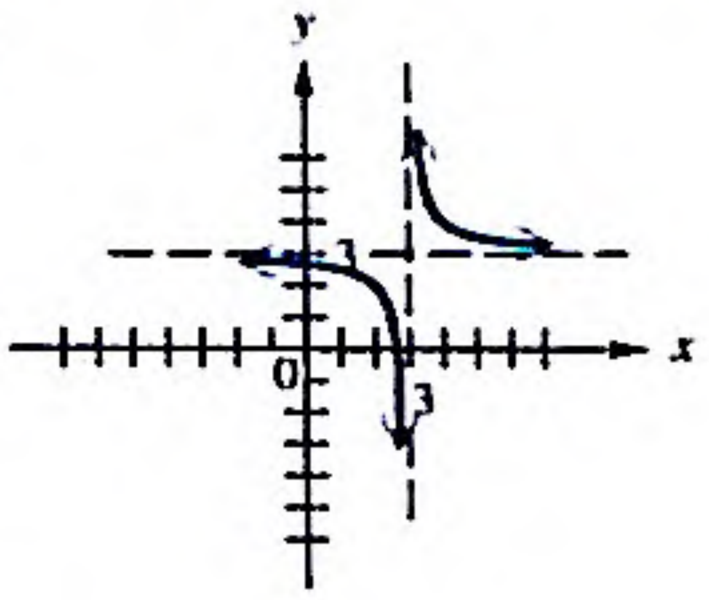
8. Is $f(x) = \frac{1}{x}$ an even or an odd function? What symmetry does its graph exhibit?

Concept Check *Use the graphs of the rational functions in choices A–D to answer each question in Exercises 9–16. There may be more than one correct choice.*

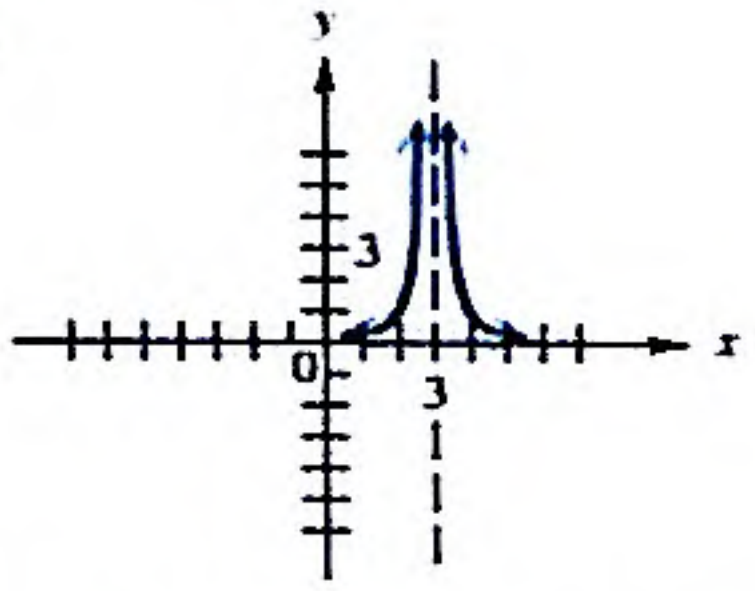
A.



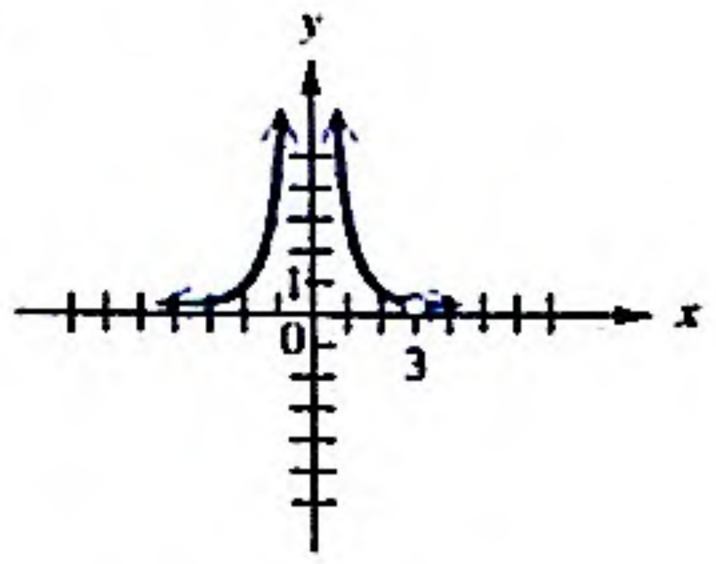
B.



C.



D.



9. Which choices have domain $(-\infty, 3) \cup (3, \infty)$?

10. Which choices have range $(-\infty, 3) \cup (3, \infty)$?

11. Which choices have range $(-\infty, 0) \cup (0, \infty)$?

12. Which choices have range $(0, \infty)$?

13. If f represents the function, only one choice has a single solution to the equation $f(x) = 3$. Which one is it?

14. Which choices have domain $(-\infty, 0) \cup (0, 3) \cup (3, \infty)$?

15. Which choices have the x-axis as a horizontal asymptote?

16. Which choices are symmetric with respect to a vertical line?

Explain how the graph of each function can be obtained from the graph of $y = \frac{1}{x}$ or $y = \frac{1}{x^2}$. Then graph f and give the (a) domain and (b) range. Determine the intervals of the domain for which the function is (c) increasing or (d) decreasing. See Examples 1–3.

17. $f(x) = \frac{2}{x}$

18. $f(x) = -\frac{3}{x}$

19. $f(x) = \frac{1}{x+2}$

20. $f(x) = \frac{1}{x-3}$

21. $f(x) = \frac{1}{x} + 1$

22. $f(x) = \frac{1}{x} - 2$

23. $f(x) = -\frac{2}{x^2}$

24. $f(x) = \frac{1}{x^2} + 3$

25. $f(x) = \frac{1}{(x-3)^2}$

26. $f(x) = \dfrac{-2}{(x-3)^2}$ **27.** $f(x) = \dfrac{-1}{(x+2)^2} - 3$ **28.** $f(x) = \dfrac{-1}{(x-4)^2} + 2$

Concept Check *Match the rational function in Column I with the appropriate description in Column II. Choices in Column II can be used only once.*

I	II
29. $f(x) = \dfrac{x+7}{x+1}$	**A.** The x-intercept is -3.
30. $f(x) = \dfrac{x+10}{x+2}$	**B.** The y-intercept is 5.
31. $f(x) = \dfrac{1}{x+4}$	**C.** The horizontal asymptote is $y = 4$.
32. $f(x) = \dfrac{-3}{x^2}$	**D.** The vertical asymptote is $x = -1$.
33. $f(x) = \dfrac{x^2-16}{x+4}$	**E.** There is a "hole" in its graph at $x = -4$.
34. $f(x) = \dfrac{4x+3}{x-7}$	**F.** The graph has an oblique asymptote.
35. $f(x) = \dfrac{x^2+3x+4}{x-5}$	**G.** The x-axis is its horizontal asymptote, and the y-axis is not its vertical asymptote.
36. $f(x) = \dfrac{x+3}{x-6}$	**H.** The x-axis is its horizontal asymptote, and the y-axis is its vertical asymptote.

Give the equations of any vertical, horizontal, or oblique asymptotes for the graph of each rational function. See Example 4.

37. $f(x) = \dfrac{3}{x-5}$ **38.** $f(x) = \dfrac{-6}{x+9}$ **39.** $f(x) = \dfrac{4-3x}{2x+1}$

40. $f(x) = \dfrac{2x+6}{x-4}$ **41.** $f(x) = \dfrac{x^2-1}{x+3}$ **42.** $f(x) = \dfrac{x^2+4}{x-1}$

43. $f(x) = \dfrac{x^2-2x-3}{2x^2-x-10}$ **44.** $f(x) = \dfrac{3x^2-6x-24}{5x^2-26x+5}$

45. $f(x) = \dfrac{x^2+1}{x^2+9}$ **46.** $f(x) = \dfrac{4x^2+25}{x^2+9}$

47. *Concept Check* Let f be the function whose graph is obtained by translating the graph of $y = \frac{1}{x}$ to the right 3 units and up 2 units.

(a) Write an equation for $f(x)$ as a quotient of two polynomials.
(b) Determine the zero(s) of f.
(c) Identify the asymptotes of the graph of $f(x)$.

48. *Concept Check* Repeat Exercise 47 with f the function whose graph is obtained by translating the graph of $y = -\frac{1}{x^2}$ to the left 3 units and up 1 unit.

49. *Concept Check* After the numerator is divided by the denominator,

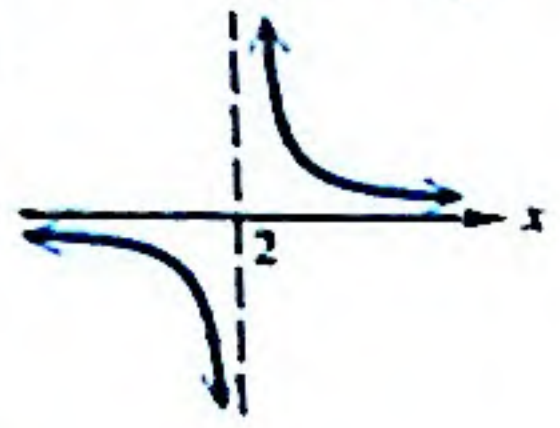
$$f(x) = \frac{x^5 + x^4 + x^2 + 1}{x^4 + 1} \quad \text{becomes} \quad f(x) = x + 1 + \frac{x^2 - x}{x^4 + 1}.$$

(a) What is the oblique asymptote of the graph of the function?
(b) Where does the graph of the function intersect its asymptote?
(c) As $x \to \infty$, does the graph of the function approach its asymptote from above or below?

50. *Concept Check* Choices A–D below show the four ways in which the graph of a rational function can approach the vertical line $x = 2$ as an asymptote. Identify the graph of each rational function defined in parts (a)–(d).

(a) $f(x) = \dfrac{1}{(x-2)^2}$ (b) $f(x) = \dfrac{1}{x-2}$ (c) $f(x) = \dfrac{-1}{x-2}$ (d) $f(x) = \dfrac{-1}{(x-2)^2}$

A.



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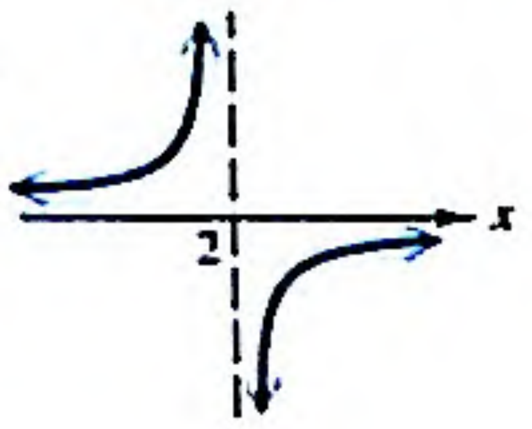

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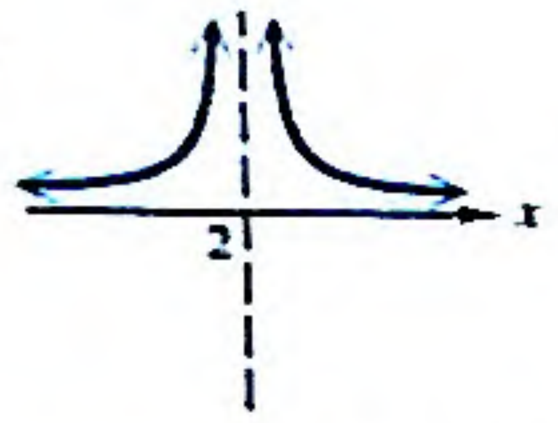

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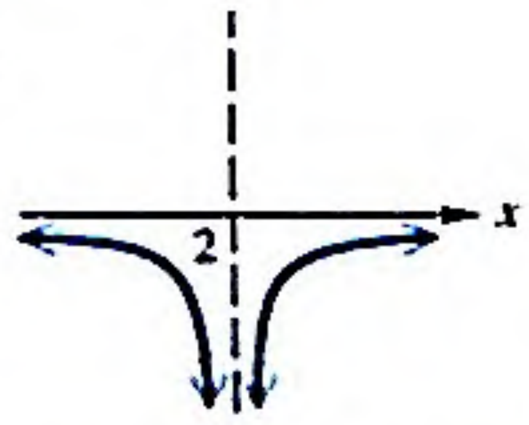


51. *Concept Check* Which function has a graph that does not have a vertical asymptote?

A. $f(x) = \dfrac{1}{x^2+2}$ B. $f(x) = \dfrac{1}{x^2-2}$ C. $f(x) = \dfrac{3}{x^2}$ D. $f(x) = \dfrac{2x+1}{x-8}$

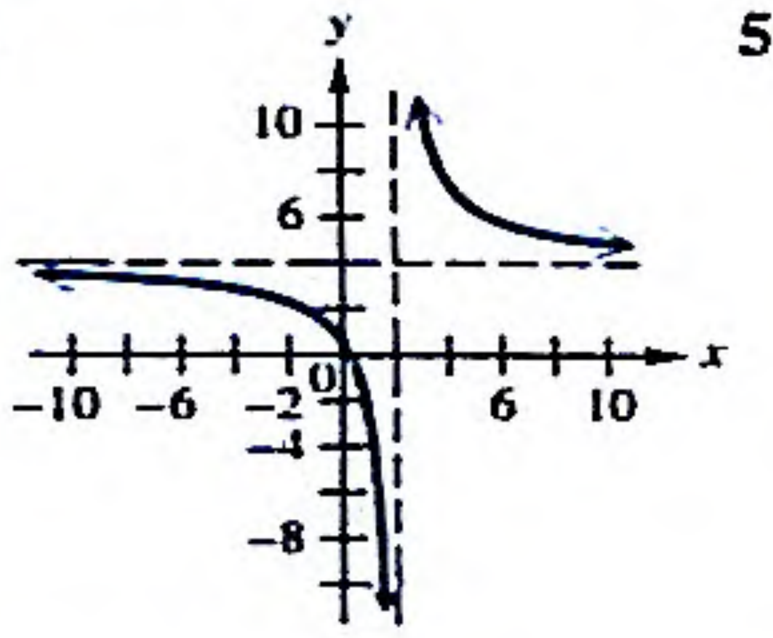
52. *Concept Check* Which function has a graph that does not have a horizontal asymptote?

A. $f(x) = \dfrac{2x-7}{x+3}$ B. $f(x) = \dfrac{3x}{x^2-9}$

C. $f(x) = \dfrac{x^2-9}{x+3}$ D. $f(x) = \dfrac{x+5}{(x+2)(x-3)}$

Identify any vertical, horizontal, or oblique asymptotes in the graph of $y = f(x)$. State the domain of f.

53.



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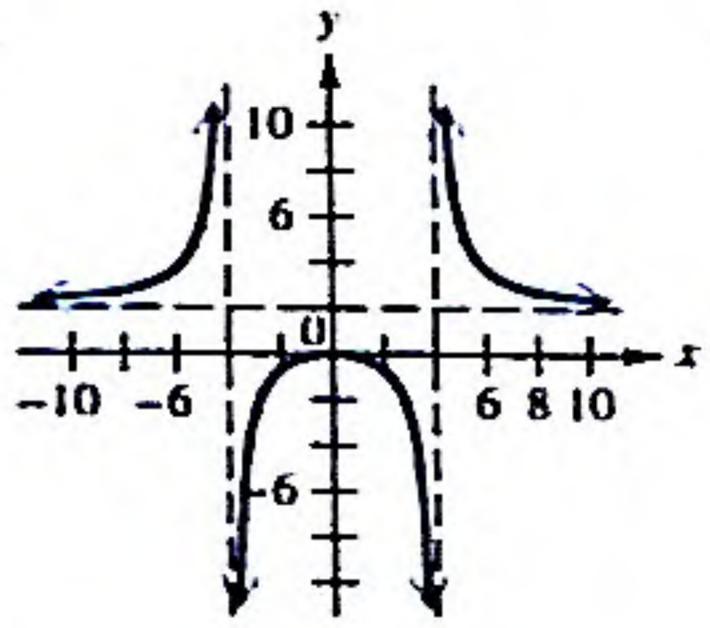

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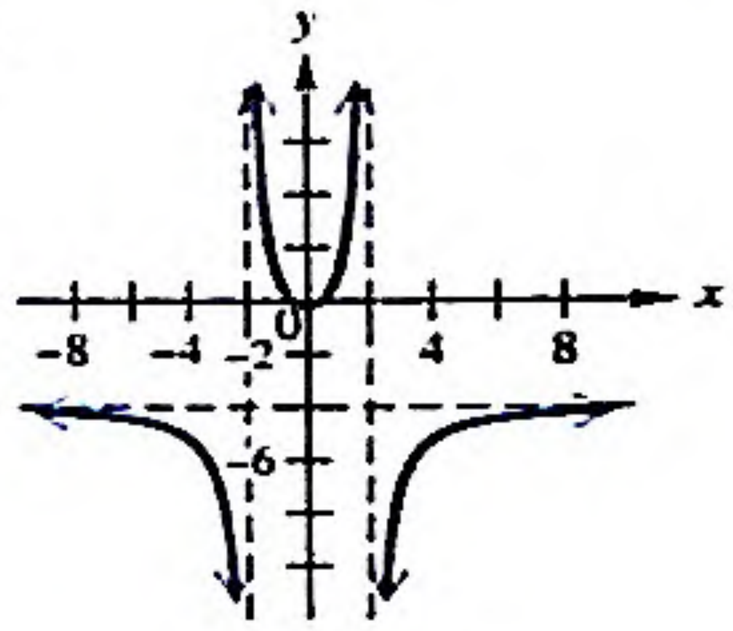

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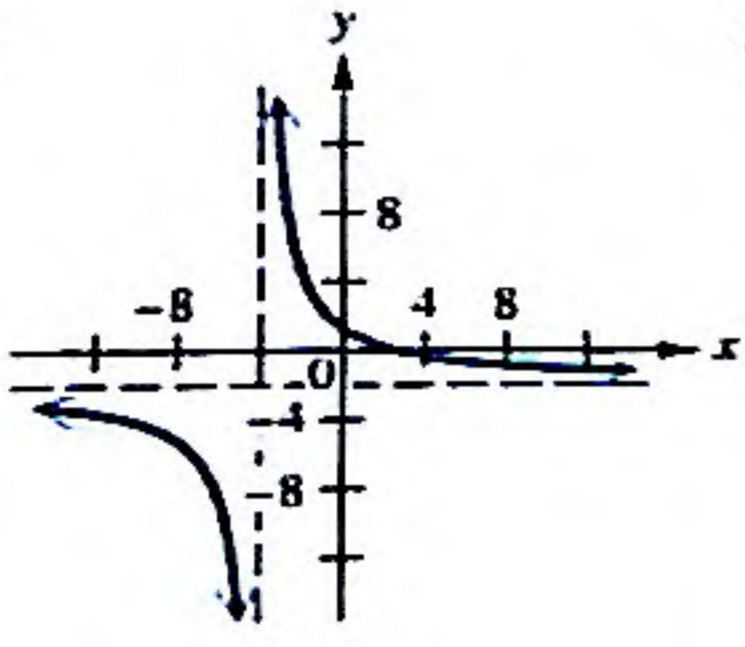

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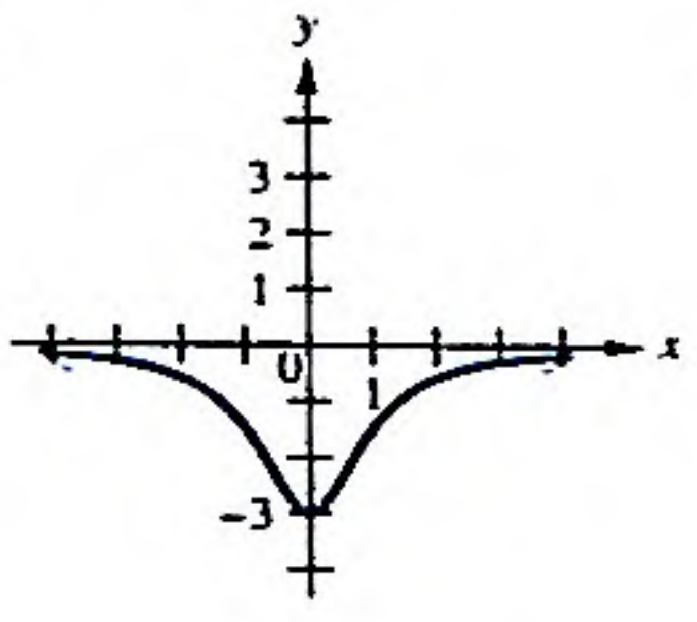

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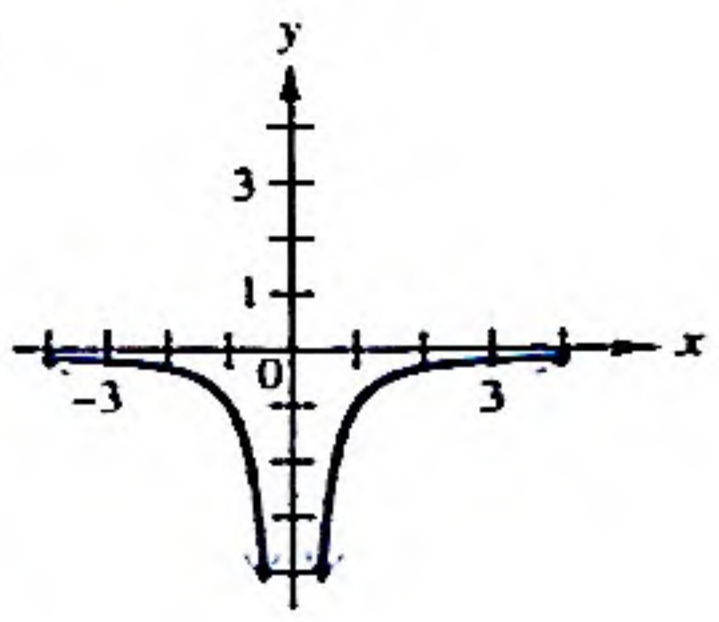

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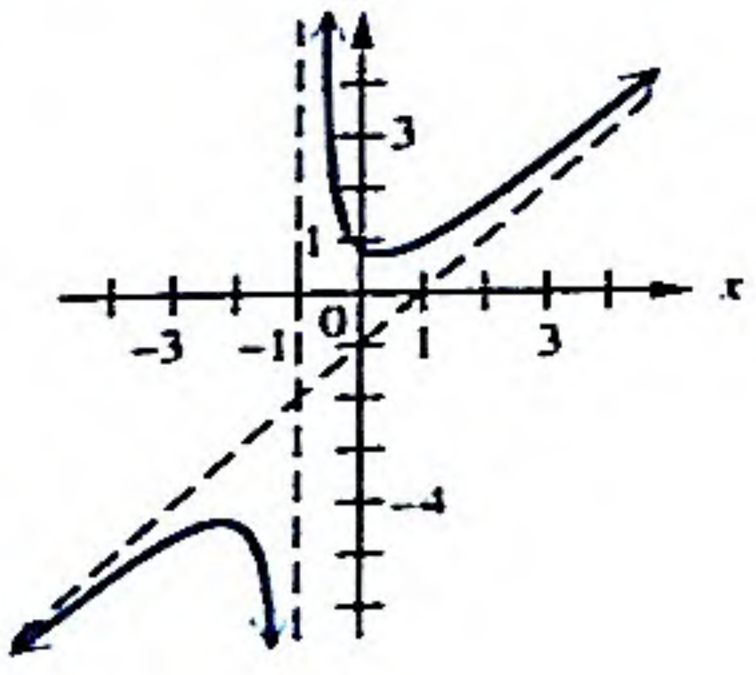

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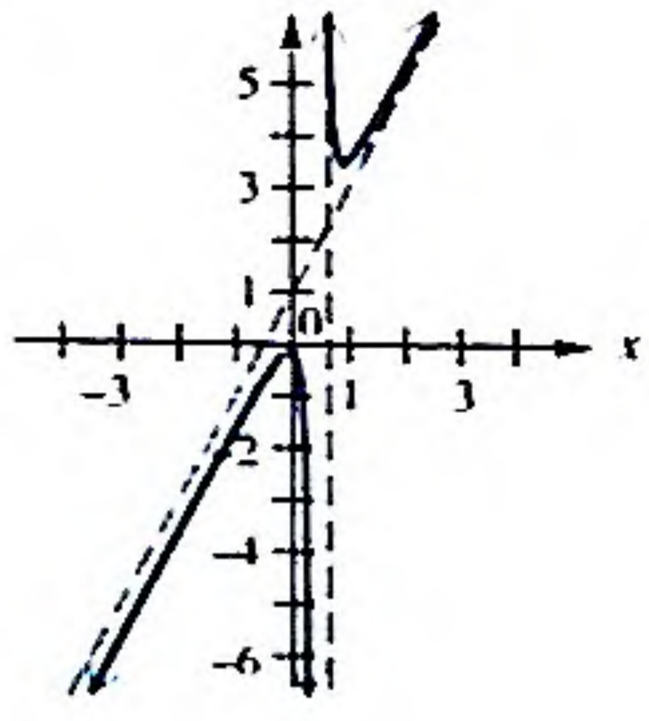


Sketch the graph of each rational function. See Examples 5–9.

61. $f(x) = \dfrac{x+1}{x-4}$

62. $f(x) = \dfrac{x-5}{x+3}$

63. $f(x) = \dfrac{x+2}{x-3}$

64. $f(x) = \dfrac{x-3}{x+4}$

65. $f(x) = \dfrac{4-2x}{8-x}$

66. $f(x) = \dfrac{6-3x}{4-x}$

67. $f(x) = \dfrac{3x}{x^2-x-2}$

68. $f(x) = \dfrac{2x+1}{x^2+6x+8}$

69. $f(x) = \dfrac{5x}{x^2-1}$

70. $f(x) = \dfrac{x}{4-x^2}$

71. $f(x) = \dfrac{(x+6)(x-2)}{(x+3)(x-4)}$

72. $f(x) = \dfrac{(x+3)(x-5)}{(x+1)(x-4)}$

73. $f(x) = \dfrac{3x^2+3x-6}{x^2-x-12}$

74. $f(x) = \dfrac{4x^2+4x-24}{x^2-3x-10}$

75. $f(x) = \dfrac{9x^2-1}{x^2-4}$

76. $f(x) = \dfrac{16x^2-9}{x^2-9}$

77. $f(x) = \dfrac{(x-3)(x+1)}{(x-1)^2}$

78. $f(x) = \dfrac{x(x-2)}{(x+3)^2}$

79. $f(x) = \dfrac{x}{x^2-9}$

80. $f(x) = \dfrac{-5}{2x+4}$

81. $f(x) = \dfrac{1}{x^2+1}$

82. $f(x) = \dfrac{(x-5)(x-2)}{x^2+9}$

83. $f(x) = \dfrac{(x+4)^2}{(x-1)(x+5)}$

84. $f(x) = \dfrac{(x+1)^2}{(x+2)(x-3)}$

85. $f(x) = \dfrac{20+6x-2x^2}{8+6x-2x^2}$

86. $f(x) = \dfrac{18+6x-4x^2}{4+6x+2x^2}$

87. $f(x) = \dfrac{x^2+1}{x+3}$

88. $f(x) = \dfrac{2x^2+3}{x-4}$

89. $f(x) = \dfrac{x^2+2x}{2x-1}$

90. $f(x) = \dfrac{x^2-x}{x+2}$

91. $f(x) = \dfrac{x^2-9}{x+3}$

92. $f(x) = \dfrac{x^2-16}{x+4}$

93. $f(x) = \dfrac{2x^2-5x-2}{x-2}$

94. $f(x) = \dfrac{x^2-5}{x-3}$

95. $f(x) = \dfrac{x^2-1}{x^2-4x+3}$

96. $f(x) = \dfrac{x^2-4}{x^2+3x+2}$

97. $f(x) = \dfrac{(x^2-9)(2+x)}{(x^2-4)(3+x)}$

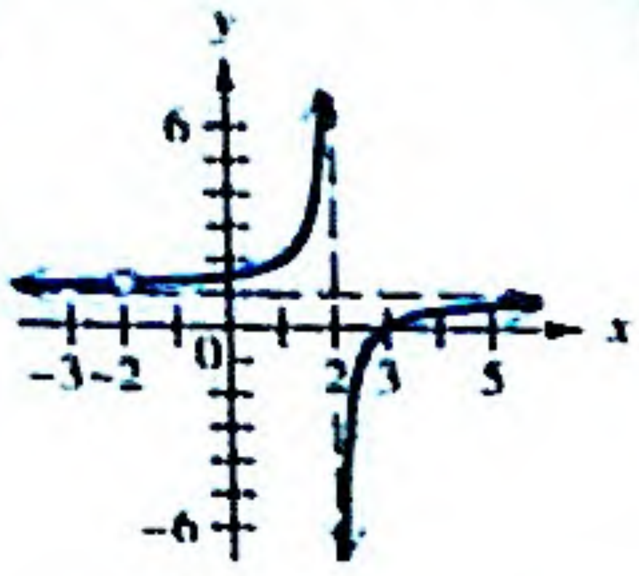
98. $f(x) = \dfrac{(x^2-16)(3+x)}{(x^2-9)(4+x)}$

99. $f(x) = \dfrac{x^4-20x^2+64}{x^4-10x^2+9}$

100. $f(x) = \dfrac{x^4-5x^2+4}{x^4-24x^2+108}$

Connecting Graphs with Equations *Find an equation for each rational function graph. (Hint: See the note preceding Example 8.)*

101.



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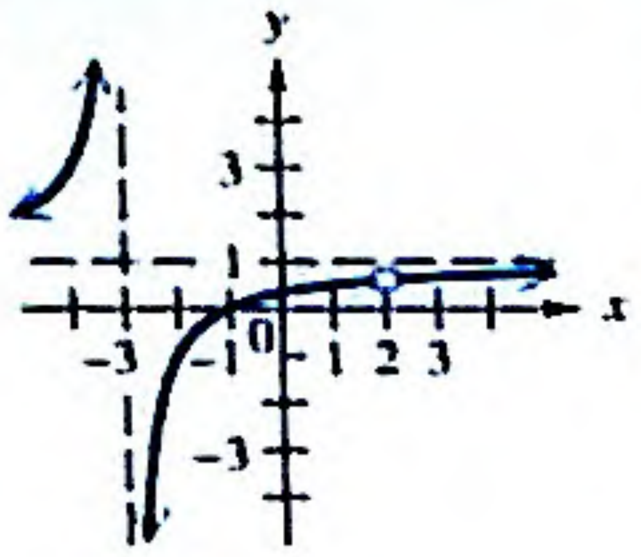

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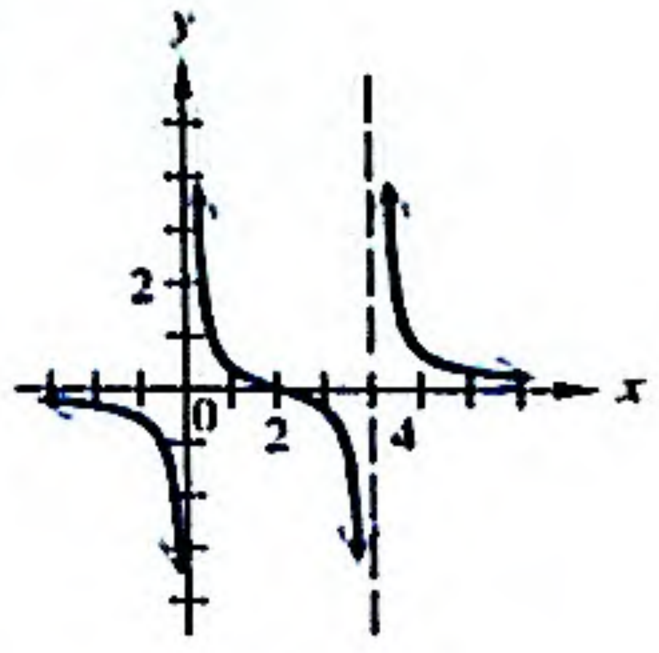

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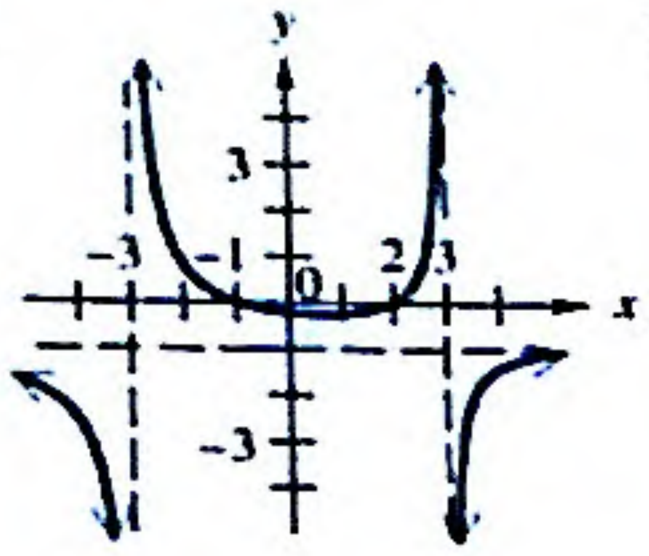

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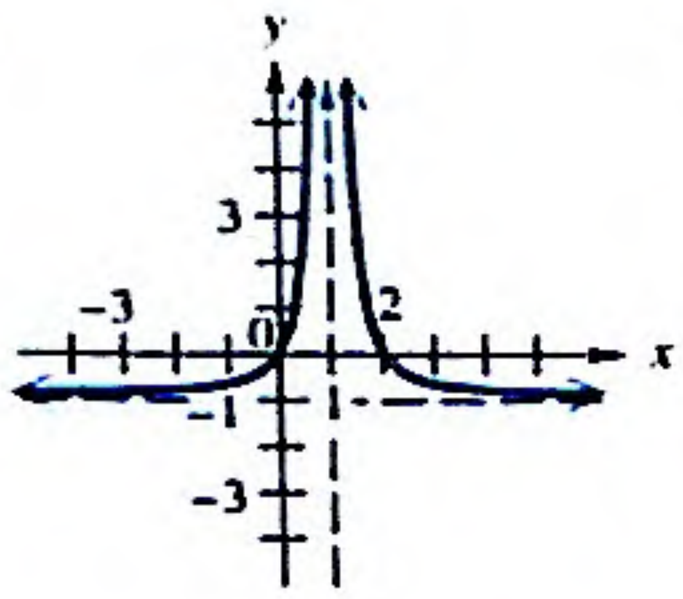

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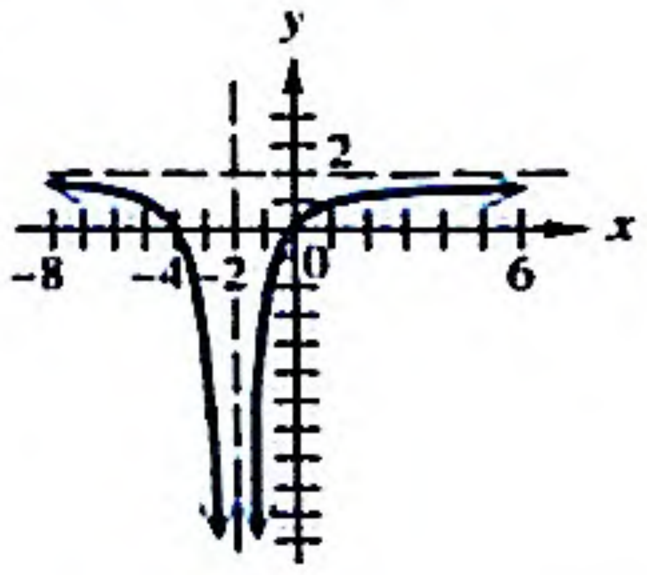


Concept Check *In Exercises 107 and 108, find a possible equation for the function with a graph having the given features.*

107. x-intercepts: -1 and 3
y-intercept: -3
vertical asymptote: $x = 1$
horizontal asymptote: $y = 1$

108. x-intercepts: 1 and 3
y-intercept: none
vertical asymptotes: $x = 0$ and $x = 2$
horizontal asymptote: $y = 1$

Use a graphing calculator to graph the rational function in the specified exercise. Then use the graph to find $f(1.25)$.

109. Exercise 61 **110.** Exercise 67 **111.** Exercise 89 **112.** Exercise 91

(Modeling) Solve each problem. See Example 10.

113. ***Traffic Intensity*** Let the average number of vehicles arriving at the gate of an amusement park per minute be equal to k, and let the average number of vehicles admitted by the park attendants be equal to r. Then, the average waiting time T (in minutes) for each vehicle arriving at the park is given by the rational function

$$T(r) = \frac{2r - k}{2r^2 - 2kr},$$

where $r > k$. (*Source:* Mannering, F., and W. Kilareski, *Principles of Highway Engineering and Traffic Analysis*, 2nd ed., John Wiley & Sons.)

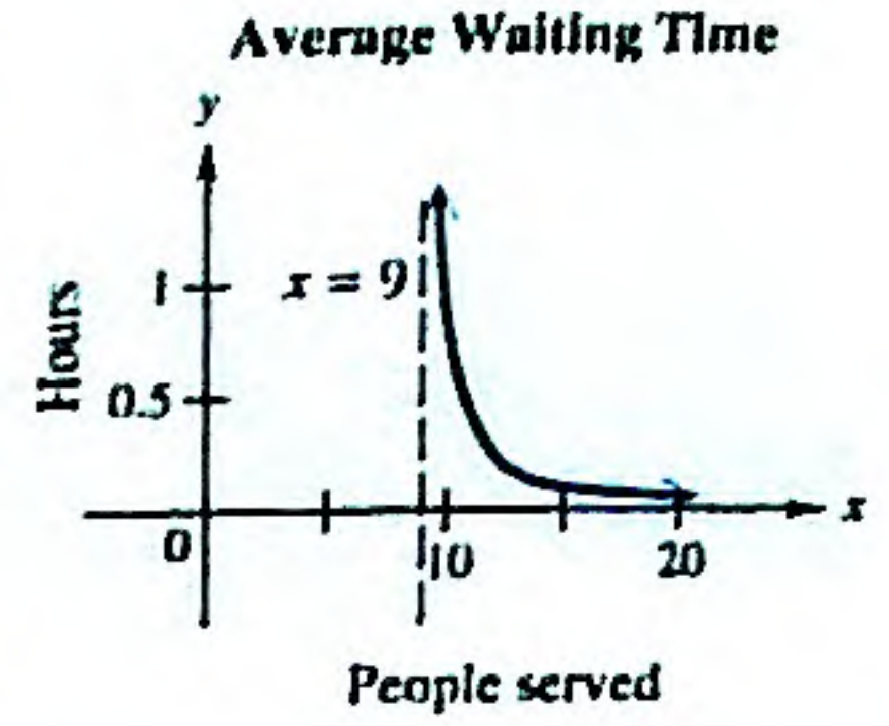
(a) It is known from experience that on Saturday afternoon $k = 25$. Use graphing to estimate the admittance rate r that is necessary to keep the average waiting time T for each vehicle to 30 sec.

(b) If one park attendant can serve 5.3 vehicles per minute, how many park attendants will be needed to keep the average wait to 30 sec?

114. ***Waiting in Line*** **Queuing theory** (also known as waiting-line theory) investigates the problem of providing adequate service economically to customers waiting in line. Suppose customers arrive at a fast-food service window at the rate of 9 people per hour. With reasonable assumptions, the average time (in hours) that a customer will wait in line before being served is modeled by

$$f(x) = \frac{9}{x(x-9)},$$

where x is the average number of people served per hour. A graph of $f(x)$ for $x > 9$ is shown in the figure on the next page.



(a) Why is the function meaningless if the average number of people served per hour is less than 9?

Suppose the average time to serve a customer is 5 min.

(b) How many customers can be served in an hour?

(c) How many minutes will a customer have to wait in line (on the average)?

(d) Suppose we want to halve the average waiting time to 7.5 min $\left(\frac{1}{8}\text{ hr}\right)$. How fast must an employee work to serve a customer (on the average)? (*Hint:* Let $f(x) = \frac{1}{8}$ and solve the equation for x. Convert your answer to minutes.) How might this reduction in serving time be accomplished?

115. ***Braking Distance*** The rational function

$$d(x) = \frac{8710x^2 - 69{,}400x + 470{,}000}{1.08x^2 - 324x + 82{,}200}$$

can be used to accurately model the braking distance for automobiles traveling at x miles per hour, where $20 \le x \le 70$. (*Source:* Mannering, F., and W. Kilareski, *Principles of Highway Engineering and Traffic Analysis*, 2nd ed., John Wiley & Sons.)

(a) Use graphing to estimate x when $d(x) = 300$.

(b) Complete the table for each value of x.

(c) If a car doubles its speed, does the braking distance double or more than double? Explain.

(d) Suppose that the automobile braking distance doubled whenever the speed doubled. What type of relationship would exist between the braking distance and the speed?

x	$d(x)$	x	$d(x)$
20		50	
25		55	
30		60	
35		65	
40		70	
45			

116. ***Braking Distance*** The grade x of a hill is a measure of its steepness. For example, if a road rises 10 ft for every 100 ft of horizontal distance, then it has an uphill grade of

$$x = \frac{10}{100}, \quad \text{or} \quad 10\%.$$

Grades are typically kept quite small—usually less than 10%. The braking distance D for a car traveling at 50 mph on a wet, uphill grade is given by

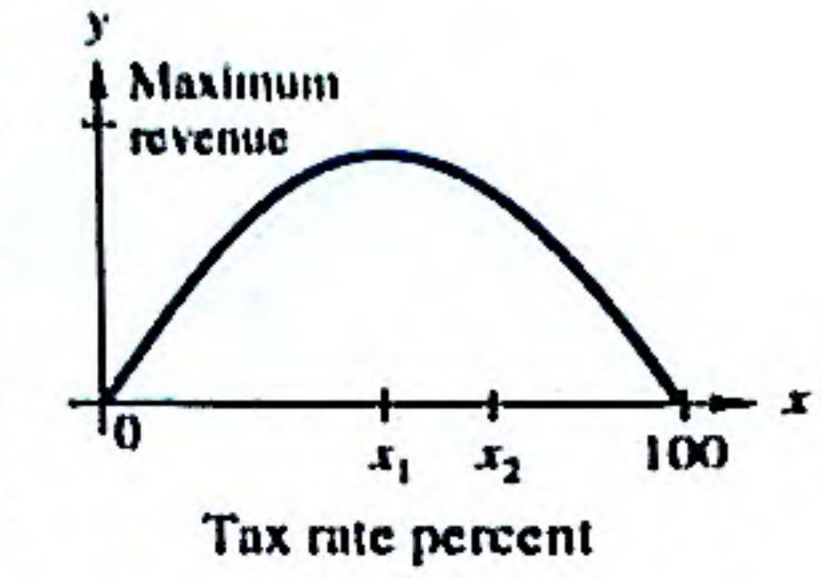
$$D(x) = \frac{2500}{30(0.3 + x)}.$$

(*Source:* Haefner, L., *Introduction to Transportation Systems*, Holt, Rinehart and Winston.)

(a) Evaluate $D(0.05)$ and interpret the result.

(b) Describe what happens to braking distance as the hill becomes steeper. Does this agree with your driving experience?

(c) Estimate the grade associated with a braking distance of 220 ft.



117. ***Tax Revenue*** Economist Arthur Laffer has been a center of controversy because of his Laffer curve, an idealized version of which is shown here. According to this curve, increasing a tax rate, say from x_1 percent to x_2 percent on the graph, can actually lead to a decrease in government revenue. All economists agree on the endpoints, 0 revenue at tax rates of both 0% and 100%, but there is much disagreement on the location of the rate x_1 that produces maximum revenue. Suppose an economist studying the Laffer curve produces the rational function

$$R(x) = \frac{80x - 8000}{x - 110},$$

where $R(x)$ is government revenue in tens of millions of dollars for a tax rate of x percent, with the function valid for $55 \le x \le 100$. Find the revenue for the following tax rates.

(a) 55% (b) 60% (c) 70% (d) 90% (e) 100%

(f) Graph R in the window $[0, 100]$ by $[0, 80]$.

118. ***Tax Revenue*** See Exercise 117. Suppose an economist determines that

$$R(x) = \frac{60x - 6000}{x - 120},$$

where $y = R(x)$ is government revenue in tens of millions of dollars for a tax rate of x percent, with $y = R(x)$ valid for $50 \le x \le 100$. Find the revenue for each tax rate.

(a) 50% (b) 60% (c) 80% (d) 100%

(e) Graph R in the window $[0, 100]$ by $[0, 50]$.

Relating Concepts

For individual or collaborative investigation *(Exercises 119–128)*

Consider the following "monster" rational function.

$$f(x) = \frac{x^4 - 3x^3 - 21x^2 + 43x + 60}{x^4 - 6x^3 + x^2 + 24x - 20}$$

Analyzing this function will synthesize many of the concepts of this and earlier chapters. Work Exercises 119–128 in order.

119. Find the equation of the horizontal asymptote.

120. Given that -4 and -1 are zeros of the numerator, factor the numerator completely.

121. (a) Given that 1 and 2 are zeros of the denominator, factor the denominator completely.

(b) Write the entire quotient for f so that the numerator and the denominator are in factored form.

122. (a) What is the common factor in the numerator and the denominator?

(b) For what value of x will there be a point of discontinuity (i.e., a "hole")?

123. What are the x-intercepts of the graph of f?

124. What is the y-intercept of the graph of f?

125. Find the equations of the vertical asymptotes

(continued)

126. Determine the point or points of intersection of the graph of f with its horizontal asymptote.

127. Sketch the graph of f.

128. Use the graph of f to solve each inequality.

(a) $f(x) < 0$ **(b)** $f(x) > 0$

Chapter 3 Quiz (Sections 3.1–3.5)

1. Graph each quadratic function. Give the vertex, axis, domain, range, and intervals of the domain over which each function is increasing or decreasing.

   (a) $f(x) = -2(x + 3)^2 - 1$ **(b)** $f(x) = 2x^2 - 8x + 3$

2. ***Height of a Projected Object*** A ball is projected directly upward from an initial height of 200 ft with an initial velocity of 64 ft per sec.

   (a) Use the projectile height function $s(t) = -16t^2 + v_0t + s_0$ to describe the height of the ball in terms of time t.

   (b) For what interval of time is the height of the ball greater than 240 ft?

Use synthetic division to decide whether the given number k is a zero of the polynomial function. If it is not, give the value of f(k).

3. $f(x) = 2x^4 + x^3 - 3x + 4$; $k = 2$ 4. $f(x) = x^2 - 4x + 5$; $k = 2 + i$

5. Find a polynomial of least degree having only real coefficients with zeros -2, 3, and $3 - i$.

Graph each polynomial function. Factor first if the function is not in factored form.

6. $f(x) = x(x - 2)^3(x + 2)^2$

7. $f(x) = 2x^4 - 9x^3 - 5x^2 + 57x - 45$

8. $f(x) = -4x^5 + 16x^4 + 13x^3 - 76x^2 - 3x + 18$

Sketch the graph of each rational function.

9. $f(x) = \dfrac{3x + 1}{x^2 + 7x + 10}$ 10. $f(x) = \dfrac{x^2 + 2x + 1}{x - 1}$

Summary Exercises on Solving Equations and Inequalities

In **Section 1.7** we solved rational inequalities by rewriting them so that 0 is on one side. We then determined the values that cause either the numerator or denominator to equal 0, and by using a test value from each interval determined by these values, we identified the solution set.

We can now solve rational inequalities by inspecting the graph of a related function. The graphs can be obtained by using technology or by using the steps for graphing a rational function given in **Section 3.5**.

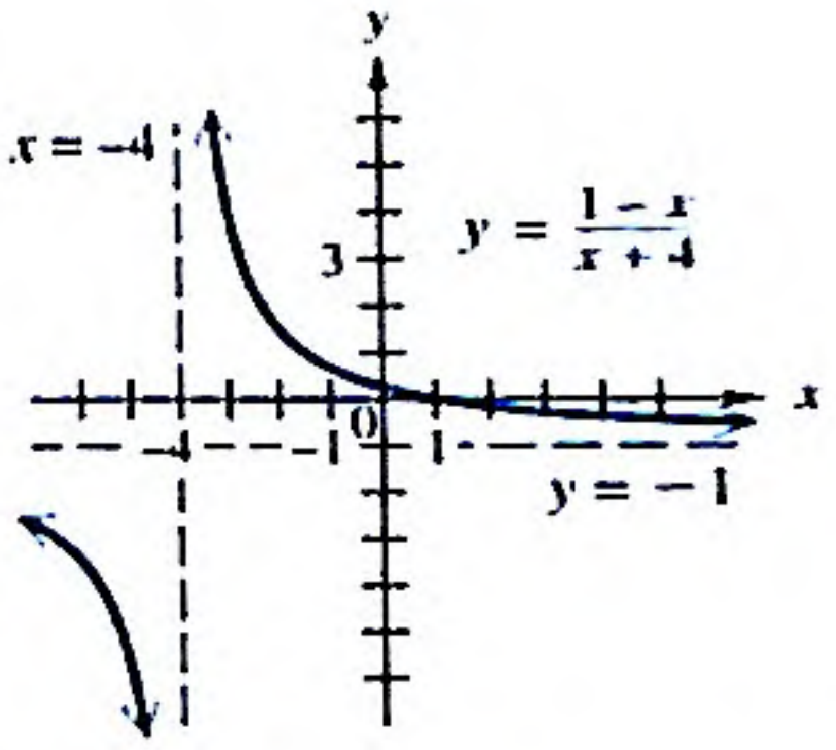
EXAMPLE Solving a Rational Inequality

Solve the inequality.

$$\frac{1-x}{x+4} \geq 0$$

SOLUTION Graph the related rational function

$$y = \frac{1-x}{x+4}.$$



The real solutions of $\frac{1-x}{x+4} \geq 0$ are the x-values for which the graph lies above or on the x-axis. This is true for all x to the right of the vertical asymptote at $x = -4$, up to and including the x-intercept 1. Therefore, the solution set of the inequality is $(-4, 1]$.

By inspecting the graph of the related function, we can also determine that the solution set of $\frac{1-x}{x+4} < 0$ is $(-\infty, -4) \cup (1, \infty)$ and that the solution set of the equation $\frac{1-x}{x+4} = 0$ is $\{1\}$, the value of the x-intercept. This graphical method may be used to solve other equations and inequalities including those defined by polynomials.

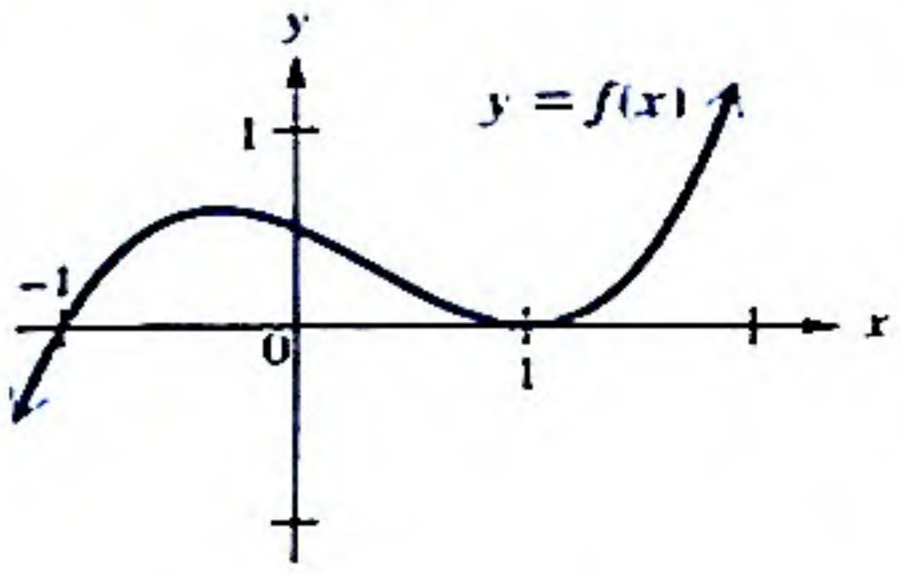
✔ *Now Try Exercise 19.*

In Exercises 1–8, solve each equation.

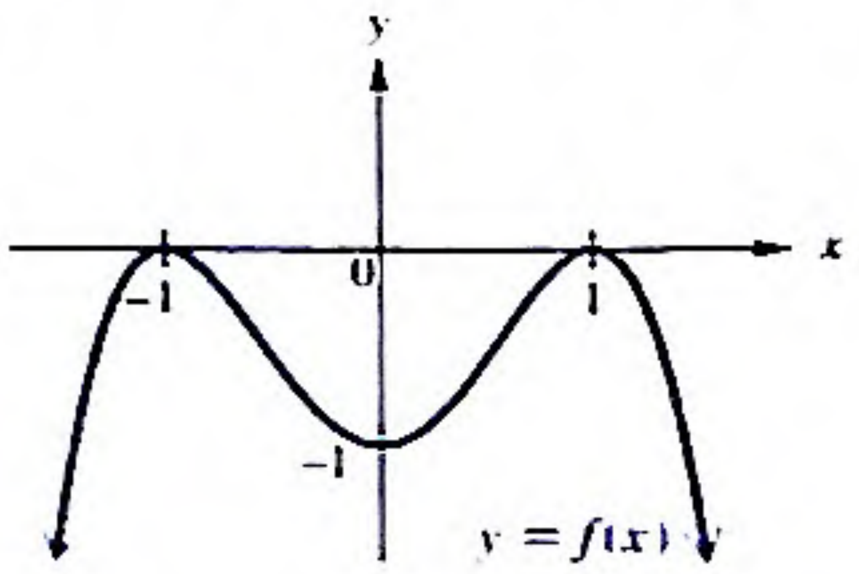
1. $\frac{5x+8}{-2} = 2x - 10$
2. $\frac{1}{5}x + 0.25x = \frac{1}{2}x - 1$
3. $(x-5)^{-4} - 13(x-5)^{-2} = -36$
4. $x = 13\sqrt{x} - 40$
5. $\sqrt{2x-5} - \sqrt{x-3} = 1$
6. $3 = \sqrt{x+2} + \sqrt{x-1}$
7. $x^{2/3} + \frac{1}{2} = \frac{3}{4}$
8. $27 - (x-4)^{3/2} = 0$

In Exercises 9–12, use the graph of the function to solve each equation or inequality.

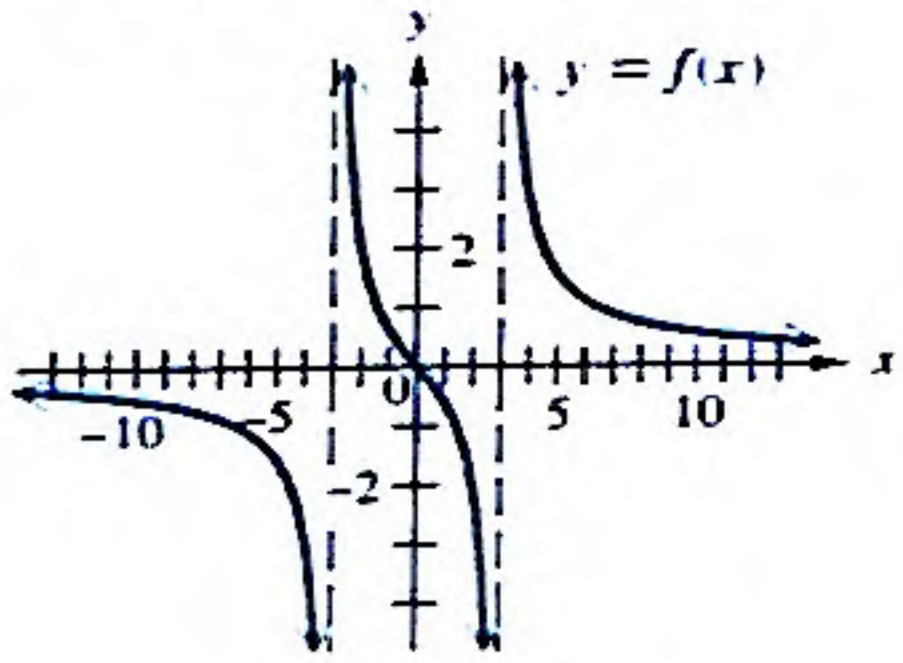
9. (a) $f(x) > 0$ (b) $f(x) \leq 0$



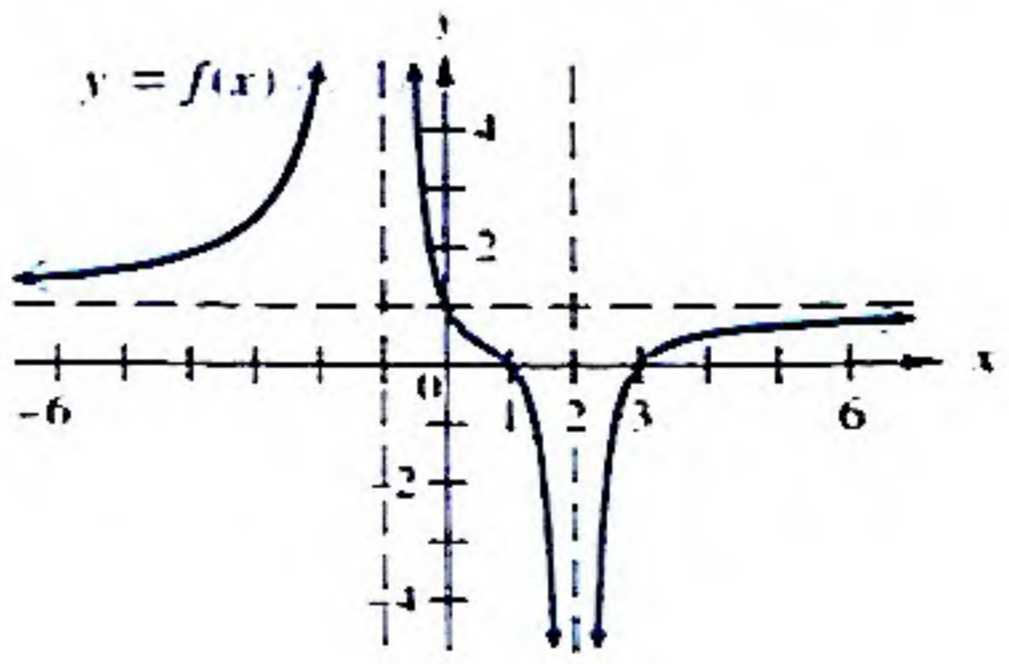
10. (a) $f(x) < 0$ (b) $f(x) > 0$



11. (a) $f(x) = 0$ (b) $f(x) > 0$



12. (a) $f(x) = 0$ (b) $f(x) \leq 0$



In Exercises 13–22, sketch the graph of an appropriate function and then use the graph to solve each equation or inequality. (Note: First identify the expression as an equation or an inequality. When appropriate, use the steps for graphing polynomial functions from Section 3.4 or rational functions from Section 3.5.)

13. $25x^2 - 20x + 4 > 0$

14. $3x + 4 \le x^2$

15. $x^4 - 2x^3 - 3x^2 + 4x + 4 = 0$

16. $x^3 + 5x^2 + 3x - 9 \ge 0$

17. $-x^4 - x^3 + 12x^2 = 0$

18. $-4x^4 + 13x^2 - 3 > 0$

19. $\dfrac{2x^2 - 13x + 15}{x^2 - 3x} \ge 0$

20. $\dfrac{x^2 + 3x - 1}{x + 1} > 3$

21. $\dfrac{x - 1}{(x - 3)^2} = 0$

22. $\dfrac{x}{x^2 - 4} > 0$

3.6 Variation

- **Direct Variation**
- **Inverse Variation**
- **Combined and Joint Variation**

To apply mathematics we often need to express relationships between quantities. For example,

- In chemistry, the ideal gas law describes how temperature, pressure, and volume are related.
- In physics, various formulas in optics describe the relationship between the focal length of a lens and the size of an image.

This section introduces some special applications of polynomial and rational functions.

Direct Variation When one quantity is a constant multiple of another quantity, the two quantities are said to *vary directly*. For example, if you work for an hourly wage of \$10, then

$$[\text{pay}] = 10[\text{hours worked}].$$

Doubling the hours doubles the pay. Tripling the hours triples the pay, and so on. This is stated more precisely as follows.

Direct Variation

y **varies directly** as x, or y is **directly proportional** to x, if there exists a nonzero real number k, called the **constant of variation,** such that for all x,

$$y = kx.$$

The direct variation equation $y = kx$ defines a linear function, where the constant of variation k is the slope of the line. For $k > 0$,

- As the value of x *increases,* the value of y *increases.*
- As the value of x *decreases,* the value of y *decreases.*

When used to describe a direct variation relationship, the phrase "directly proportional" is sometimes abbreviated to just "proportional."

CAUTION Note that *and* in the expression "y varies jointly as x and z" translates as the product $y = kxz$. ***The word "and" does not indicate addition here.***

EXAMPLE 3 Solving a Joint Variation Problem

The area of a triangle varies jointly as the lengths of the base and the height. A triangle with base 10 ft and height 4 ft has area 20 ft². Find the area of a triangle with base 3 ft and height 8 ft. (See **Figure 60.**)

SOLUTION

Step 1 Let $\mathcal{A}$ represent the area, b the base, and h the height of the triangle. Then, for some number k,

$$\mathcal{A} = kbh. \quad \text{$\mathcal{A}$ varies jointly as b and h.}$$

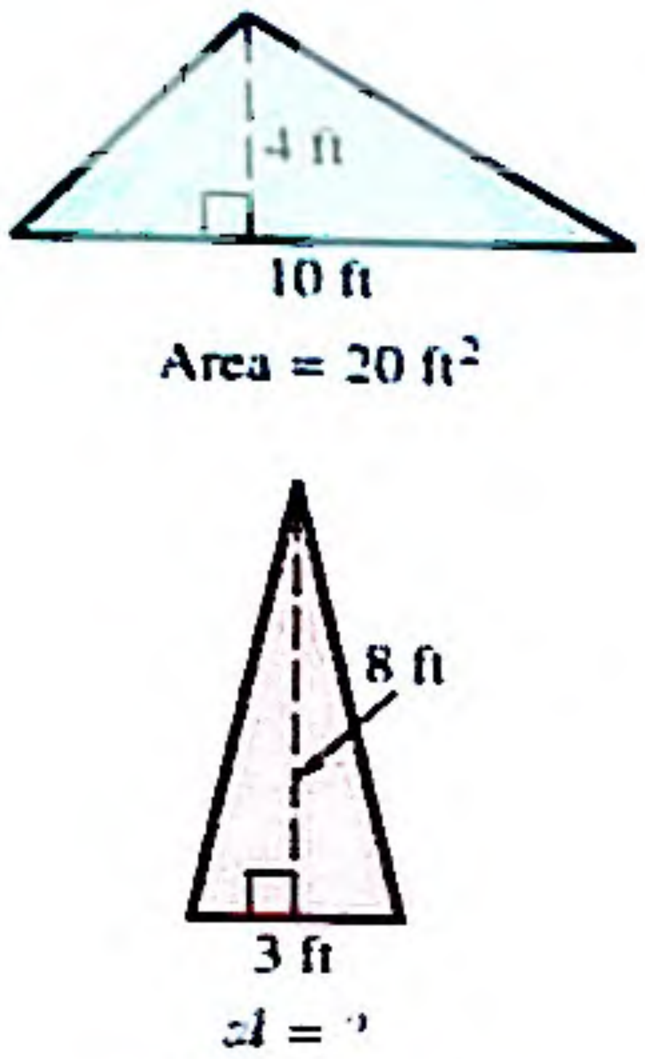


Figure 60

Step 2 Since $\mathcal{A}$ is 20 when b is 10 and h is 4, substitute and solve for k.

$$20 = k(10)(4)$$

$$\frac{1}{2} = k$$

Step 3 The relationship among the variables is the familiar formula for the area of a triangle,

$$\mathcal{A} = \frac{1}{2}bh.$$

Step 4 To find $\mathcal{A}$ when $b = 3$ ft and $h = 8$ ft, substitute into the formula.

$$\mathcal{A} = \frac{1}{2}(3)(8) = 12 \text{ ft}^2$$

✔ ***Now Try Exercise 33.***

EXAMPLE 4 Solving a Combined Variation Problem

The number of vibrations per second (the pitch) of a steel guitar string varies directly as the square root of the tension and inversely as the length of the string. If the number of vibrations per second is 50 when the tension is 225 newtons and the length is 0.60 m, find the number of vibrations per second when the tension is 196 newtons and the length is 0.65 m.

SOLUTION Let n represent the number of vibrations per second, T represent the tension, and L represent the length of the string. Then, from the information in the problem, write the variation equation (Step 1).

$$n = \frac{k\sqrt{T}}{L} \quad \text{n varies directly as the square root of T and inversely as L.}$$

Substitute the given values for n, T, and L to find k (Step 2).

$$50 = \frac{k\sqrt{225}}{0.60} \quad \text{Let } n = 50,\ T = 225,\ L = 0.60.$$

$$30 = k\sqrt{225} \quad \text{Multiply by 0.60.}$$

$$30 = 15k \quad \sqrt{225} = 15$$

$$k = 2 \quad \text{Divide by 15. Interchange sides.}$$

Substitute for k to find the relationship among the variables (Step 3).

$$n = \frac{2\sqrt{T}}{L}$$

Now use the second set of values for T and L to find n (Step 4).

$$n = \frac{2\sqrt{196}}{0.65} \approx 43 \quad \text{Let } T = 196,\ L = 0.65.$$

The number of vibrations per second is approximately 43.

✔ *Now Try Exercise 37.*

3.6 Exercises

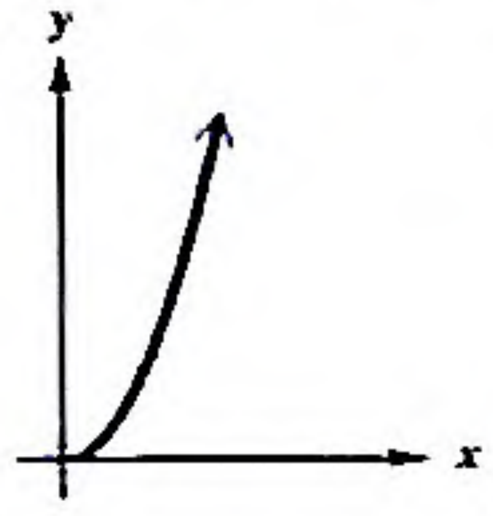
Concept Check *Write each formula as an English phrase using the word* varies *or* proportional.

1. $C = 2\pi r$, where C is the circumference of a circle of radius r
2. $d = \frac{1}{5}s$, where d is the approximate distance (in miles) from a storm, and s is the number of seconds between seeing lightning and hearing thunder
3. $r = \frac{d}{t}$, where r is the speed when traveling d miles in t hours
4. $d = \dfrac{1}{4\pi n r^2}$, where d is the distance a gas atom of radius r travels between collisions, and n is the number of atoms per unit volume
5. $s = kx^3$, where s is the strength of a muscle that has length x
6. $f = \dfrac{mv^2}{r}$, where f is the centripetal force of an object of mass m moving along a circle of radius r at velocity v

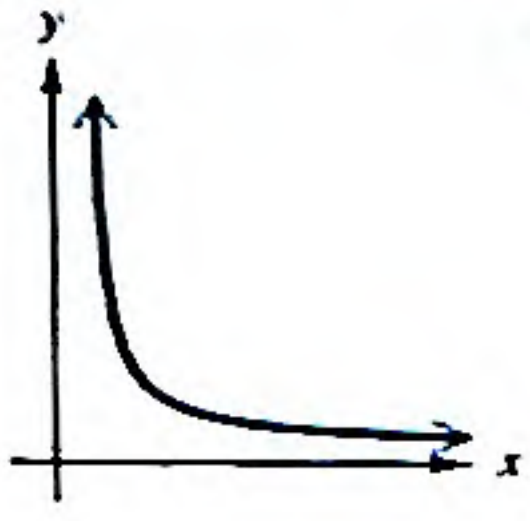
Concept Check *Match each statement in Exercises 7–10 with its corresponding graph in choices A–D. In each case, $k > 0$.*

7. y varies directly as x. $(y = kx)$
8. y varies inversely as x. $\left(y = \frac{k}{x}\right)$
9. y varies directly as the second power of x. $(y = kx^2)$
10. x varies directly as the second power of y. $(x = ky^2)$

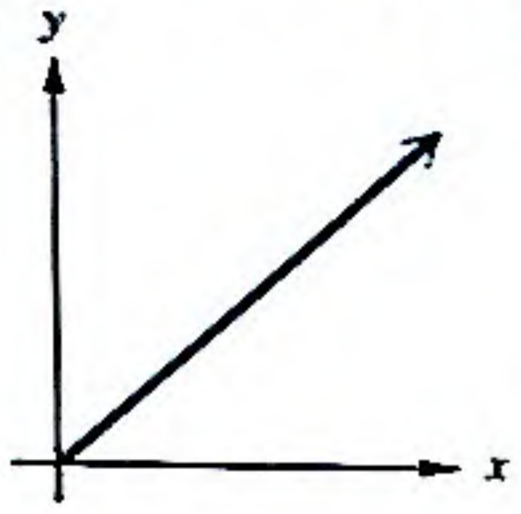
A.



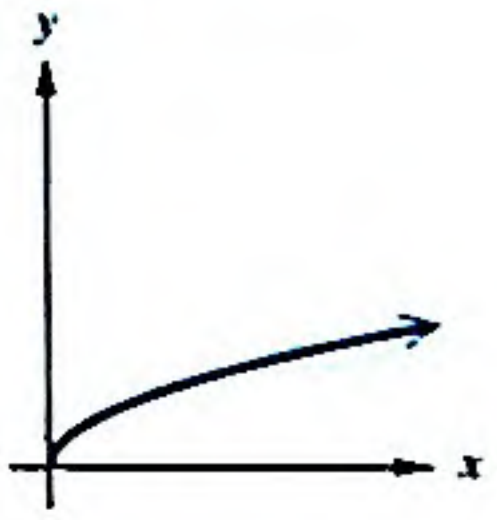
B.



C.



D.



Solve each variation problem. See Examples 1–4.

11. If y varies directly as x, and $y = 20$ when $x = 4$, find y when $x = -6$.
12. If y varies directly as x, and $y = 9$ when $x = 30$, find y when $x = 40$.
13. If m varies jointly as x and y, and $m = 10$ when $x = 2$ and $y = 14$, find m when $x = 21$ and $y = 8$.
14. If m varies jointly as z and p, and $m = 10$ when $z = 2$ and $p = 7.5$, find m when $z = 6$ and $p = 9$.

15. If y varies inversely as x, and $y = 10$ when $x = 3$, find y when $x = 20$.

16. If y varies inversely as x, and $y = 20$ when $x = \frac{1}{4}$, find y when $x = 15$.

17. Suppose r varies directly as the square of m, and inversely as s. If $r = 12$ when $m = 6$ and $s = 4$, find r when $m = 6$ and $s = 20$.

18. Suppose p varies directly as the square of z, and inversely as r. If $p = \frac{32}{5}$ when $z = 4$ and $r = 10$, find p when $z = 3$ and $r = 32$.

19. Let a be directly proportional to m and n^2, and inversely proportional to y^3. If $a = 9$ when $m = 4$, $n = 9$, and $y = 3$, find a when $m = 6$, $n = 2$, and $y = 5$.

20. If y varies directly as x, and inversely as m^2 and r^2, and $y = \frac{5}{3}$ when $x = 1$, $m = 2$, and $r = 3$, find y when $x = 3$, $m = 1$, and $r = 8$.

Solve each problem. See Examples 1–4.

21. *Circumference of a Circle* The circumference of a circle varies directly as the radius. A circle with radius 7 in. has circumference 43.96 in. Find the circumference of the circle if the radius changes to 11 in.

22. *Pressure Exerted by a Liquid* The pressure exerted by a certain liquid at a given point varies directly as the depth of the point beneath the surface of the liquid. The pressure at 10 ft is 50 pounds per square inch (psi). What is the pressure at 15 ft?

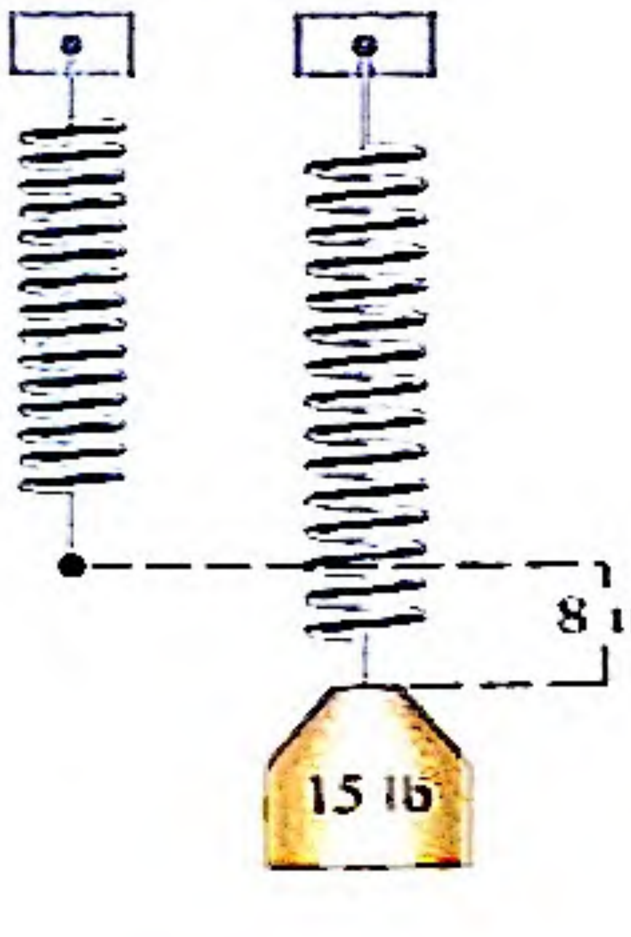
23. *Resistance of a Wire* The resistance in ohms of a platinum wire temperature sensor varies directly as the temperature in kelvins (K). If the resistance is 646 ohms at a temperature of 190 K, find the resistance at a temperature of 250 K.

24. *Weight on the Moon* The weight of an object on Earth is directly proportional to the weight of that same object on the moon. A 200-lb astronaut would weigh 32 lb on the moon. How much would a 50-lb dog weigh on the moon?

25. *Distance to the Horizon* The distance that a person can see to the horizon on a clear day from a point above the surface of Earth varies directly as the square root of the height at that point. If a person 144 m above the surface of Earth can see 18 km to the horizon, how far can a person see to the horizon from a point 64 m above the surface?

26. *Water Emptied by a Pipe* The amount of water emptied by a pipe varies directly as the square of the diameter of the pipe. For a certain constant water flow, a pipe emptying into a canal will allow 200 gal of water to escape in an hour. The diameter of the pipe is 6 in. How much water would a 12-in. pipe empty into the canal in an hour, assuming the same water flow?

27. *Hooke's Law for a Spring* Hooke's law for an elastic spring states that the distance a spring stretches varies directly as the force applied. If a force of 15 lb stretches a certain spring 8 in., how much will a force of 30 lb stretch the spring?



28. *Current in a Circuit* The current in a simple electrical circuit varies inversely as the resistance. If the current is 50 amps when the resistance is 10 ohms, find the current if the resistance is 5 ohms.

29. *Speed of a Pulley* The speed of a pulley varies inversely as its diameter. One kind of pulley, with diameter 3 in., turns at 150 revolutions per minute. Find the speed of a similar pulley with diameter 5 in.

30. *Weight of an Object* The weight of an object varies inversely as the square of its distance from the center of Earth. If an object 8000 mi from the center of Earth weighs 90 lb, find its weight when it is 12,000 mi from the center of Earth.

31. *Current Flow* In electric current flow, it is found that the resistance offered by a fixed length of wire of a given material varies inversely as the square of the diameter of the wire. If a wire 0.01 in. in diameter has a resistance of 0.4 ohm, what is the resistance of a wire of the same length and material with diameter 0.03 in., to the nearest ten-thousandth of an ohm?

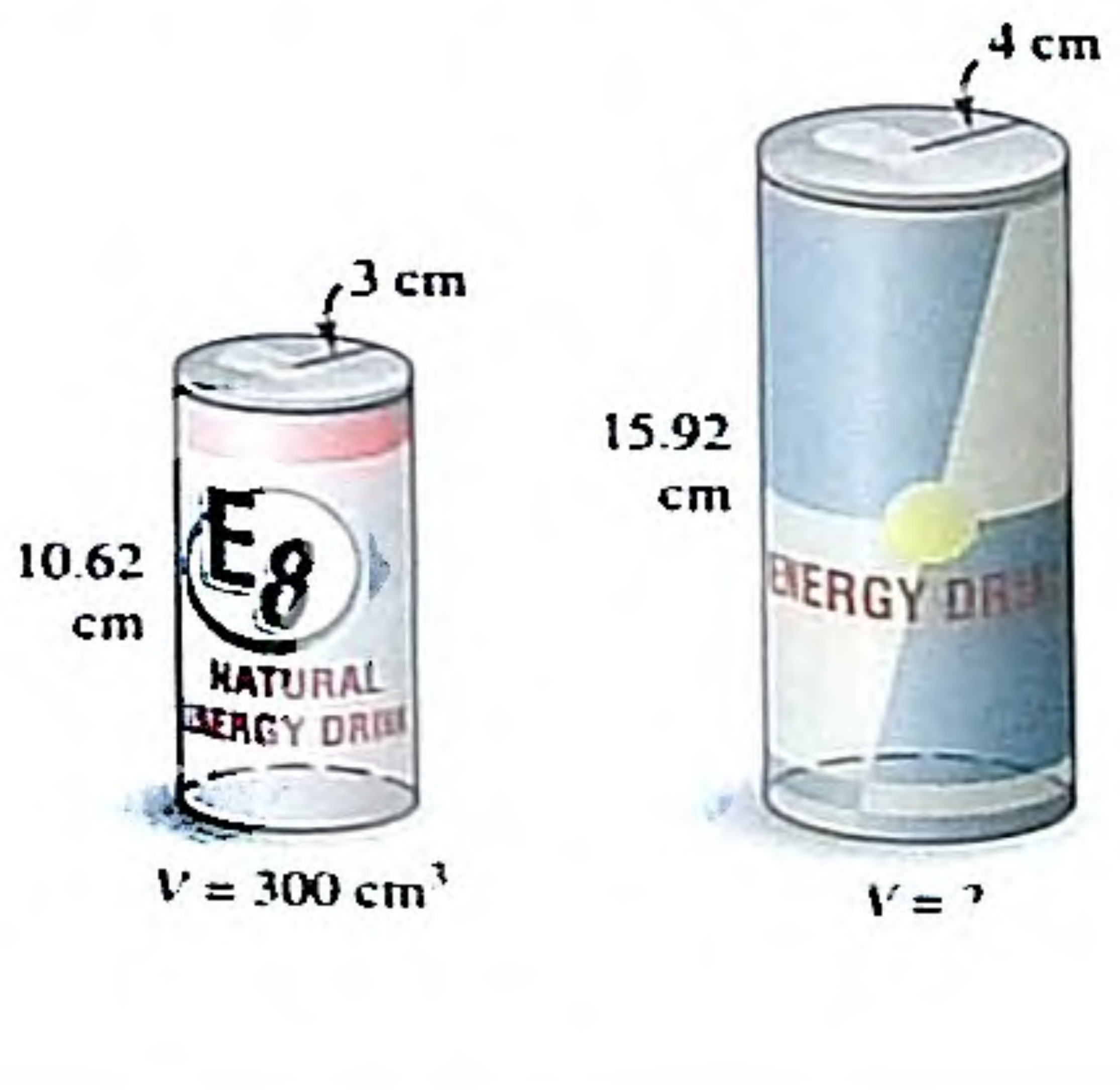
32. *Illumination* The illumination produced by a light source varies inversely as the square of the distance from the source. The illumination of a light source at 5 m is 70 candela. What is the illumination 12 m from the source?

33. *Simple Interest* Simple interest varies jointly as principal and time. If $1000 invested for 2 yr earned $70, find the amount of interest earned by $5000 for 5 yr.

34. *Volume of a Gas* The volume of a gas varies inversely as the pressure and directly as the temperature in kelvins (K). If a certain gas occupies a volume of 1.3 L at 300 K and a pressure of 18 newtons, find the volume at 340 K and a pressure of 24 newtons.

35. *Force of Wind* The force of the wind blowing on a vertical surface varies jointly as the area of the surface and the square of the velocity. If a wind of 40 mph exerts a force of 50 lb on a surface of $\frac{1}{2}$ ft^2, how much force will a wind of 80 mph place on a surface of 2 ft^2?

36. *Volume of a Cylinder* The volume of a right circular cylinder is jointly proportional to the square of the radius of the circular base and to the height. If the volume is 300 cm^3 when the height is 10.62 cm and the radius is 3 cm, find the volume to the nearest tenth of a cylinder with radius 4 cm and height 15.92 cm.



37. ***Sports Arena Construction*** The roof of a new sports arena rests on round concrete pillars. The maximum load a cylindrical column of circular cross section can hold varies directly as the fourth power of the diameter and inversely as the square of the height. The arena has 9-m-tall columns that are 1 m in diameter and will support a load of 8 metric tons. How many metric tons will be supported by a column 12 m high and $\frac{2}{3}$ m in diameter?

Load = 8 metric tons

38. ***Sports Arena Construction*** The sports arena in **Exercise 37** requires a horizontal beam 16 m long, 24 cm wide, and 8 cm high. The maximum load of a horizontal beam that is supported at both ends varies directly as the width of the beam and the square of its height and inversely as the length between supports. If a beam of the same material 8 m long, 12 cm wide, and 15 cm high can support a maximum of 400 kg, what is the maximum load the beam in the arena will support?

39. ***Period of a Pendulum*** The period of a pendulum varies directly as the square root of the length of the pendulum and inversely as the square root of the acceleration due to gravity. Find the period when the length is 121 cm and the acceleration due to gravity is 980 cm per second squared, if the period is 6π seconds when the length is 289 cm and the acceleration due to gravity is 980 cm per second squared.

40. ***Long-Distance Phone Calls*** The number of long-distance phone calls between two cities in a certain time period varies directly as the populations p_1 and p_2 of the cities and inversely as the distance between them. If 10,000 calls are made between two cities 500 mi apart, having populations of 50,000 and 125,000, find the number of calls between two cities 800 mi apart, having populations of 20,000 and 80,000.

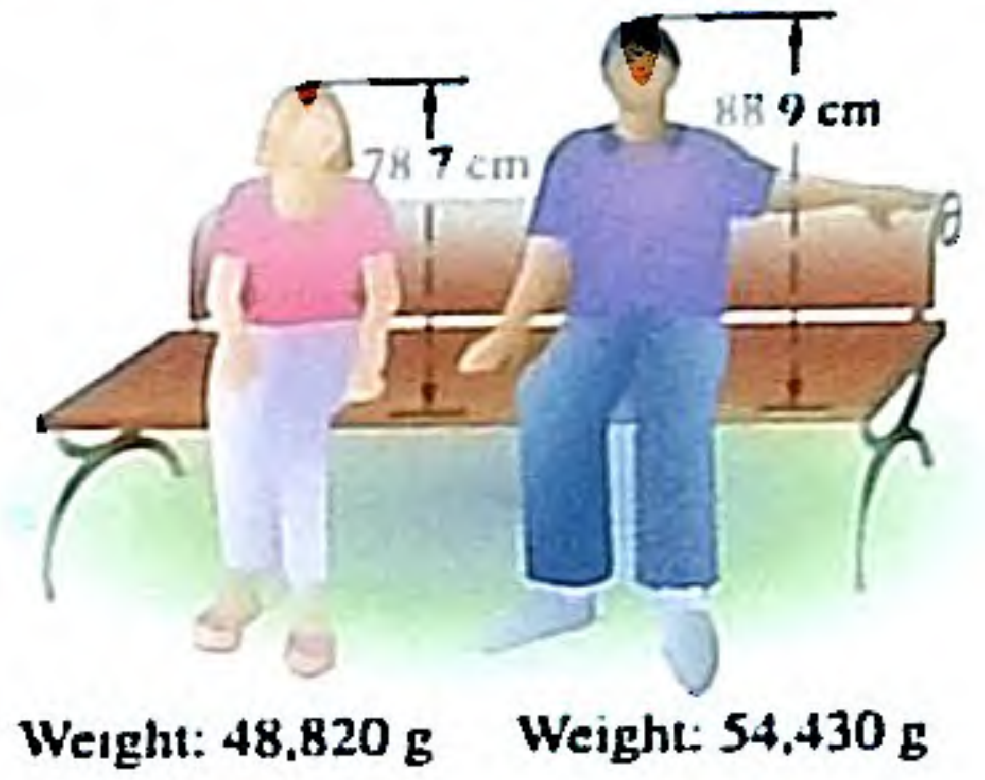
41. ***Body Mass Index*** The federal government has developed the **body mass index** (BMI) to determine ideal weights. A person's BMI is directly proportional to his or her weight in pounds and inversely proportional to the square of his or her height in inches. (A BMI of 19 to 25 corresponds to a healthy weight.) A 6-foot-tall person weighing 177 lb has BMI 24. Find the BMI (to the nearest whole number) of a person whose weight is 130 lb and whose height is 66 in.

42. ***Poiseuille's Law*** According to Poiseuille's law, the resistance to flow of a blood vessel, R, is directly proportional to the length, l, and inversely proportional to the fourth power of the radius, r. If $R = 25$ when $l = 12$ and $r = 0.2$, find R to the nearest hundredth as r increases to 0.3, while l is unchanged.

43. ***Stefan-Boltzmann Law*** The Stefan-Boltzmann law says that the radiation of heat R from an object is directly proportional to the fourth power of the kelvin temperature of the object. For a certain object, $R = 213.73$ at room temperature (293 K). Find R to the nearest hundredth if the temperature increases to 335 K.

44. ***Nuclear Bomb Detonation*** Suppose the effects of detonating a nuclear bomb will be felt over a distance from the point of detonation that is directly proportional to the cube root of the yield of the bomb. Suppose a 100-kiloton bomb has certain effects to a radius of 3 km from the point of detonation. Find the distance to the nearest tenth that the effects would be felt for a 1500-kiloton bomb.

45. *Malnutrition Measure* A measure of malnutrition, called the **pelidisi**, varies directly as the cube root of a person's weight in grams and inversely as the person's sitting height in centimeters. A person with a pelidisi below 100 is considered to be undernourished, while a pelidisi greater than 100 indicates overfeeding. A person who weighs 48,820 g with a sitting height of 78.7 cm has a pelidisi of 100. Find the pelidisi (to the nearest whole number) of a person whose weight is 54,430 g and whose sitting height is 88.9 cm. Is this individual undernourished or overfed?



46. *Photography* Variation occurs in a formula from photography. In

$$L = \frac{25F^2}{st},$$

the luminance, L, varies directly as the square of the F-stop, F, and inversely as the product of the film ASA number, s, and the shutter speed, t.

(a) What would an appropriate F-stop be for 200 ASA film and a shutter speed of $\frac{1}{250}$ sec when 500 footcandles of light is available?

(b) If 125 footcandles of light is available and an F-stop of 2 is used with 200 ASA film, what shutter speed should be used?

Concept Check Work each problem.

47. For $k > 0$, if y varies directly as x, then when x increases, y ________, and when x decreases, y ________.

48. For $k > 0$, if y varies inversely as x, then when x increases, y ________, and when x decreases, y ________.

49. What happens to y if y varies inversely as x, and x is doubled?

50. What happens to y if y varies directly as x, and x is halved?

51. Suppose y is directly proportional to x, and x is replaced by $\frac{1}{3}x$. What happens to y?

52. Suppose y is inversely proportional to x, and x is tripled. What happens to y?

53. Suppose p varies directly as r^3 and inversely as t^2. If r is halved and t is doubled, what happens to p?

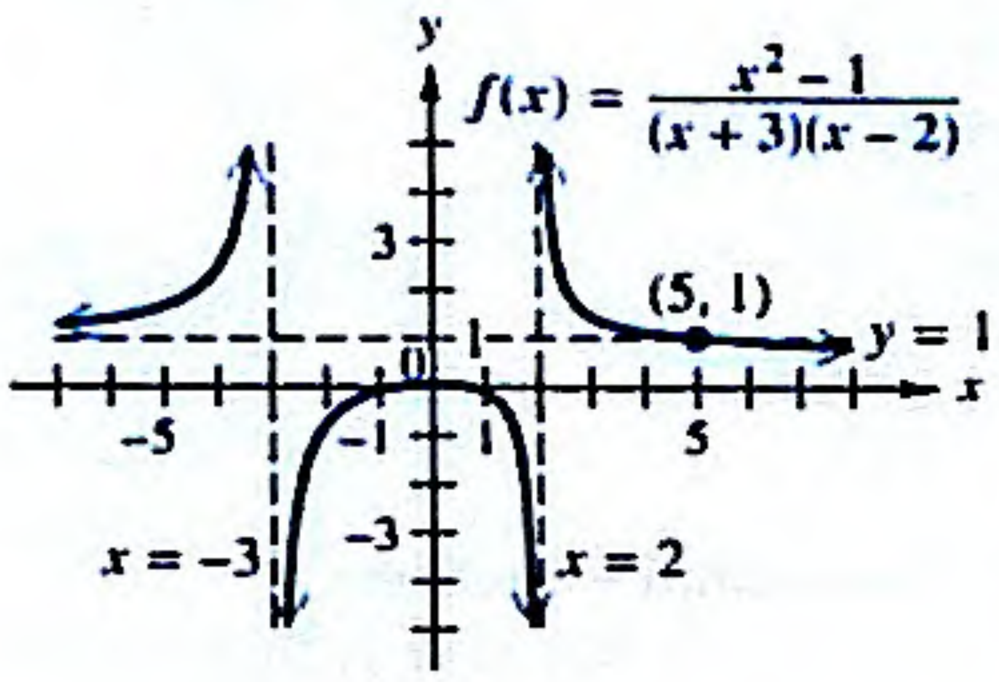
54. Suppose m varies directly as p^2 and q^4. If p doubles and q triples, what happens to m?

Concepts **Examples**

3.5 Rational Functions: Graphs, Applications, and Models

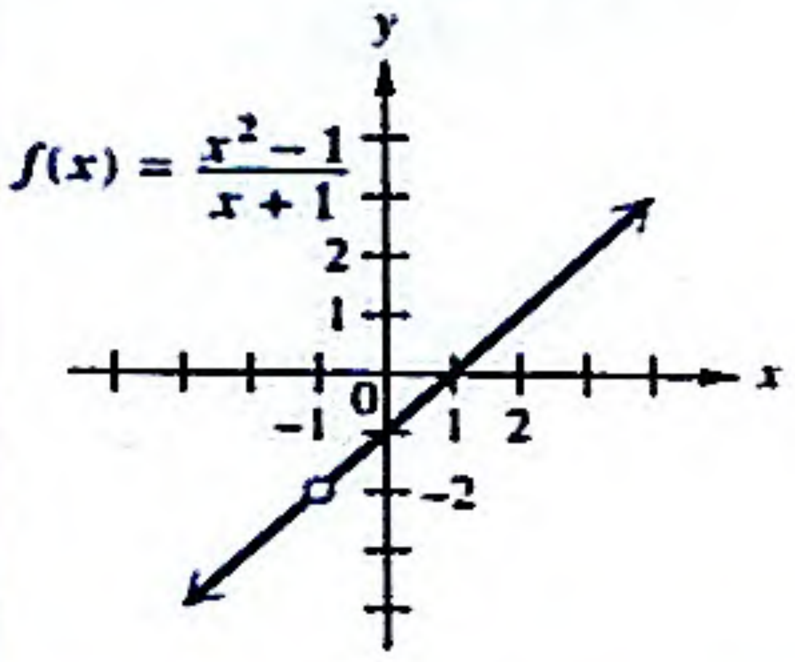
Graphing Rational Functions
To graph a rational function in lowest terms, find the asymptotes and intercepts. Determine whether the graph intersects a nonvertical asymptote. Plot a few points, as necessary, to complete the sketch.

Graph $f(x) = \dfrac{x^2 - 1}{(x + 3)(x - 2)}$.



Point of Discontinuity
If a rational function is not written in lowest terms, there may be a "hole" in the graph instead of an asymptote.

Graph $f(x) = \dfrac{x^2 - 1}{x + 1}$.



$$\begin{aligned} f(x) &= \frac{x^2 - 1}{x + 1} \\ &= \frac{(x + 1)(x - 1)}{x + 1} \\ &= x - 1, \quad x \neq -1 \end{aligned}$$

3.6 Variation

Direct Variation
y varies directly as the nth power of x if for all x there exists a nonzero real number k such that

$$y = kx^n.$$

The area of a circle varies directly as the square of the radius.

$$\mathcal{A} = kr^2 \quad (k = \pi)$$

Inverse Variation
y varies inversely as the nth power of x if for all x there exists a nonzero real number k such that

$$y = \frac{k}{x^n}.$$

Pressure of a gas varies inversely as volume.

$$P = \frac{k}{V}$$

Joint Variation
For real numbers m and n, y varies jointly as the nth power of x and the mth power of z if for all x and z there exists a nonzero real number k such that

$$y = kx^n z^m.$$

The area of a triangle varies jointly as its base and its height.

$$\mathcal{A} = kbh \quad (k = \tfrac{1}{2})$$

Chapter 3 Review Exercises

Graph each quadratic function. Give the vertex, axis, x-intercepts, y-intercept, domain, range, and the intervals of the domain for which each function is increasing or decreasing.

1. $f(x) = 3(x + 4)^2 - 5$
2. $f(x) = -\frac{2}{3}(x - 6)^2 + 7$
3. $f(x) = -3x^2 - 12x - 1$
4. $f(x) = 4x^2 - 4x + 3$

Concept Check In Exercises 5–8, consider the function

$$f(x) = a(x - h)^2 + k, \quad \text{for } a > 0.$$

5. What are the coordinates of the lowest point of its graph?
6. What is the y-intercept of its graph?
7. Under what conditions will its graph have one or more x-intercepts? For these conditions, express the x-intercept(s) in terms of a, h, and k.
8. If a is positive, what is the least value of $ax^2 + bx + c$ in terms of a, b, and c?

(Modeling) Solve each problem.

9. ***Area of a Rectangle*** Use a quadratic function to find the dimensions of the rectangular region of maximum area that can be enclosed with 180 m of fencing, if no fencing is needed along one side of the region.

10. ***Height of a Projectile*** A projectile is fired vertically upward, and its height $s(t)$ in feet after t seconds is given by the function

$$s(t) = -16t^2 + 800t + 600.$$

(a) From what height was the projectile fired?
(b) After how many seconds will it reach its maximum height?
(c) What is the maximum height it will reach?
(d) Between what two times (in seconds, to the nearest tenth) will it be more than 5000 ft above the ground?
(e) After how many seconds, to the nearest tenth, will the projectile hit the ground?

11. ***Food Bank Volunteers*** During the course of a year, the number of volunteers available to run a food bank each month is modeled by $V(x)$, where

$$V(x) = 2x^2 - 32x + 150$$

between the months of January and August. Here x is time in months, with $x = 1$ representing January. From August to December, $V(x)$ is modeled by

$$V(x) = 31x - 226.$$

Find the number of volunteers in each of the following months.

(a) January (b) May (c) August (d) October (e) December
(f) Sketch a graph of $y = V(x)$ for January through December. In what month are the fewest volunteers available?

12. ***Concentration of Atmospheric CO_2*** In 1990, the International Panel on Climate Change (IPCC) stated that if current trends of burning fossil fuel and deforestation were to continue, then future amounts of atmospheric carbon dioxide in parts per million (ppm) would increase, as shown in the table.

Year	Carbon Dioxide
1990	353
2000	375
2075	590
2175	1090
2275	2000

Source: IPCC

(a) Let $x = 0$ represent 1990, $x = 10$ represent 2000, and so on. Find a function of the form

$$f(x) = a(x - h)^2 + k$$

that models the data. Use $(0, 353)$ as the vertex and $(285, 2000)$ as another point to determine a.

(b) Use the function to predict the amount of carbon dioxide in 2300.

Consider the function $f(x) = -2.64x^2 + 5.47x + 3.54$ *for Exercises 13–16.*

13. Use the discriminant to explain how you can determine the number of x-intercepts the graph of $f(x)$ will have even before graphing it on your calculator. (See Section 1.4.)

14. Graph the function in the standard viewing window of your calculator, and use the calculator to solve the equation $f(x) = 0$. Express solutions as approximations to the nearest hundredth.

15. Use your answer to Exercise 14 and the graph of f to solve

(a) $f(x) > 0$, and (b) $f(x) < 0$.

16. Use the capabilities of your calculator to find the coordinates of the vertex of the graph. Express coordinates to the nearest hundredth.

Use synthetic division to perform each division.

17. $\dfrac{x^3 + x^2 - 11x - 10}{x - 3}$

18. $\dfrac{3x^3 + 8x^2 + 5x + 10}{x + 2}$

19. $\dfrac{2x^3 - x + 6}{x + 4}$

20. $\dfrac{3x^3 + 6x^2 - 8x + 3}{x + 3}$

Express $f(x)$ *in the form* $f(x) = (x - k)q(x) + r$ *for the given value of* k.

21. $5x^3 - 3x^2 + 2x - 6$; $k = 2$

22. $-3x^3 + 5x - 6$; $k = -1$

Use synthetic division to find $f(2)$.

23. $f(x) = -x^3 + 5x^2 - 7x + 1$

24. $f(x) = 2x^3 - 3x^2 + 7x - 12$

25. $f(x) = 5x^4 - 12x^2 + 2x - 8$

26. $f(x) = x^5 + 4x^2 - 2x - 4$

Use synthetic division to determine whether k *is a zero of the function.*

27. $f(x) = x^3 + 2x^2 + 3x + 2$; $k = -1$

28. $f(x) = 2x^3 + 5x^2 + 30$; $k = -4$

29. ***Concept Check*** If $f(x)$ is a polynomial function with real coefficients, and if $7 + 2i$ is a zero of the function, then what other complex number must also be a zero?

30. ***Concept Check*** Suppose the polynomial function f has a zero at $x = -3$. Which of the following statements *must* be true?

A. 3 is an x-intercept of the graph of f.

B. 3 is a y-intercept of the graph of f.

C. $x - 3$ is a factor of $f(x)$.

D. $f(-3) = 0$

Find a polynomial function with real coefficients and least degree having the given zeros.

31. $-1, 4, 7$

32. $8, 2, 3$

33. $\sqrt{3}, -\sqrt{3}, 2, 3$

34. $-2 + \sqrt{5}, -2 - \sqrt{5}, -2, 1$

35. $2, 4, -i$

36. $0, 5, 1 + 2i$

Find all rational zeros of each function.

37. $f(x) = 2x^3 - 9x^2 - 6x + 5$

38. $f(x) = 8x^4 - 14x^3 - 29x^2 - 4x + 3$

In Exercises 39–41, show that the polynomial function has a real zero as described in parts (a) and (b). In Exercises 39 and 40, then work part (c).

39. $f(x) = 3x^3 - 8x^2 + x + 2$

(a) between -1 and 0

(b) between 2 and 3

(c) Find the zero in part (b) to three decimal places.

40. $f(x) = 4x^3 - 37x^2 + 50x + 60$

(a) between 2 and 3

(b) between 7 and 8

(c) Find the zero in part (b) to three decimal places.

41. $f(x) = 6x^4 + 13x^3 - 11x^2 - 3x + 5$

(a) no zero greater than 1

(b) no zero less than -3

Solve each problem.

42. Use Descartes' rule of signs to determine the different possibilities for the numbers of positive, negative, and nonreal complex zeros of

$$f(x) = x^3 + 3x^2 - 4x - 2.$$

43. Is $x + 1$ a factor of $f(x) = x^3 + 2x^2 + 3x + 2$?

44. Find a polynomial function f with real coefficients of degree 4 with 3, 1, and $-1 + 3i$ as zeros, and $f(2) = -36$.

45. Find a polynomial function f of degree 3 with -2, 1, and 4 as zeros, and $f(2) = 16$.

46. Find all zeros of $f(x) = x^4 - 3x^3 - 8x^2 + 22x - 24$, given that $1 + i$ is a zero.

47. Find all zeros of $f(x) = 2x^4 - x^3 + 7x^2 - 4x - 4$, given that 1 and $-2i$ are zeros.

48. Find a value of k such that $x - 4$ is a factor of

$$f(x) = x^3 - 2x^2 + kx + 4.$$

49. Find a value of k such that when the polynomial $x^3 - 3x^2 + kx - 4$ is divided by $x - 2$, the remainder is 5.

50. ***Concept Check*** Give an example of a fourth-degree polynomial function having exactly two distinct real zeros, and then sketch its graph.

51. ***Concept Check*** Give an example of a cubic polynomial function having exactly one real zero, and then sketch its graph.

52. Give the maximum number of turning points of the graph of each function.

(a) $f(x) = x^5 - 9x^2$ (b) $f(x) = 4x^3 - 6x^2 + 2$

53. ***Concept Check*** Suppose the leading term of a polynomial function is $10x^7$. What can you conclude about each of the following features of the graph of the function?

(a) domain (b) range (c) end behavior (d) number of zeros
(e) number of turning points

54. ***Concept Check*** Repeat Exercise 53 for a polynomial function with leading term $-9x^6$.

Sketch the graph of each polynomial function.

55. $f(x) = (x - 2)^2(x + 3)$

56. $f(x) = -2x^3 + 7x^2 - 2x - 3$

57. $f(x) = 2x^3 + x^2 - x$

58. $f(x) = x^4 - 3x^2 + 2$

59. $f(x) = x^4 + x^3 - 3x^2 - 4x - 4$

60. $f(x) = -2x^4 + 7x^3 - 4x^2 - 4x$

Concept Check *For each polynomial function, identify its graph from choices A–F.*

61. $f(x) = (x-2)^2(x-5)$

62. $f(x) = -(x-2)^2(x-5)$

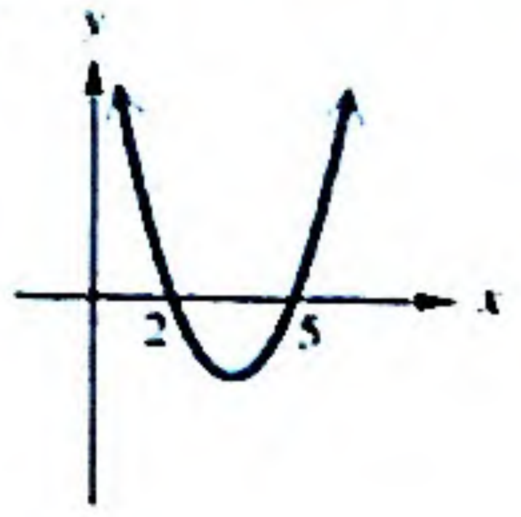
63. $f(x) = (x-2)^2(x-5)^2$

64. $f(x) = (x-2)(x-5)$

65. $f(x) = -(x-2)(x-5)$

66. $f(x) = -(x-2)^2(x-5)^2$

A.



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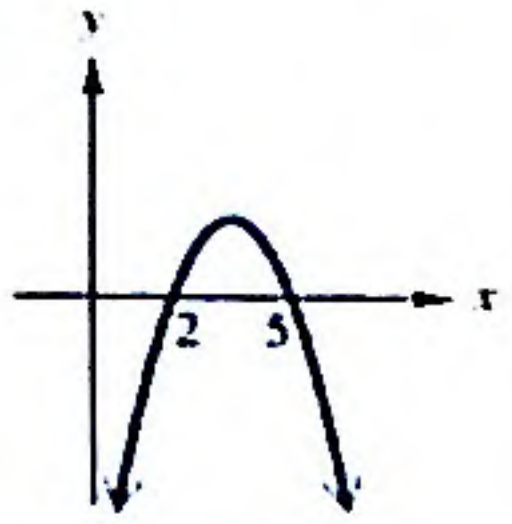

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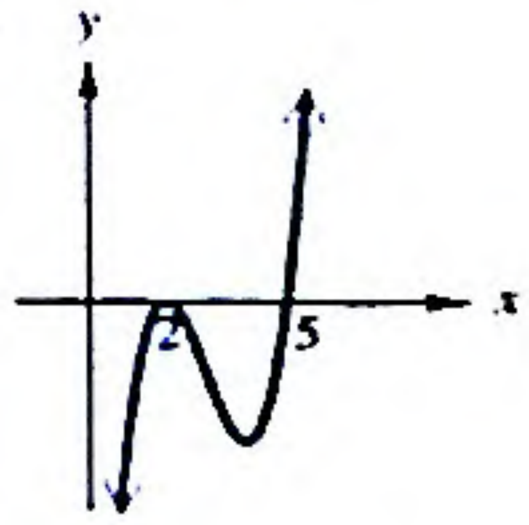

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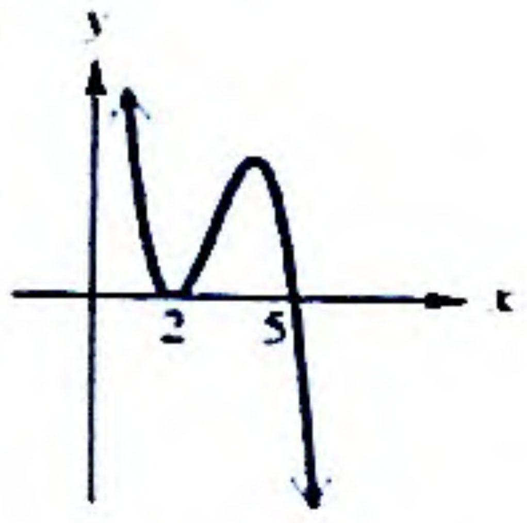

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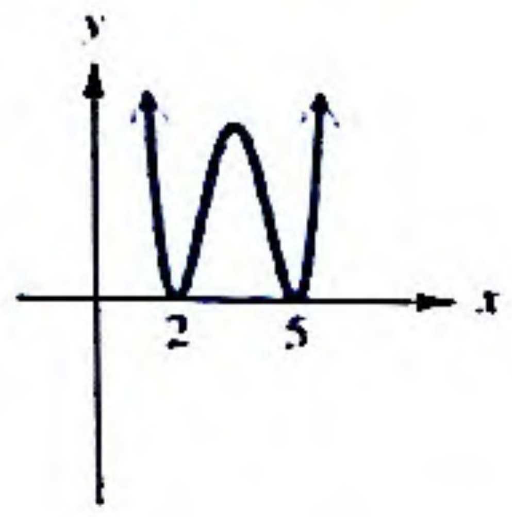
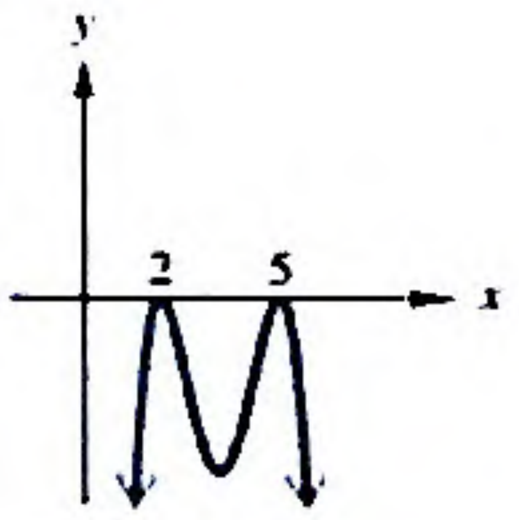


F.



Use a graphing calculator to graph each polynomial function in the viewing window specified. Then determine the real zeros to as many decimal places as the calculator will provide.

67. $f(x) = x^3 - 8x^2 + 2x + 5$; window: $[-10, 10]$ by $[-60, 60]$

68. $f(x) = x^4 - 4x^3 - 5x^2 + 14x - 15$; window: $[-10, 10]$ by $[-60, 60]$

Solve each problem.

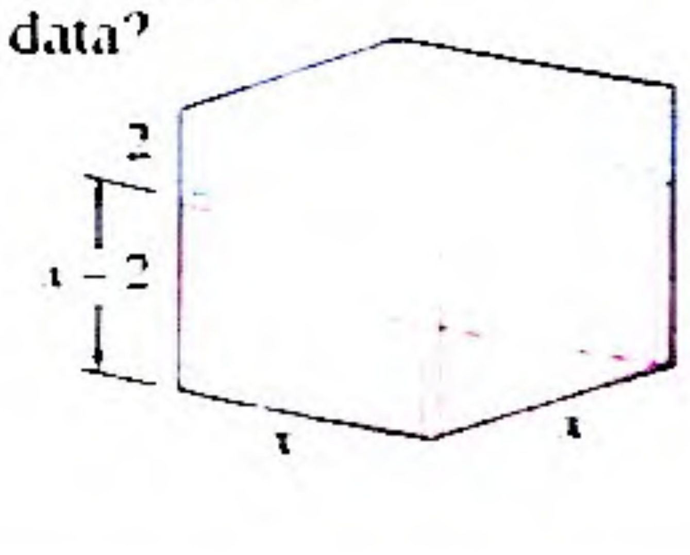
69. *(Modeling) Medicare Beneficiary Spending* Out-of-pocket spending projections for a typical Medicare beneficiary as a share of his or her income are given in the table. Let $x = 0$ represent 1990, so $x = 8$ represents 1998. Use a graphing calculator to do the following.

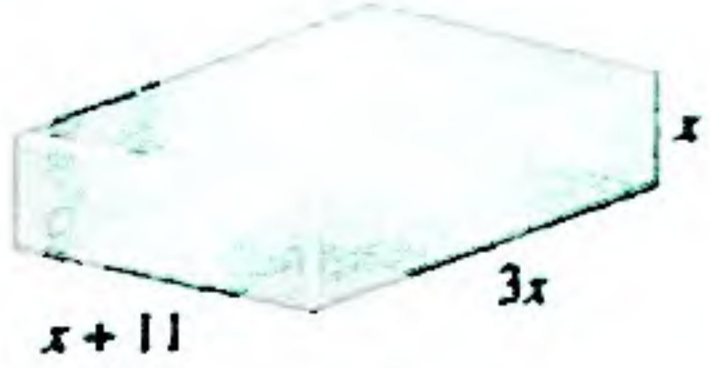
Year	Percent of Income
1998	18.6
2000	19.3
2005	21.7
2010	24.7
2015	27.5
2020	28.3
2025	28.6

Source: Urban Institute's Analysis of 1998 Medicare Trustees' Report.

(a) Graph the data points.
(b) Find a quadratic function to model the data.
(c) Find a cubic function to model the data.
(d) Graph each function in the same viewing window as the data points.
(e) Compare the two functions. Which is a better fit for the data?

70. *Dimensions of a Cube* After a 2-in. slice is cut off the top of a cube, the resulting solid has a volume of 32 in.³ Find the dimensions of the original cube.





71. ***Dimensions of a Box*** The width of a rectangular box is three times its height, and its length is 11 in. more than its height. Find the dimensions of the box if its volume is 720 in.^{3}.

72. The function $f(x) = \frac{1}{x}$ is negative at $x = -1$ and positive at $x = 1$ but has no zero between -1 and 1. Explain why this does not contradict the intermediate value theorem.

Graph each rational function.

73. $f(x) = \dfrac{4}{x - 1}$

74. $f(x) = \dfrac{4x - 2}{3x + 1}$

75. $f(x) = \dfrac{6x}{x^2 + x - 2}$

76. $f(x) = \dfrac{2x}{x^2 - 1}$

77. $f(x) = \dfrac{x^2 + 4}{x + 2}$

78. $f(x) = \dfrac{x^2 - 1}{x}$

79. $f(x) = \dfrac{-2}{x^2 + 1}$

80. $f(x) = \dfrac{4x^2 - 9}{2x + 3}$

81. ***Concept Check***

(a) Sketch the graph of a function that does not intersect its horizontal asymptote $y = 1$, has the line $x = 3$ as a vertical asymptote, and has x-intercepts 2 and 4.

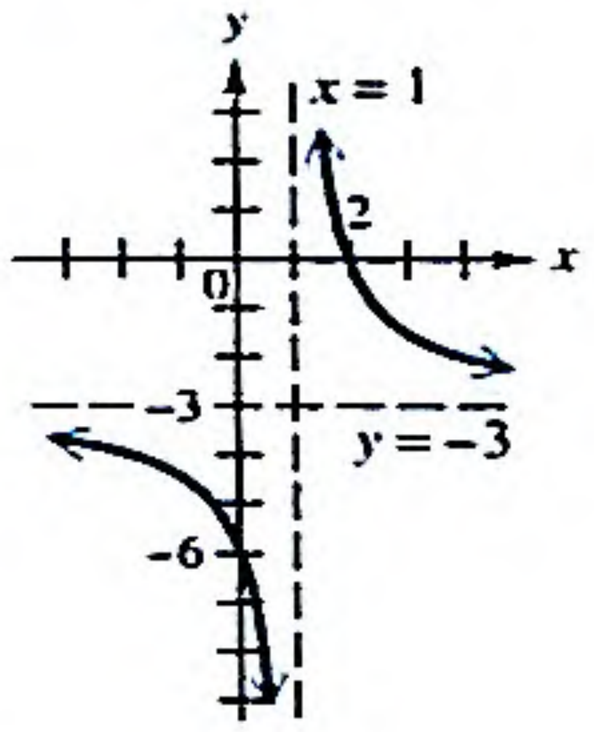
(b) Find an equation for a possible corresponding rational function.

82. ***Concept Check***

(a) Sketch the graph of a function that is never negative and has the lines $x = -1$ and $x = 1$ as vertical asymptotes, the x-axis as a horizontal asymptote, and 0 as an x-intercept.

(b) Find an equation for a possible corresponding rational function.

83. ***Connecting Graphs with Equations*** Find an equation for the rational function graphed here.



(Modeling) Solve each problem.

84. ***Antique-Car Competition*** Antique-car owners often enter their cars in a concours d'elegance in which a maximum of 100 points can be awarded to a particular car. Points are awarded for the general attractiveness of the car. The function

$$C(x) = \frac{10x}{49(101 - x)}$$

models the cost, in thousands of dollars, of restoring a car so that it will win x points.

(a) Graph the function in the window $[0, 101]$ by $[0, 10]$.

(b) How much would an owner expect to pay to restore a car in order to earn 95 points?

85. ***Environmental Pollution*** In situations involving environmental pollution, a cost-benefit model expresses cost as a function of the percentage of pollutant removed from the environment. Suppose a cost-benefit model is expressed as

$$C(x) = \frac{6.7x}{100 - x},$$

where $C(x)$ is cost in thousands of dollars of removing x percent of a pollutant.

(a) Graph the function in the window $[0, 100]$ by $[0, 100]$.
(b) How much would it cost to remove 95% of the pollutant?

Solve each variation problem.

86. If x varies directly as y, and $x = 12$ when $y = 4$, find x when $y = 12$.

87. If x varies directly as y, and $x = 20$ when $y = 14$, find y when $x = 50$.

88. If z varies inversely as w, and $z = 10$ when $w = \frac{1}{2}$, find z when $w = 10$.

89. If t varies inversely as s, and $t = 3$ when $s = 5$, find s when $t = 20$.

90. p varies jointly as q and r^2, and $p = 100$ when $q = 2$ and $r = 3$. Find p when $q = 5$ and $r = 2$.

91. f varies jointly as g^2 and h, and $f = 50$ when $g = 5$ and $h = 4$. Find f when $g = 3$ and $h = 6$.

92. ***Pressure in a Liquid*** The pressure on a point in a liquid is directly proportional to the distance from the surface to the point. In a certain liquid, the pressure at a depth of 4 m is 60 kg per m^2. Find the pressure at a depth of 10 m.

93. ***Skidding Car*** The force needed to keep a car from skidding on a curve varies inversely as the radius r of the curve and jointly as the weight of the car and the square of the speed. It takes 3000 lb of force to keep a 2000-lb car from skidding on a curve of radius 500 ft at 30 mph. What force will keep the same car from skidding on a curve of radius 800 ft at 60 mph?

94. ***Power of a Windmill*** The power a windmill obtains from the wind varies directly as the cube of the wind velocity. If a wind of 10 km per hr produces 10,000 units of power, how much power is produced by a wind of 15 km per hr?

Chapter 3 Test

1. Sketch the graph of the quadratic function $f(x) = -2x^2 + 6x - 3$. Give the intercepts, vertex, axis, domain, range, and the intervals of the domain for which the function is increasing or decreasing.

2. *(Modeling) Height of a Projectile* A small rocket is fired directly upward, and its height s in feet after t seconds is given by the function

$$s(t) = -16t^2 + 88t + 48$$

(a) Determine the time at which the rocket reaches its maximum height.
(b) Determine the maximum height.
(c) Between what two times (in seconds, to the nearest tenth) will the rocket be more than 100 ft above ground level?
(d) After how many seconds will the rocket return to the ground?

Use synthetic division to perform each division.

3. $\dfrac{3x^3 + 4x^2 - 9x + 6}{x + 2}$

4. $\dfrac{2x^3 - 11x^2 + 25}{x - 5}$

5. Use synthetic division to determine $f(5)$, if $f(x) = 2x^3 - 9x^2 + 4x + 8$.

6. Use the factor theorem to determine whether the polynomial $x - 3$ is a factor of

$$6x^4 - 11x^3 - 35x^2 + 34x + 24.$$

If it is, what is the other factor? If it is not, explain why.

7. Find all zeros of $f(x)$, given that $f(x) = x^3 + 8x^2 + 25x + 26$ and -2 is one zero.

8. Find a fourth degree polynomial function $f(x)$ having only real coefficients, -1, 2, and i as zeros, and $f(3) = 80$.

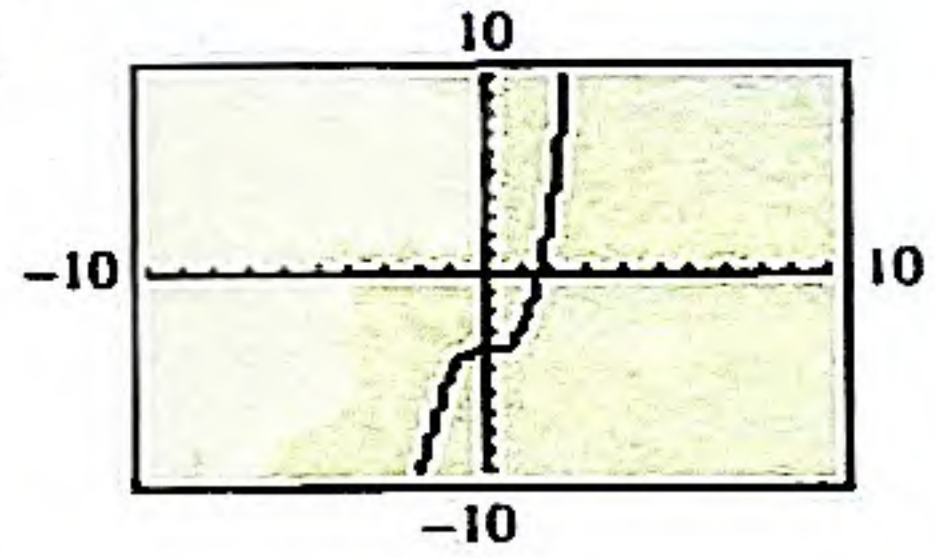
9. Why can't the polynomial function $f(x) = x^4 + 8x^2 + 12$ have any real zeros?

10. Consider the function defined by $f(x) = x^3 - 5x^2 + 2x + 7$.
    (a) Use the intermediate value theorem to show that f has a zero between 1 and 2.
    (b) Use Descartes' rule of signs to determine the different possibilities for the numbers of positive, negative, and nonreal complex zeros.
    (c) Use a graphing calculator to find all real zeros to as many decimal places as the calculator will give.

11. Graph the functions $f(x) = x^4$ and $g(x) = -2(x + 5)^4 + 3$ on the same axes. How can the graph of g be obtained by a transformation of the graph of f?

12. Use end behavior to determine which one of the following graphs is that of $f(x) = -x^7 + x - 4$.

A.



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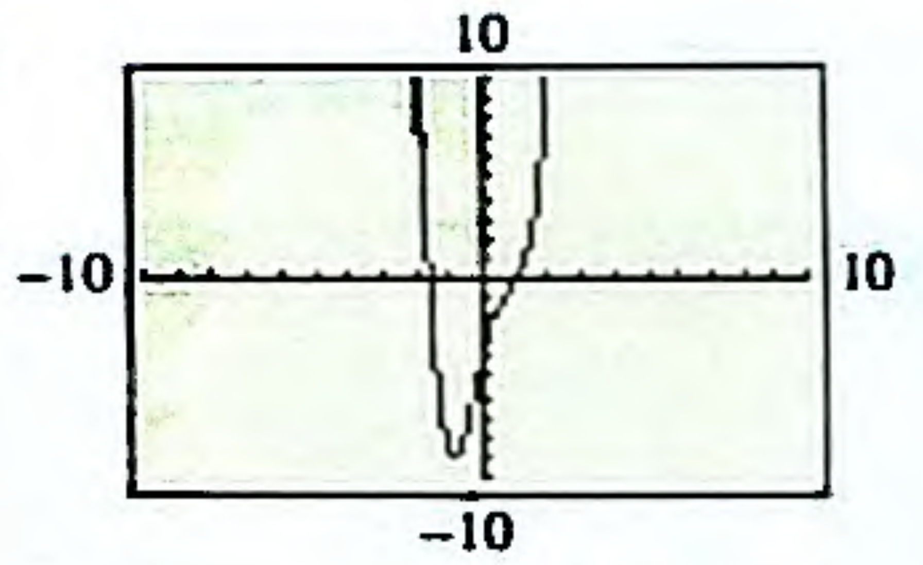

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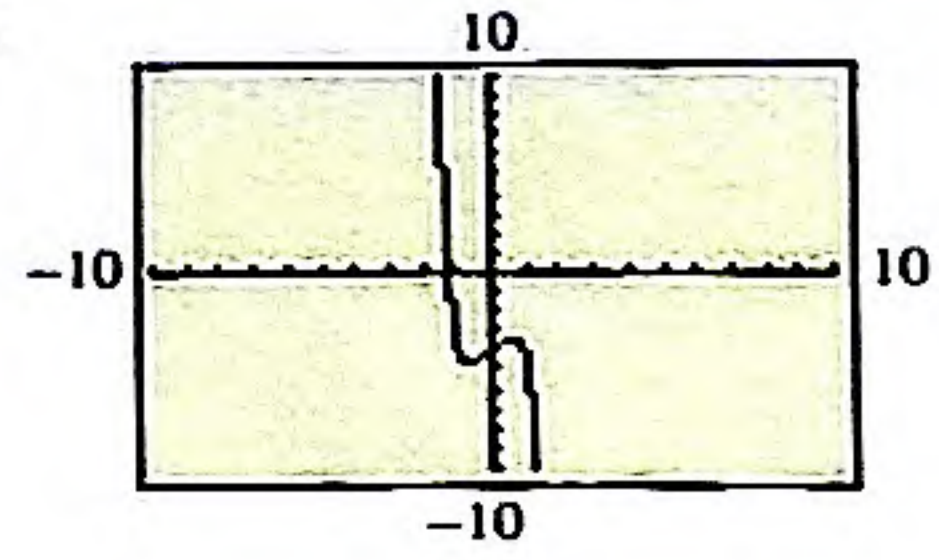

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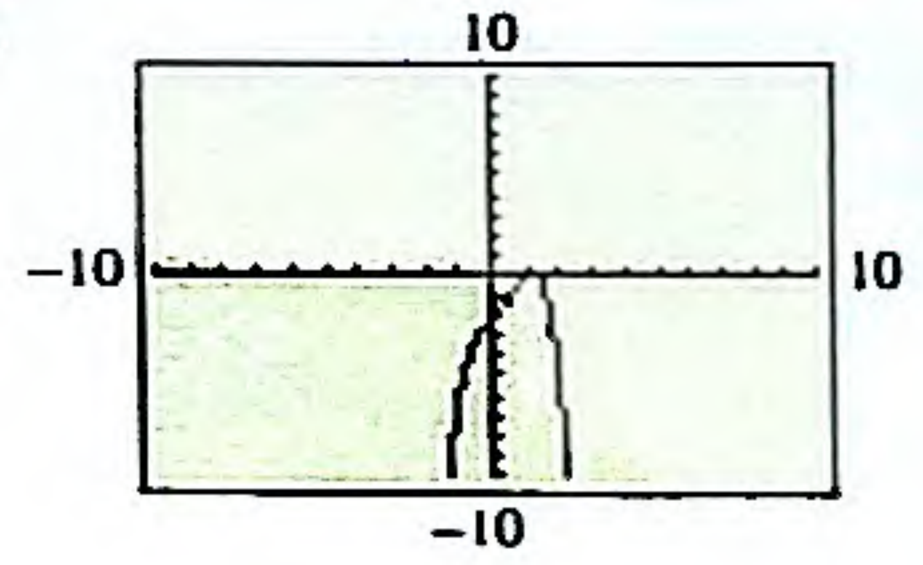


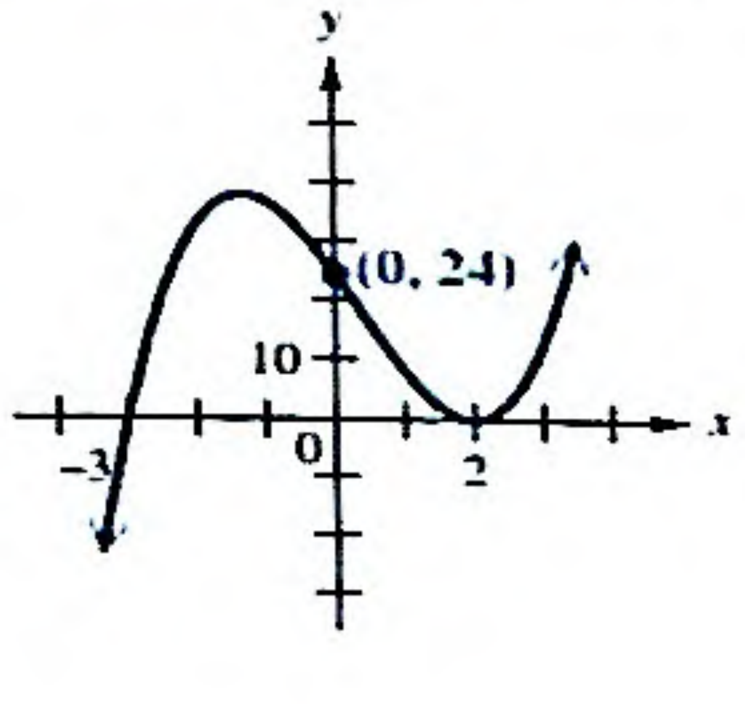
Graph each polynomial function.

13. $f(x) = x^3 - 5x^2 + 3x + 9$

14. $f(x) = 2x^2(x - 2)^2$

15. $f(x) = -x^3 - 4x^2 + 11x + 30$

16. ***Connecting Graphs with Equations*** Find a cubic polynomial function having the graph shown.



17. *(Modeling) Oil Pressure* The pressure of oil in a reservoir tends to drop with time. Engineers found that the change in pressure is modeled by

$$f(t) = 1.06t^3 - 24.6t^2 + 180t,$$

for t (in years) in the interval $[0, 15]$.

(a) What was the change after 2 yr?

(b) For what time periods, to the nearest tenth of a year, is the amount of change in pressure increasing? decreasing? Use a graph to decide.

Graph each rational function.

18. $f(x) = \dfrac{3x - 1}{x - 2}$

19. $f(x) = \dfrac{x^2 - 1}{x^2 - 9}$

20. Consider the rational function $f(x) = \dfrac{2x^2 + x - 6}{x - 1}$.

(a) Determine the equation of the oblique asymptote.

(b) Determine the x-intercepts.

(c) Determine the y-intercept.

(d) Determine the equation of the vertical asymptote.

(e) Sketch the graph.

21. If y varies directly as the square root of x, and $y = 12$ when $x = 4$, find y when $x = 100$.

22. *Weight On and Above Earth* The weight w of an object varies inversely as the square of the distance d between the object and the center of Earth. If a man weighs 90 kg on the surface of Earth, how much would he weigh 800 km above the surface? (*Hint:* The radius of Earth is about 6400 km.)

4

Inverse, Exponential, and Logarithmic Functions

The magnitudes of earthquakes, the loudness of sounds, and the growth or decay of some populations are examples of quantities that are described by *exponential functions* and their inverses, *logarithmic functions.*

4.1 Inverse Functions

4.2 Exponential Functions

4.3 Logarithmic Functions

Summary Exercises on Inverse, Exponential, and Logarithmic Functions

4.4 Evaluating Logarithms and the Change-of-Base Theorem

Chapter 4 Quiz

4.5 Exponential and Logarithmic Equations

4.6 Applications and Models of Exponential Growth and Decay

Summary Exercises on Functions: Domains and Defining Equations

Graphs of f and f^{-1} are shown in **Figures 10** and **11**. The line $y = x$ is included on the graphs to show that the graphs of f and f^{-1} are mirror images with respect to this line.

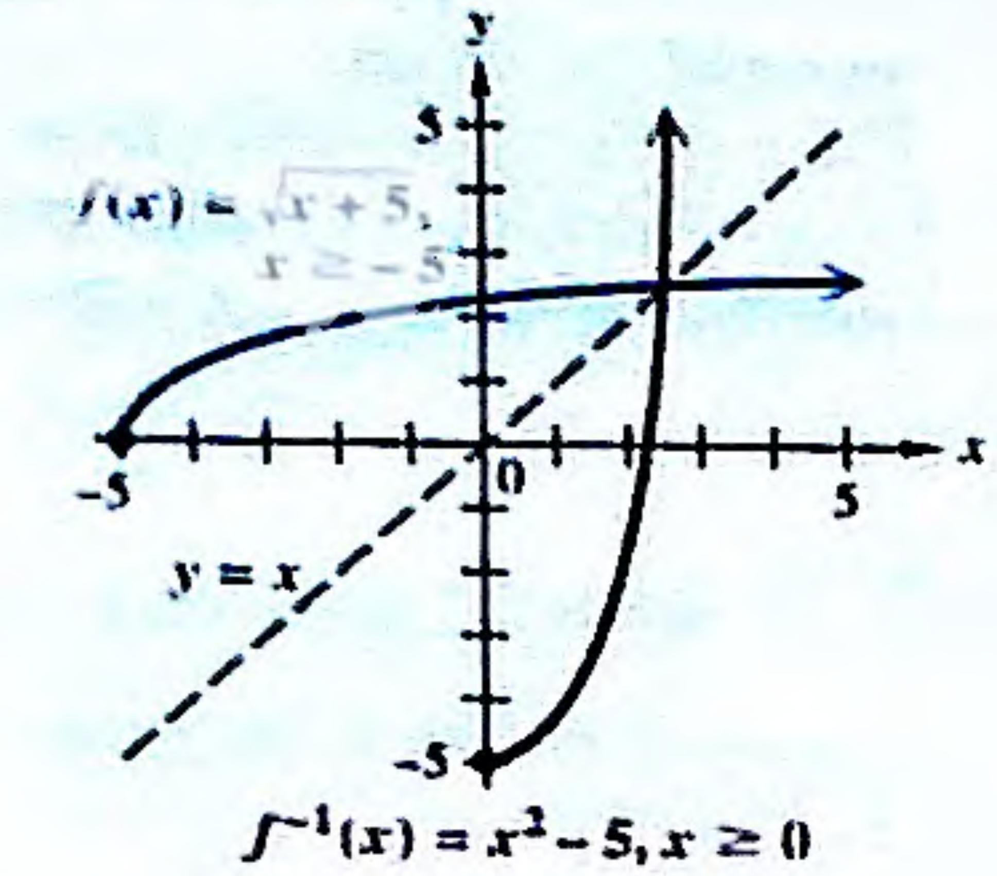


Figure 10

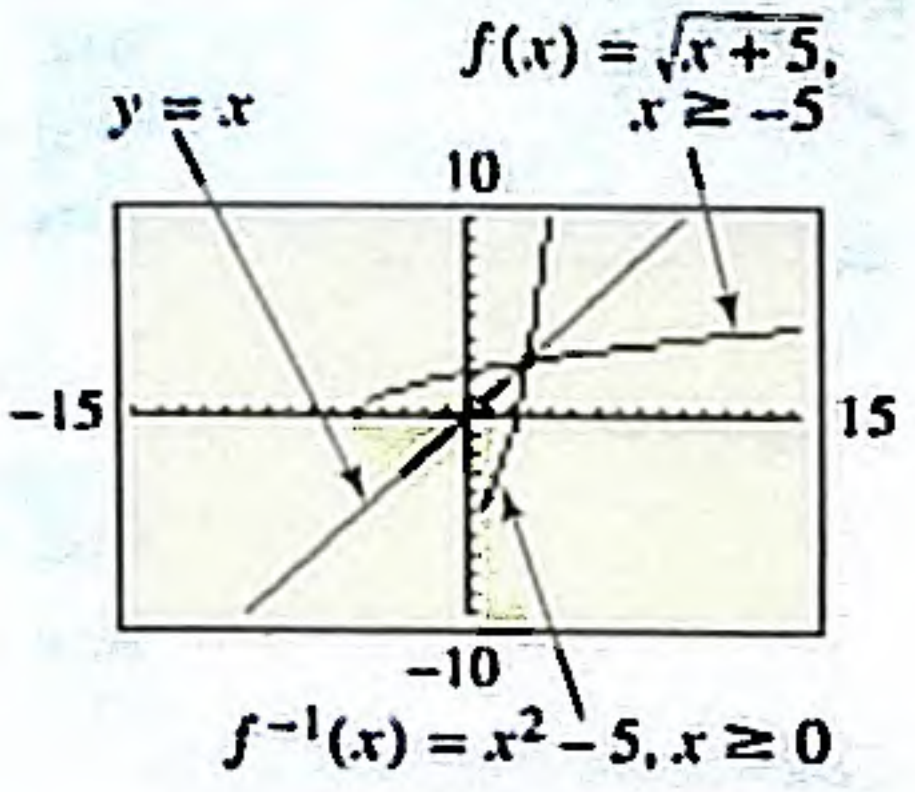


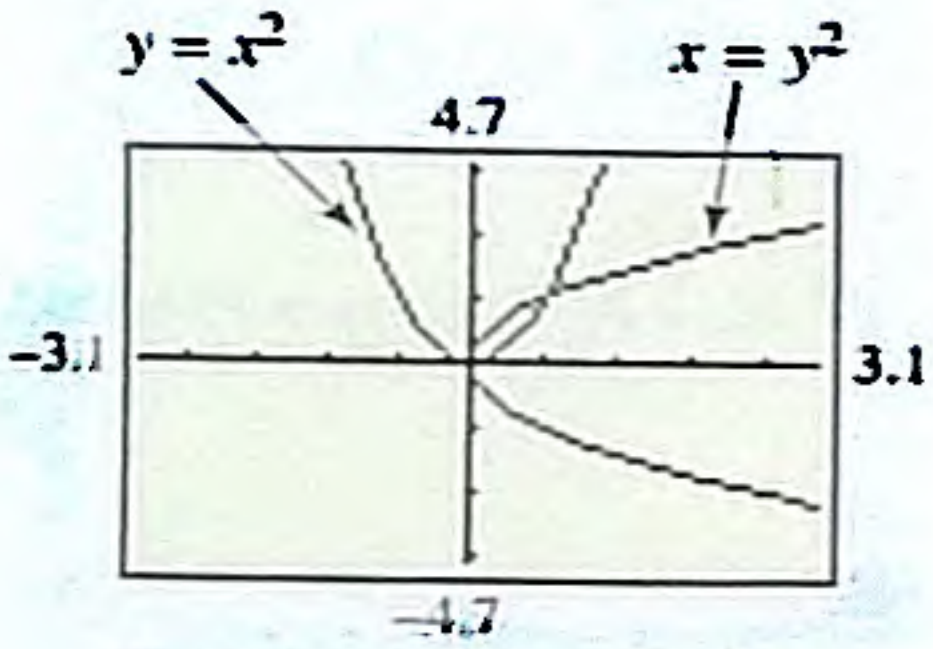
Figure 11

✔ *Now Try Exercise 75.*

Important Facts about Inverses

1. If f is one-to-one, then f^{-1} exists.
2. The domain of f is the range of f^{-1}, and the range of f is the domain of f^{-1}.
3. If the point (a, b) lies on the graph of f, then (b, a) lies on the graph of f^{-1}. The graphs of f and f^{-1} are reflections of each other across the line $y = x$.
4. To find the equation for f^{-1}, replace $f(x)$ with y, interchange x and y, and solve for y. This gives $f^{-1}(x)$.

Some graphing calculators have the capability of "drawing" the reflection of a graph across the line $y = x$. This feature does not require that the function be one-to-one, however, so the resulting figure may not be the graph of a function. See **Figure 12**. *Again, it is necessary to understand the mathematics to interpret results correctly.* ∎



Despite the fact that $y = x^2$ is not one-to-one, the calculator will draw its "inverse," $x = y^2$.

Figure 12

An Application of Inverse Functions to Cryptography A one-to-one function and its inverse can be used to make information secure. The function is used to encode a message, and its inverse is used to decode the coded message. In practice, complicated functions are used. We illustrate the process with a simple function in Example 9.

EXAMPLE 9 Using Functions to Encode and Decode a Message

Use the one-to-one function $f(x) = 3x + 1$ and the following numerical values assigned to each letter of the alphabet to encode and decode the message BE MY FACEBOOK FRIEND.

A	1	H	8	O	15	V	22
B	2	I	9	P	16	W	23
C	3	J	10	Q	17	X	24
D	4	K	11	R	18	Y	25
E	5	L	12	S	19	Z	26
F	6	M	13	T	20		
G	7	N	14	U	21		

A	1	N	14
B	2	O	15
C	3	P	16
D	4	Q	17
E	5	R	18
F	6	S	19
G	7	T	20
H	8	U	21
I	9	V	22
J	10	W	23
K	11	X	24
L	12	Y	25
M	13	Z	26

SOLUTION The message BE MY FACEBOOK FRIEND would be encoded as

$$\begin{array}{ccccccccc} 7 & 16 & 40 & 76 & 19 & 4 & 10 & 16 & 7 \\ 46 & 46 & 34 & 19 & 55 & 28 & 16 & 43 & 13 \end{array}$$

because

B corresponds to 2 and $f(2) = 3(2) + 1 = 7$,

E corresponds to 5 and $f(5) = 3(5) + 1 = 16$, and so on.

Using the inverse $f^{-1}(x) = \frac{1}{3}x - \frac{1}{3}$ to decode yields

$$f^{-1}(7) = \frac{1}{3}(7) - \frac{1}{3} = 2, \quad \text{which corresponds to B,}$$

$$f^{-1}(16) = \frac{1}{3}(16) - \frac{1}{3} = 5, \quad \text{which corresponds to E,} \quad \text{and so on.}$$

✓ *Now Try Exercise 97.*

4.1 Exercises

***Concept Check** In Exercises 1 and 2, answer the question and then write a short explanation.*

1. The table shows the number of registered passenger cars in the United States for the years 2004–2008.

If this correspondence is considered to be a function that pairs each year with the number of registered passenger cars, is it one-to-one? If not, explain why.

Year	Registered Passenger Cars (in thousands)
2004	136,431
2005	136,568
2006	135,400
2007	135,933
2008	137,080

Source: U.S. Federal Highway Administration.

2. The table shows the number of hazardous waste sites on the National Priority List for the Superfund program in 2008 for seven states in the top ten.

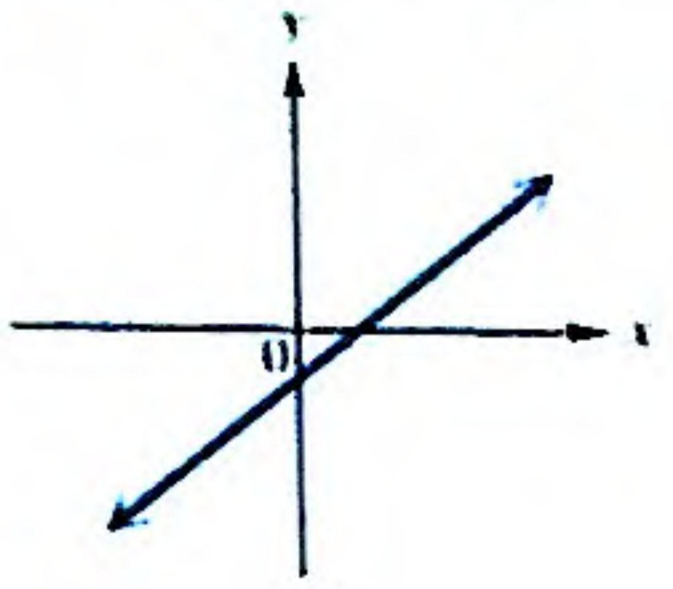
If this correspondence is considered to be a function that pairs each state with its number of hazardous waste sites, is it one-to-one? If not, explain why.

State	Number of Sites
New Jersey	116
California	97
Pennsylvania	96
New York	86
Florida	52
Illinois	49
Texas	49

Source: U.S. Environmental Protection Agency

Decide whether each function as graphed or defined is one-to-one. See Examples 1 and 2.

3.



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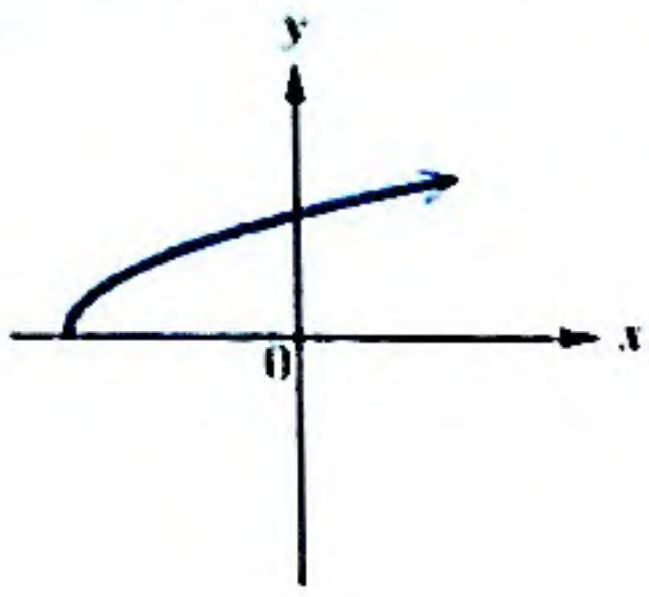

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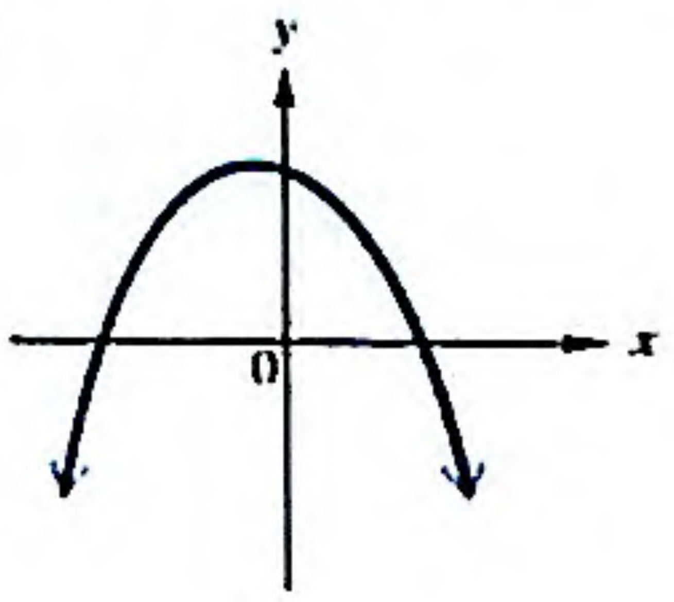

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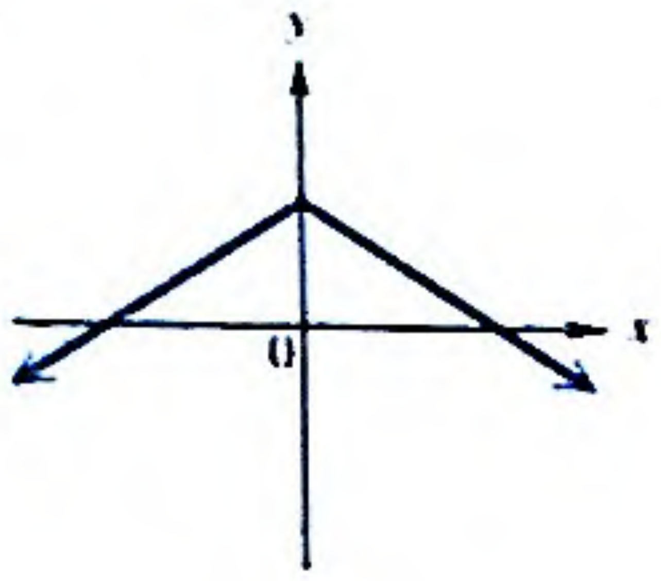

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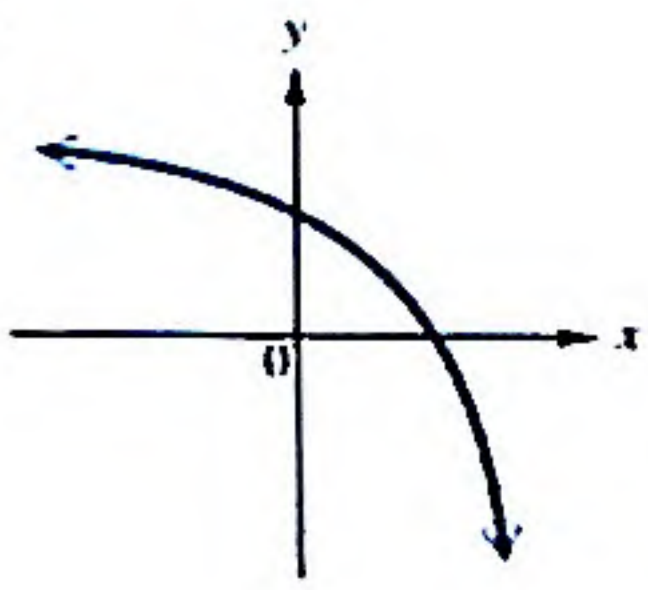

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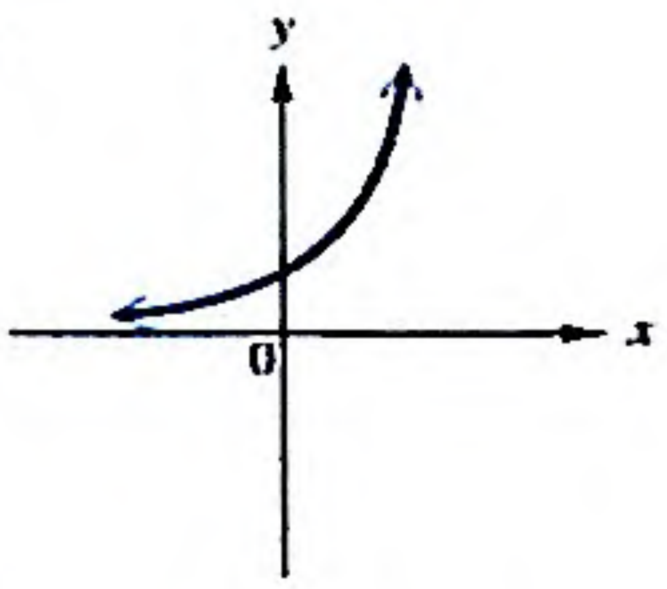


9. $y = 2x - 8$

10. $y = 4x + 20$

11. $y = \sqrt{36 - x^2}$

12. $y = -\sqrt{100 - x^2}$

13. $y = 2x^3 - 1$

14. $y = 3x^3 - 6$

15. $y = \dfrac{-1}{x + 2}$

16. $y = \dfrac{4}{x - 8}$

17. $y = 2(x + 1)^2 - 6$

18. $y = -3(x - 6)^2 + 8$

19. $y = \sqrt[3]{x + 1} - 3$

20. $y = -\sqrt[3]{x + 2} - 8$

21. Explain why a constant function, such as $f(x) = 3$, defined over the set of real numbers, cannot be one-to-one.

22. (a) Explain why a polynomial function of even degree cannot have an inverse.
(b) Explain why a polynomial function of odd degree *may* not be one-to-one.

Concept Check *Answer each of the following.*

23. For a function to have an inverse, it must be ____________.

24. If two functions f and g are inverses, then $(f \circ g)(x) =$ ____________, and ____________ $= x$.

25. The domain of f is equal to the ____________ of f^{-1}, and the range of f is equal to the ____________ of f^{-1}.

26. If the point (a, b) lies on the graph of f, and f has an inverse, then the point ____________ lies on the graph of f^{-1}

27. *True* or *false*. If $f(x) = x^2$, then $f^{-1}(x) = \sqrt{x}$.

28. If a function f has an inverse, then the graph of f^{-1} may be obtained by reflecting the graph of f across the line with equation ____________

29. If a function f has an inverse and $f(-3) = 6$, then $f^{-1}(6) =$ ____________.

30. If $f(-4) = 16$ and $f(4) = 16$, then f ____________ (does/does not) have an inverse because ____________.

Concept Check *In Exercises 31–36, an everyday activity is described. Keeping in mind that an inverse operation "undoes" what an operation does, describe each inverse activity.*

31. tying your shoelaces

32. starting a car

33. entering a room

34. climbing the stairs

35. screwing in a light bulb

36. filling a cup

Decide whether the given functions are inverses. See Example 4.

37.

x	$f(x)$
3	−4
2	−6
5	8
1	9
4	3

x	$g(x)$
−4	3
−6	2
8	5
9	1
3	4

38.

x	$f(x)$
−2	−8
−1	−1
0	0
1	1
2	8

x	$g(x)$
8	−2
1	−1
0	0
−1	1
−8	2

39. $f = \{(2, 5), (3, 5), (4, 5)\}$; $g = \{(5, 2)\}$

40. $f = \{(1, 1), (3, 3), (5, 5)\}$; $g = \{(1, 1), (3, 3), (5, 5)\}$

Use the definition of inverses to determine whether f and g are inverses. See Example 3.

41. $f(x) = 2x + 4, \quad g(x) = \frac{1}{2}x - 2$

42. $f(x) = 3x + 9, \quad g(x) = \frac{1}{3}x - 3$

43. $f(x) = -3x + 12, \quad g(x) = -\frac{1}{3}x - 12$

44. $f(x) = -4x + 2, \quad g(x) = -\frac{1}{4}x - 2$

45. $f(x) = \frac{x+1}{x-2}, \quad g(x) = \frac{2x+1}{x-1}$

46. $f(x) = \frac{x-3}{x+4}, \quad g(x) = \frac{4x+3}{1-x}$

47. $f(x) = \frac{2}{x+6}, \quad g(x) = \frac{6x+2}{x}$

48. $f(x) = \frac{-1}{x+1}, \quad g(x) = \frac{1-x}{x}$

49. $f(x) = x^2 + 3, \quad x \geq 0; \quad g(x) = \sqrt{x-3}, \quad x \geq 3$

50. $f(x) = \sqrt{x+8}, \quad x \geq -8; \quad g(x) = x^2 - 8, \quad x \geq 0$

If the function is one-to-one, find its inverse. See Example 4.

51. $\{(-3, 6), (2, 1), (5, 8)\}$

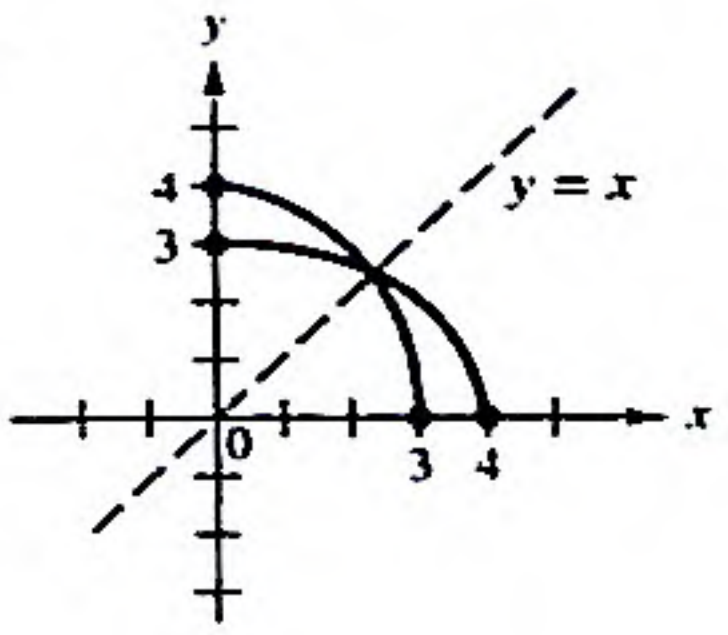
52. $\left\{(3, -1), (5, 0), (0, 5), \left(4, \frac{2}{3}\right)\right\}$

53. $\{(1, -3), (2, -7), (4, -3), (5, -5)\}$

54. $\{(6, -8), (3, -4), (0, -8), (5, -4)\}$

Decide whether each pair of functions graphed are inverses. See Example 7.

55.



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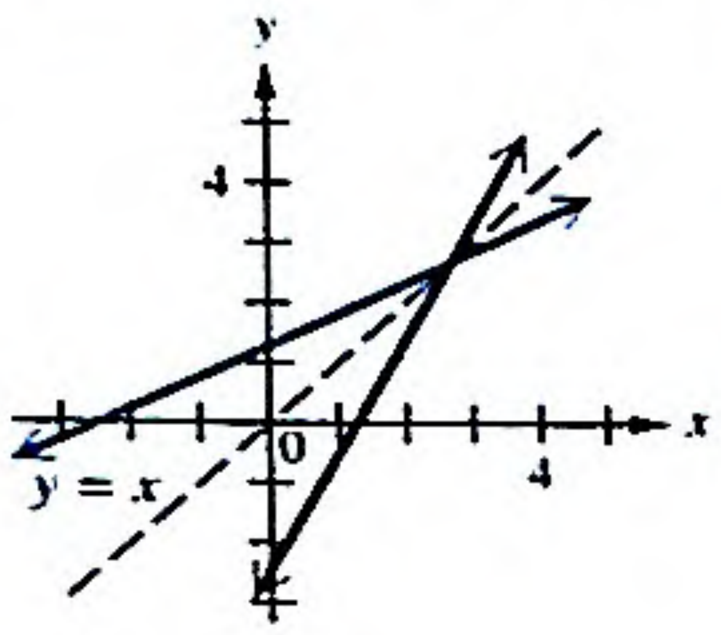

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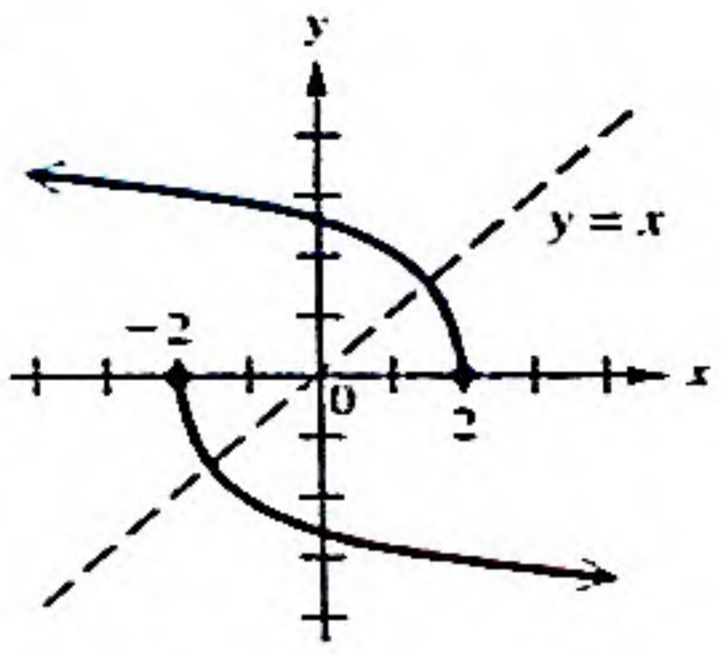

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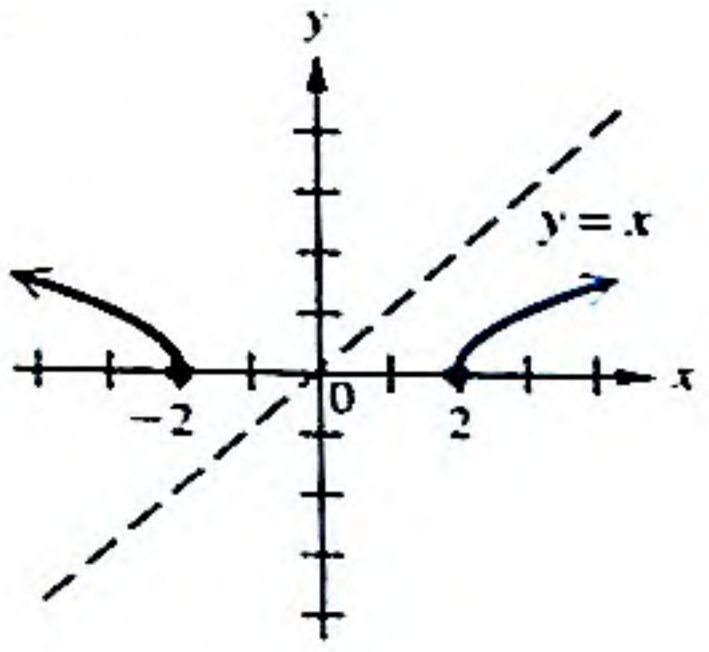


For each function as defined that is one-to-one, (a) write an equation for the inverse function in the form $y = f^{-1}(x)$, (b) graph f and f^{-1} on the same axes, and (c) give the domain and the range of f and f^{-1}. If the function is not one-to-one, say so. **See Examples 5–8.**

59. $y = 3x - 4$

60. $y = 4x - 5$

61. $f(x) = -4x + 3$

62. $f(x) = -6x - 8$

63. $f(x) = x^3 + 1$

64. $f(x) = -x^3 - 2$

65. $y = x^2 + 8$

66. $y = -x^2 + 2$

67. $y = \frac{1}{x}, \quad x \neq 0$

68. $y = \frac{4}{x}, \quad x \neq 0$

69. $f(x) = \frac{1}{x - 3}, \quad x \neq 3$

70. $f(x) = \frac{1}{x + 2}, \quad x \neq -2$

71. $f(x) = \frac{x + 1}{x - 3}, \quad x \neq 3$

72. $f(x) = \frac{x + 2}{x - 1}, \quad x \neq 1$

73. $f(x) = \frac{2x + 6}{x - 3}, \quad x \neq 3$

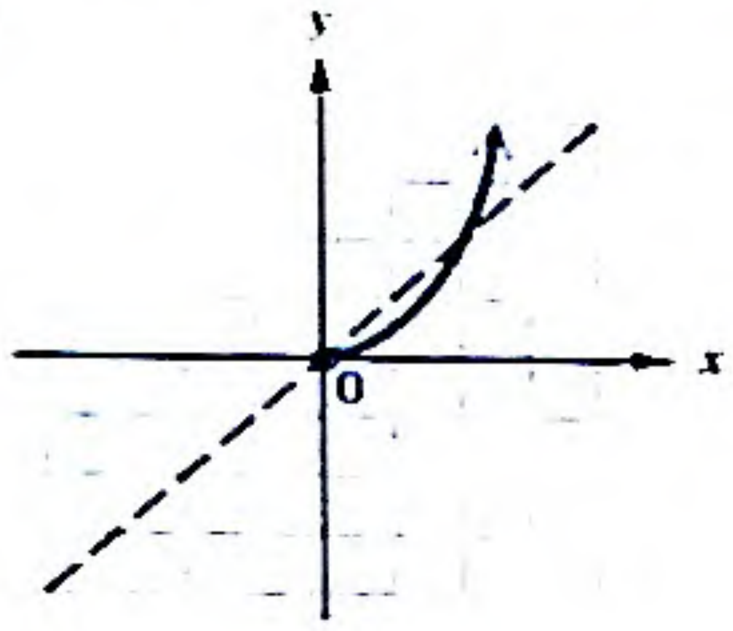
74. $f(x) = \frac{-3x + 12}{x - 6}, \quad x \neq 6$

75. $f(x) = \sqrt{6 + x}, \quad x \geq -6$

76. $f(x) = -\sqrt{x^2 - 16}, \quad x \geq 4$

Graph the inverse of each one-to-one function. **See Example 7.**

77.



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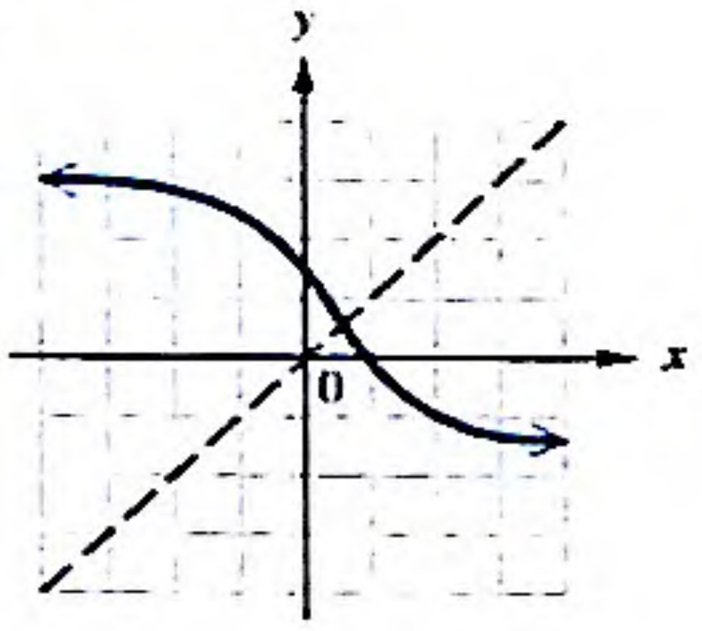

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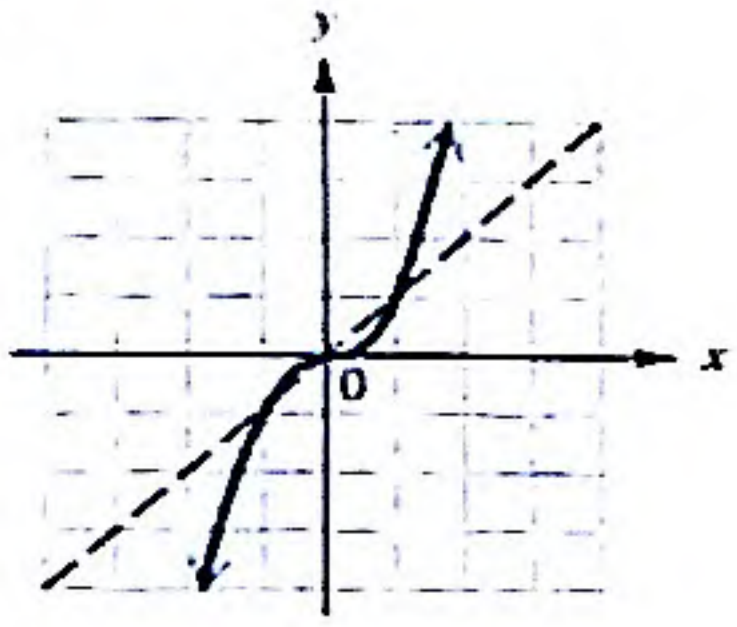

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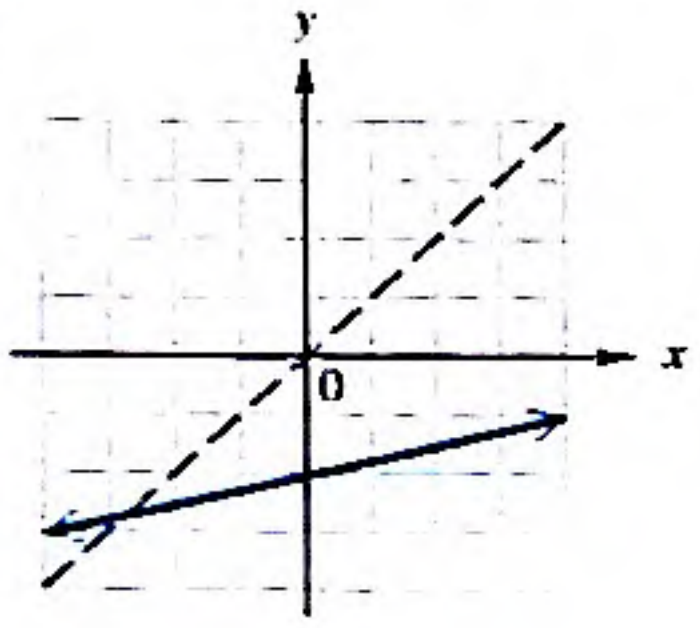

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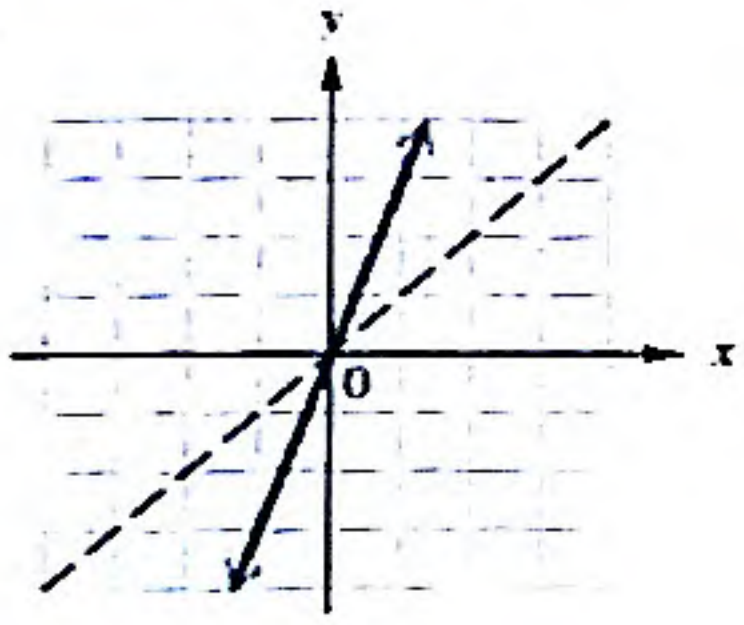

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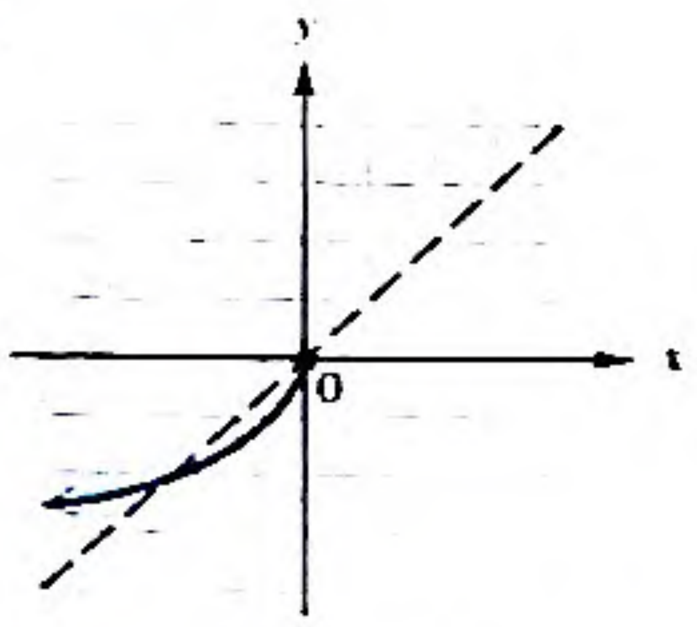


Concept Check The graph of a function f is shown in the figure. Use the graph to find each value.

83. $f^{-1}(4)$

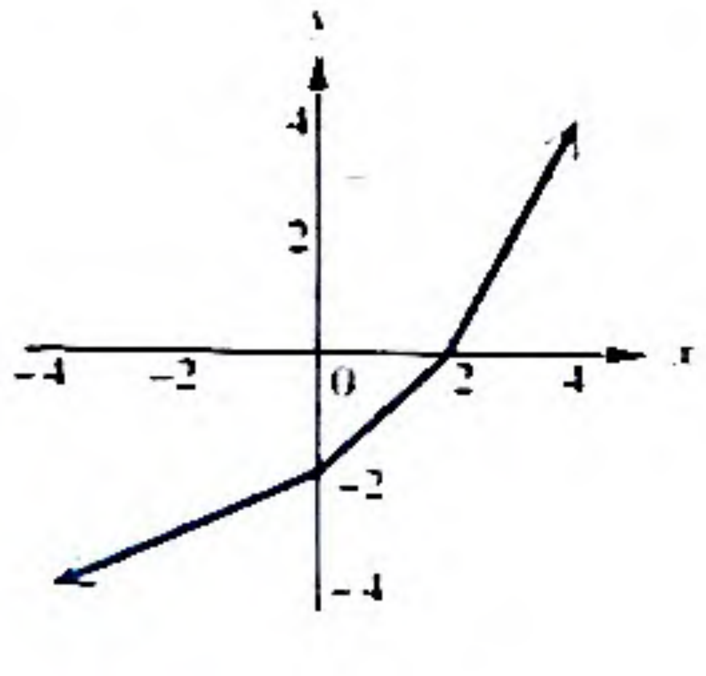
84. $f^{-1}(2)$

85. $f^{-1}(0)$

86. $f^{-1}(-2)$

87. $f^{-1}(-3)$

88. $f^{-1}(-4)$



Concept Check Answer each of the following.

89. Suppose $f(x)$ is the number of cars that can be built for x dollars. What does $f^{-1}(1000)$ represent?

90. Suppose $f(r)$ is the volume (in cubic inches) of a sphere of radius r inches. What does $f^{-1}(5)$ represent?

91. If a line has slope a, what is the slope of its reflection across the line $y = x$?

92. For a one-to-one function f, find $(f^{-1} \circ f)(2)$, where $f(2) = 3$.

Use a graphing calculator to graph each function defined as follows, using the given viewing window. Use the graph to decide which functions are one-to-one. If a function is one-to-one, give the equation of its inverse.

93. $f(x) = 6x^3 + 11x^2 - 6$; $[-3, 2]$ by $[-10, 10]$

94. $f(x) = x^4 - 5x^2$; $[-3, 3]$ by $[-8, 8]$

95. $f(x) = \dfrac{x-5}{x+3}$, $x \neq -3$; $[-8, 8]$ by $[-6, 8]$

96. $f(x) = \dfrac{-x}{x-4}$, $x \neq 4$; $[-1, 8]$ by $[-6, 6]$

Use the alphabet coding assignment given in ***Example 9*** *for Exercises 97–100.*

97. The function $f(x) = 3x - 2$ was used to encode a message as

37 25 19 61 13 34 22 1 55 1 52 52 25 64 13 10.

Find the inverse function and determine the message.

98. The function $f(x) = 2x - 9$ was used to encode a message as

−5 9 5 5 9 27 15 29 −1 21 19 31 −3 27 41.

Find the inverse function and determine the message.

99. Encode the message SEND HELP, using the one-to-one function $f(x) = x^3 - 1$. Give the inverse function that the decoder will need when the message is received.

100. Encode the message SAILOR BEWARE, using the one-to-one function $f(x) = (x + 1)^3$. Give the inverse function that the decoder will need when the message is received.

4.2 Exponential Functions

- Exponents and Properties
- Exponential Functions
- Exponential Equations
- Compound Interest
- The Number *e* and Continuous Compounding
- Exponential Models

Exponents and Properties Recall the definition of $a^{m/n}$: If a is a real number, m is an integer, n is a positive integer, and $\sqrt[n]{a}$ is a real number, then

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m. \quad \text{(Section R.7)}$$

For example,

$$16^{3/4} = \left(\sqrt[4]{16}\right)^3 = 2^3 = 8,$$

$$27^{-1/3} = \frac{1}{27^{1/3}} = \frac{1}{\sqrt[3]{27}} = \frac{1}{3}, \quad \text{and} \quad 64^{-1/2} = \frac{1}{64^{1/2}} = \frac{1}{\sqrt{64}} = \frac{1}{8}.$$

EXAMPLE 11 Using Data to Model Exponential Growth

Data from recent past years indicate that future amounts of carbon dioxide in the atmosphere may grow according to the table. Amounts are given in parts per million.

Year	Carbon Dioxide (ppm)
1990	353
2000	375
2075	590
2175	1090
2275	2000

Source: International Panel on Climate Change (IPCC).

(a) Make a scatter diagram of the data. Do the carbon dioxide levels appear to grow exponentially?

(b) One model for the data is the function

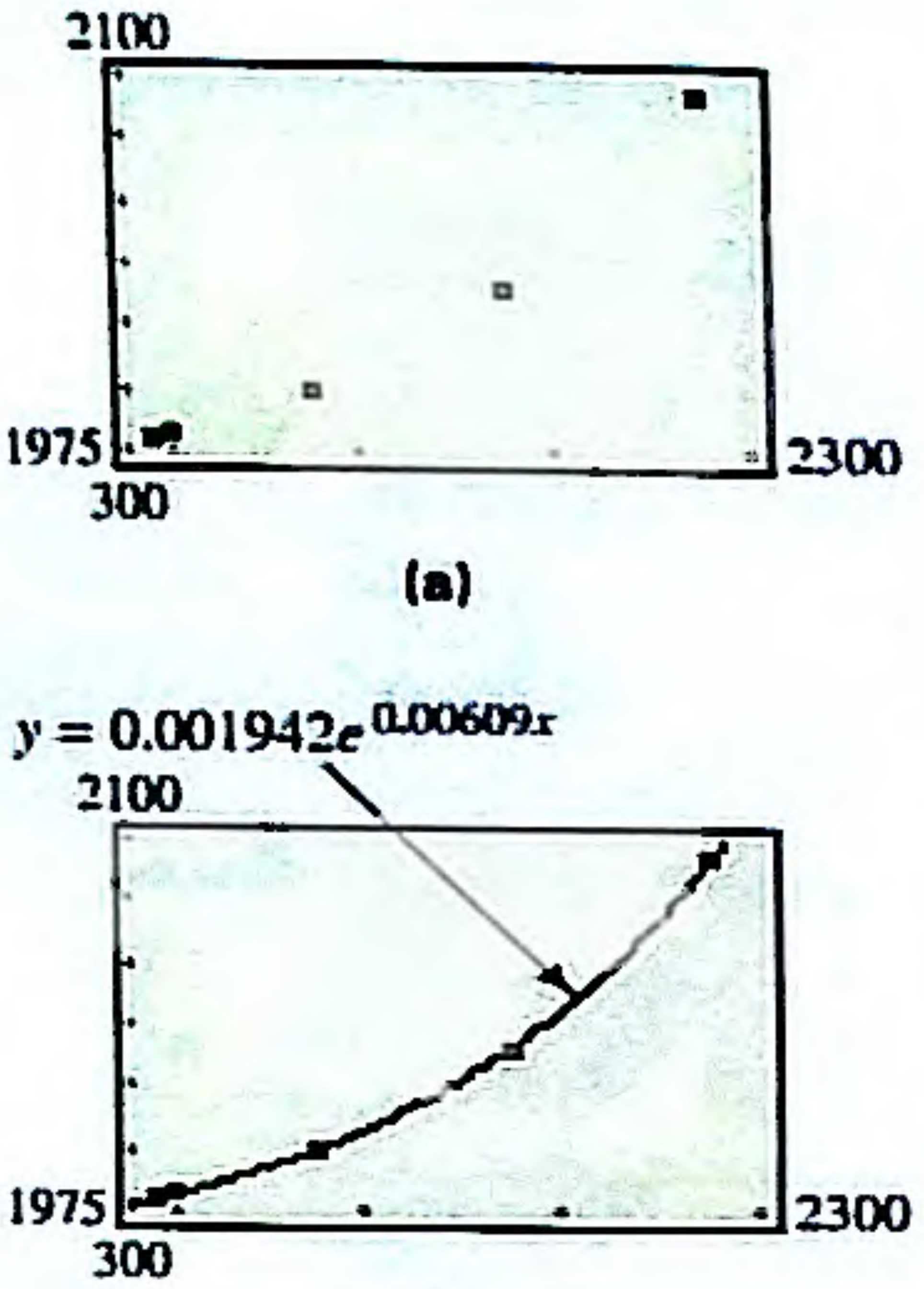
$$y = 0.001942e^{0.00609x},$$

where x is the year and $1990 \le x \le 2275$. Use a graph of this model to estimate when future levels of carbon dioxide will double and triple over the preindustrial level of 280 ppm.

SOLUTION

(a) We show a calculator graph for the data in **Figure 22(a)**. The data appear to resemble the graph of an increasing exponential function.

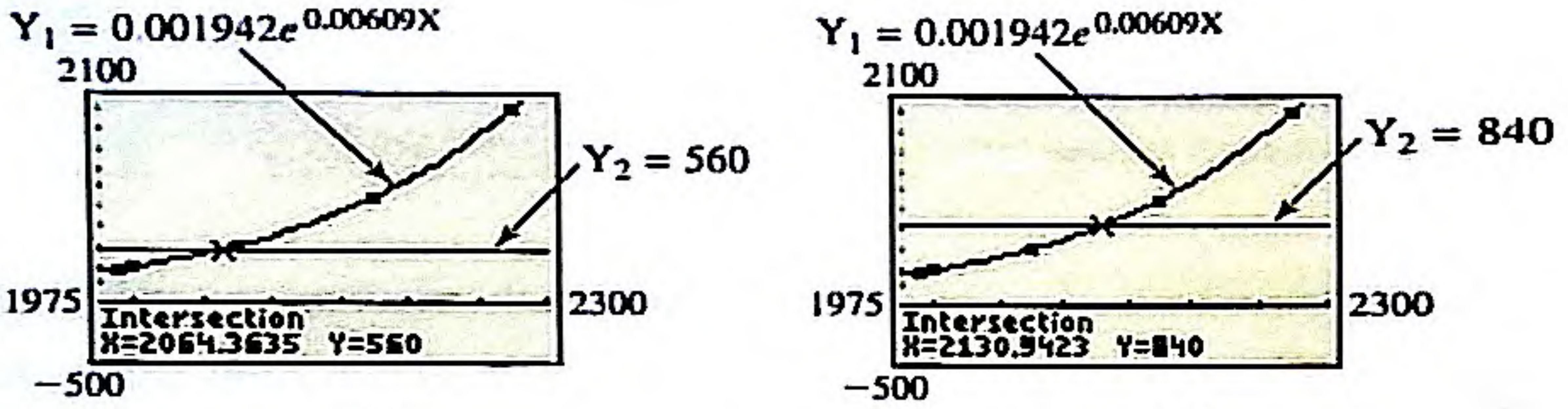
(b) A graph of $y = 0.001942e^{0.00609x}$ in **Figure 22(b)** shows that it is very close to the data points. We graph $y = 2 \cdot 280 = 560$ in **Figure 23(a)** and $y = 3 \cdot 280 = 840$ in **Figure 23(b)** on the same coordinate axes as the given function, and we use the calculator to find the intersection points.



(a)

(b)

Figure 22



```
Intersection
X=2064.3635  Y=560
```

```
Intersection
X=2130.9423  Y=840
```

(a)

(b)

Figure 23

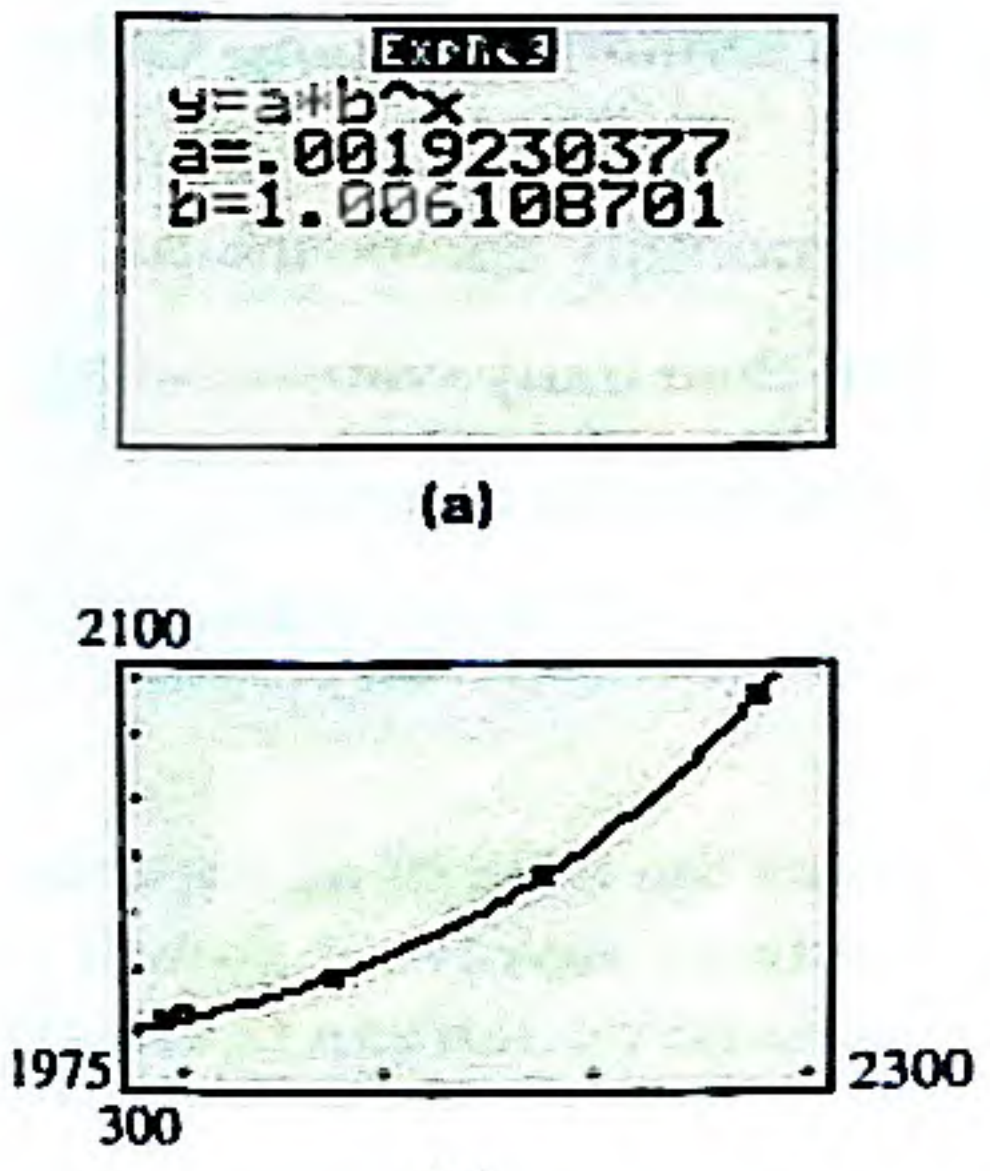
The graph of the function intersects the horizontal lines at approximately 2064.4 and 2130.9. According to this model, carbon dioxide levels will have doubled by 2064 and tripled by 2131.

✓ *Now Try Exercise 97.*

Graphing calculators are capable of fitting exponential curves to scatter diagrams like the one found in **Example 11. Figure 24(a)** shows how the TI-83/84 Plus displays another (different) equation for the atmospheric carbon dioxide example, approximated as follows.

$$y = 0.001923(1.006109)^x$$

Notice that this calculator form differs from the model in **Example 11. Figure 24(b)** shows the data points and the graph of this exponential regression equation. ■



```
ExpReg
y=a*b^x
a=.0019230377
b=1.006108701
```

(a)

(b)

Figure 24

4.2 Exercises

For $f(x) = 3^x$ and $g(x) = \left(\frac{1}{4}\right)^x$, find each of the following. In Exercises 13–16, round the answer to the nearest thousandth. See Example 1.

1. $f(2)$

2. $f(3)$

3. $f(-2)$

4. $f(-3)$

5. $g(2)$

6. $g(3)$

7. $g(-2)$

8. $g(-3)$

9. $f\left(\frac{3}{2}\right)$

10. $f\left(-\frac{5}{2}\right)$

11. $g\left(\frac{3}{2}\right)$

12. $g\left(-\frac{5}{2}\right)$

13. $f(2.34)$

14. $f(-1.68)$

15. $g(-1.68)$

16. $g(2.34)$

Graph each function. See Example 2.

17. $f(x) = 3^x$

18. $f(x) = 4^x$

19. $f(x) = \left(\frac{1}{3}\right)^x$

20. $f(x) = \left(\frac{1}{4}\right)^x$

21. $f(x) = \left(\frac{3}{2}\right)^x$

22. $f(x) = \left(\frac{5}{3}\right)^x$

23. $f(x) = \left(\frac{1}{10}\right)^{-x}$

24. $f(x) = \left(\frac{1}{6}\right)^{-x}$

25. $f(x) = 4^{-x}$

26. $f(x) = 10^{-x}$

27. $f(x) = 2^{|x|}$

28. $f(x) = 2^{-|x|}$

Sketch the graph of $f(x) = 2^x$. Then refer to it and use the techniques of Chapter 2 to graph each function as defined. See Example 3.

29. $f(x) = 2^x + 1$

30. $f(x) = 2^x - 4$

31. $f(x) = 2^{x+1}$

32. $f(x) = 2^{x-4}$

33. $f(x) = -2^{x+2}$

34. $f(x) = -2^{x-3}$

35. $f(x) = 2^{-x}$

36. $f(x) = -2^{-x}$

37. $f(x) = 2^{x-1} + 2$

38. $f(x) = 2^{x+3} + 1$

39. $f(x) = 2^{x+2} - 4$

40. $f(x) = 2^{x-3} - 1$

Sketch the graph of $f(x) = \left(\frac{1}{3}\right)^x$. Then refer to it and use the techniques of Chapter 2 to graph each function as defined. See Example 3.

41. $f(x) = \left(\frac{1}{3}\right)^x - 2$

42. $f(x) = \left(\frac{1}{3}\right)^x + 4$

43. $f(x) = \left(\frac{1}{3}\right)^{x+2}$

44. $f(x) = \left(\frac{1}{3}\right)^{x-4}$

45. $f(x) = \left(\frac{1}{3}\right)^{-x+1}$

46. $f(x) = \left(\frac{1}{3}\right)^{-x-2}$

47. $f(x) = \left(\frac{1}{3}\right)^{-x}$

48. $f(x) = -\left(\frac{1}{3}\right)^{-x}$

49. $f(x) = \left(\frac{1}{3}\right)^{x-2} + 2$

50. $f(x) = \left(\frac{1}{3}\right)^{x-1} + 3$

51. $f(x) = \left(\frac{1}{3}\right)^{x+2} - 1$

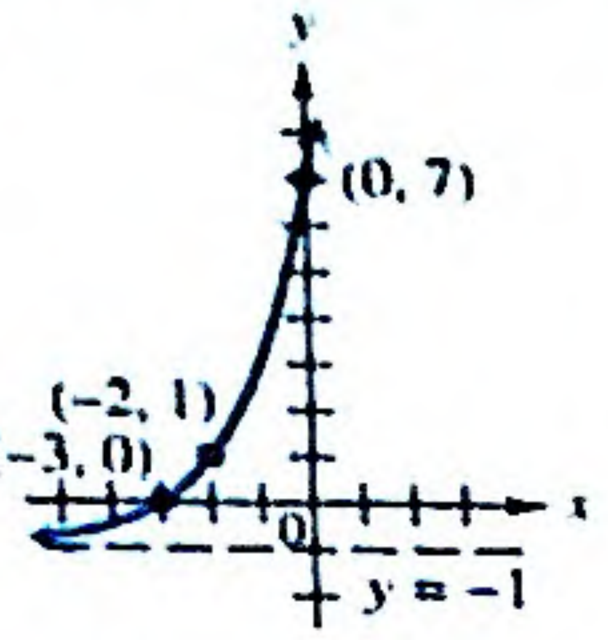
52. $f(x) = \left(\frac{1}{3}\right)^{x+3} - 2$

53. ***Concept Check*** Fill in the blank: The graph of $f(x) = a^{-x}$ is the same as that of $g(x) = (\_\_\_\_\_\_)^x$.

54. ***Concept Check*** Fill in the blanks: If $a > 1$, then the graph of $f(x) = a^x$ __________ (rises/falls) from left to right. If $0 < a < 1$, then the graph of $g(x) = a^x$ __________ (rises/falls) from left to right.

Connecting Graphs with Equations *Write an equation for the graph given. Each represents an exponential function f, with base 2 or 3, translated and/or reflected.*

55.



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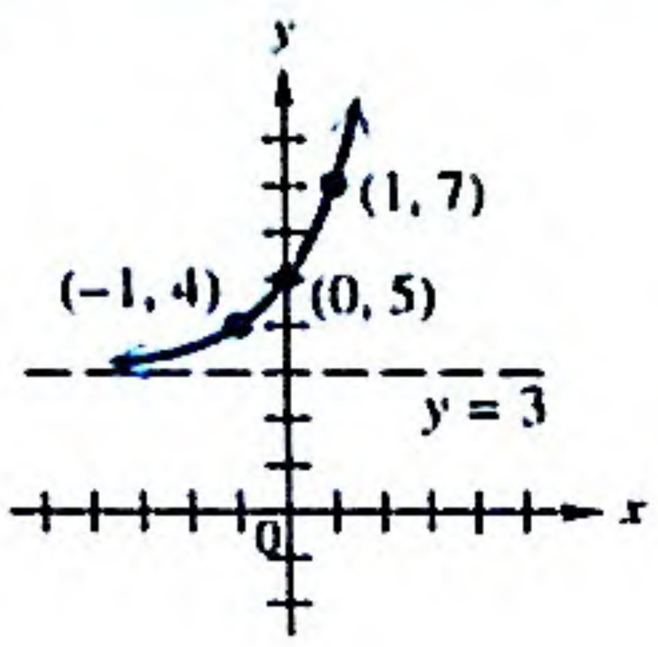

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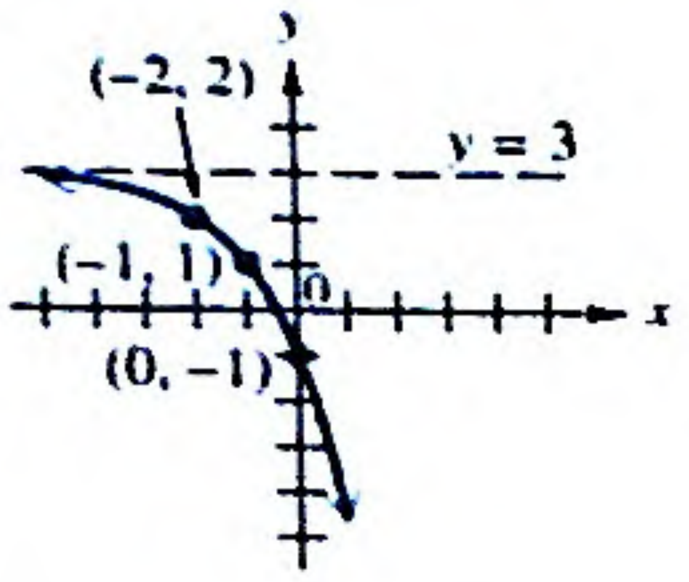

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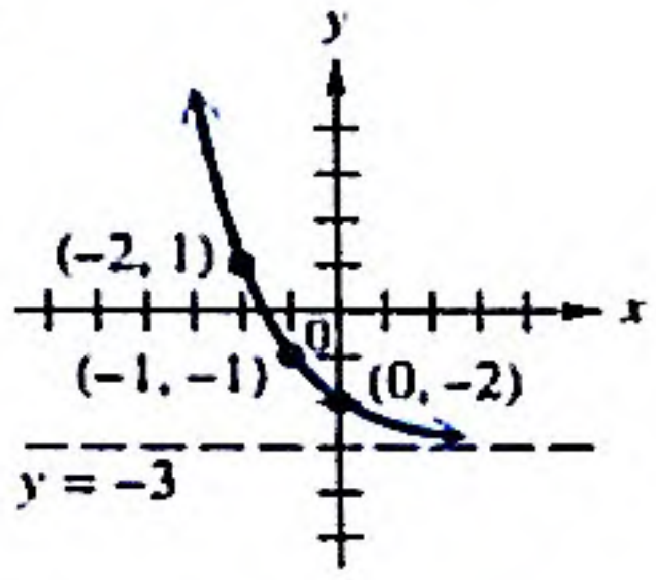

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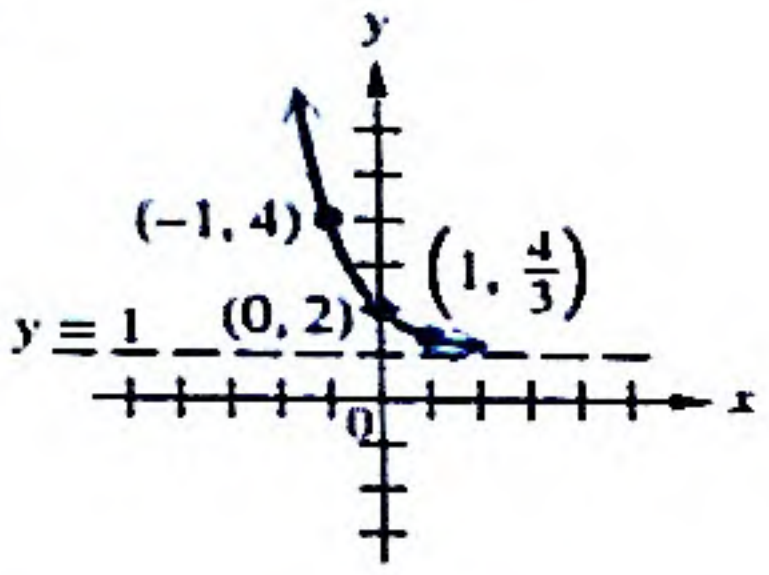

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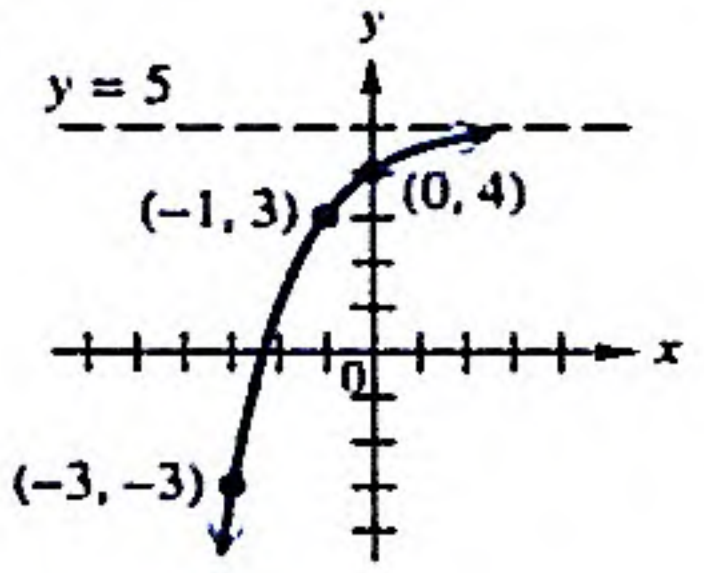


Solve each equation. See Examples 4–6.

61. $4^x = 2$

62. $125^x = 5$

63. $\left(\frac{5}{2}\right)^x = \frac{4}{25}$

64. $\left(\frac{2}{3}\right)^x = \frac{9}{4}$

65. $2^{3-2x} = 8$

66. $5^{2+2x} = 25$

67. $e^{4x-1} = (e^2)^x$

68. $e^{3-x} = (e^3)^{-x}$

69. $27^{4x} = 9^{x+1}$

70. $32^{2x} = 16^{x-1}$

71. $4^{x-2} = 2^{3x+3}$

72. $2^{6-3x} = 8^{x+1}$

73. $\left(\frac{1}{e}\right)^{-x} = \left(\frac{1}{e^2}\right)^{x+1}$

74. $e^{x-1} = \left(\frac{1}{e^4}\right)^{x+1}$

75. $(\sqrt{2})^{x+4} = 4^x$

76. $(\sqrt[3]{5})^{-x} = \left(\frac{1}{5}\right)^{x+2}$

77. $\frac{1}{27} = x^{-3}$

78. $\frac{1}{32} = x^{-5}$

79. $x^{2/3} = 4$

80. $x^{2/5} = 16$

81. $x^{5/2} = 32$

82. $x^{3/2} = 27$

83. $x^{-6} = \frac{1}{64}$

84. $x^{-4} = \frac{1}{256}$

85. $x^{5/3} = -243$

86. $x^{7/5} = -128$

Solve each problem involving compound interest. See Examples 7–9.

87. *Future Value* Find the future value and interest earned if \$8906.54 is invested for 9 yr at 5% compounded

(a) semiannually (b) continuously

88. ***Future Value*** Find the future value and interest earned if \$56,780 is invested at 5.3% compounded
    (a) quarterly for 23 quarters (b) continuously for 15 yr.

89. ***Present Value*** Find the present value that will grow to \$25,000 if interest is 6% compounded quarterly for 11 quarters.

90. ***Present Value*** Find the present value that will grow to \$45,000 if interest is 3.6% compounded monthly for 1 yr.

91. ***Present Value*** Find the present value that will grow to \$5000 if interest is 3.5% compounded quarterly for 10 yr.

92. ***Interest Rate*** Find the required annual interest rate to the nearest tenth of a percent for \$65,000 to grow to \$65,325 if interest is compounded monthly for 6 months.

93. ***Interest Rate*** Find the required annual interest rate to the nearest tenth of a percent for \$1200 to grow to \$1500 if interest is compounded quarterly for 5 yr.

94. ***Interest Rate*** Find the required annual interest rate to the nearest tenth of a percent for \$5000 to grow to \$8400 if interest is compounded quarterly for 8 yr.

Solve each problem. See Example 10.

95. ***Comparing Loans*** Bank A is lending money at 6.4% interest compounded annually. The rate at Bank B is 6.3% compounded monthly, and the rate at Bank C is 6.35% compounded quarterly. At which bank will you pay the *least* interest?

96. ***Future Value*** Suppose \$10,000 is invested at an annual rate of 5% for 10 yr. Find the future value if interest is compounded as follows.
    (a) annually (b) quarterly (c) monthly (d) daily (365 days)

(Modeling) Solve each problem. See Example 11.

97. ***Atmospheric Pressure*** The atmospheric pressure (in millibars) at a given altitude (in meters) is shown in the table.

Altitude	Pressure	Altitude	Pressure
0	1013	6000	472
1000	899	7000	411
2000	795	8000	357
3000	701	9000	308
4000	617	10,000	265
5000	541		

Source: Miller, A. and J. Thompson, *Elements of Meteorology*, Fourth Edition, Charles E. Merrill Publishing Company, Columbus, Ohio.

(a) Use a graphing calculator to make a scatter diagram of the data for atmospheric pressure P at altitude x.

(b) Would a linear or an exponential function fit the data better?

(c) The function

$$P(x) = 1013e^{-0.0001341x}$$

approximates the data. Use a graphing calculator to graph P and the data on the same coordinate axes.

(d) Use P to predict the pressures at 1500 m and 11,000 m, and compare them to the actual values of 846 millibars and 227 millibars, respectively.

98. ***World Population Growth*** Since 2000, world population in millions closely fits the exponential function

$$y = 6084e^{0.0120x},$$

where x is the number of years since 2000. (*Source:* U.S. Census Bureau.)

(a) The world population was about 6853 million in 2010. How closely does the function approximate this value?
(b) Use this model to predict the population in 2020.
(c) Use this model to predict the population in 2030.
(d) Explain why this model may not be accurate for 2030.

99. ***Deer Population*** The exponential growth of the deer population in Massachusetts can be calculated using the model

$$f(x) = 50{,}000(1 + 0.06)^x,$$

where 50,000 is the initial deer population and 0.06 is the rate of growth. $f(x)$ is the total population after x years have passed.

(a) Predict the total population after 4 yr.
(b) If the initial population was 30,000 and the growth rate was 0.12, approximately how many deer would be present after 3 yr?
(c) How many additional deer can we expect in 5 yr if the initial population is 45,000 and the current growth rate is 0.08?

100. ***Employee Training*** A person learning certain skills involving repetition tends to learn quickly at first. Then learning tapers off and skill acquisition approaches some upper limit. Suppose the number of symbols per minute that a person using a keyboard can type is given by

$$f(t) = 250 - 120(2.8)^{-0.5t},$$

where t is the number of months the operator has been in training. Find each value.

(a) $f(2)$ **(b)** $f(4)$ **(c)** $f(10)$
(d) What happens to the number of symbols per minute after several months of training?

Use a graphing calculator to find the solution set of each equation. Approximate the solution(s) to the nearest tenth.

101. $5e^{3x} = 75$ **102.** $6^{-x} = 1 - x$ **103.** $3x + 2 = 4^x$ **104.** $x = 2^x$

105. A function of the form $f(x) = x^r$, where r is a constant, is called a **power function.** Discuss the difference between an exponential function and a power function.

106. ***Concept Check*** If $f(x) = a^x$ and $f(3) = 27$, find each value of $f(x)$.

(a) $f(1)$ **(b)** $f(-1)$ **(c)** $f(2)$ **(d)** $f(0)$

Concept Check *Give an equation of the form* $f(x) = a^x$ *to define the exponential function whose graph contains the given point.*

107. $(3, 8)$ **108.** $(3, 125)$ **109.** $(-3, 64)$ **110.** $(-2, 36)$

Concept Check *Use properties of exponents to write each function in the form* $f(t) = ka^t$, *where k is a constant. (Hint. Recall that* $a^{x+y} = a^x \cdot a^y$.*)*

111. $f(t) = 3^{2t+3}$ **112.** $f(t) = 2^{3t+2}$ **113.** $f(t) = \left(\frac{1}{3}\right)^{1-2t}$ **114.** $f(t) = \left(\frac{1}{2}\right)^{1-2t}$

In calculus, it is shown that

$$e^x = 1 + x + \frac{x^2}{2 \cdot 1} + \frac{x^3}{3 \cdot 2 \cdot 1} + \frac{x^4}{4 \cdot 3 \cdot 2 \cdot 1} + \frac{x^5}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} + \cdots.$$

By using more terms, one can obtain a more accurate approximation for e^x.

115. Use the terms shown, and replace x with 1 to approximate $e^1 = e$ to three decimal places. Check your result with a calculator.

116. Use the terms shown, and replace x with -0.05 to approximate $e^{-0.05}$ to four decimal places. Check your result with a calculator.

Relating Concepts

For individual or collaborative investigation *(Exercises 117–122)*

Assume $f(x) = a^x$, where $a > 1$. Work these exercises in order.

117. Is f a one-to-one function? If so, based on **Section 4.1,** what kind of related function exists for f?

118. If f has an inverse function f^{-1}, sketch f and f^{-1} on the same set of axes.

119. If f^{-1} exists, find an equation for $y = f^{-1}(x)$ using the method described in **Section 4.1.** You need not solve for y.

120. If $a = 10$, what is the equation for $y = f^{-1}(x)$? (You need not solve for y.)

121. If $a = e$, what is the equation for $y = f^{-1}(x)$? (You need not solve for y.)

122. If the point (p, q) is on the graph of f, then the point ______ is on the graph of f^{-1}.

4.3 Logarithmic Functions

- **Logarithms**
- **Logarithmic Equations**
- **Logarithmic Functions**
- **Properties of Logarithms**

Logarithms The previous section dealt with exponential functions of the form $y = a^x$ for all positive values of a, where $a \neq 1$. The horizontal line test shows that exponential functions are one-to-one and thus have inverse functions. The equation defining the inverse of a function is found by interchanging x and y in the equation that defines the function. Starting with $y = a^x$ and interchanging x and y yields

$$x = a^y.$$

Here y is the exponent to which a must be raised in order to obtain x. We call this exponent a **logarithm,** symbolized by the abbreviation "log." The expression $\log_a x$ represents the logarithm in this discussion. The number a is called the base of the logarithm, and x is called the **argument** of the expression. It is read "logarithm with base a of x," or "logarithm of x with base a," or "base a logarithm of x."

Logarithm

For all real numbers y and all positive numbers a and x, where $a \neq 1$,

$$y = \log_a x \quad \text{is equivalent to} \quad x = a^y.$$

The expression $\log_a x$ represents the exponent to which the base a must be raised in order to obtain x.

Recall that for inverse functions f and g, $(f \circ g)(x) = (g \circ f)(x) = x$. We can use this property with exponential and logarithmic functions to state two more properties. If $f(x) = a^x$ and $g(x) = \log_a x$, then

$$(f \circ g)(x) = a^{\log_a x} \quad \text{and} \quad (g \circ f)(x) = \log_a(a^x).$$

Theorem on Inverses

For $a > 0$, $a \neq 1$, the following properties hold.

$$a^{\log_a x} = x \text{ (for } x > 0) \quad \text{and} \quad \log_a a^x = x$$

The following are examples of applications of this theorem.

$$7^{\log_7 10} = 10, \quad \log_5 5^3 = 3, \quad \text{and} \quad \log_r r^{k+1} = k + 1$$

The second statement in the theorem will be useful in Sections 4.5 and 4.6 when we solve other logarithmic and exponential equations.

4.3 Exercises

Concept Check *In Exercises 1 and 2, match the logarithm in Column I with its value in Column II. Remember that* $\log_a x$ *is the exponent to which a must be raised in order to obtain x.*

	I	II
1.	(a) $\log_2 16$	A. 0
	(b) $\log_3 1$	B. $\frac{1}{2}$
	(c) $\log_{10} 0.1$	C. 4
	(d) $\log_2 \sqrt{2}$	D. -3
	(e) $\log_e \frac{1}{e^2}$	E. -1
	(f) $\log_{1/2} 8$	F. -2

	I	II
2.	(a) $\log_3 81$	A. -2
	(b) $\log_3 \frac{1}{3}$	B. -1
	(c) $\log_{10} 0.01$	C. 0
	(d) $\log_6 \sqrt{6}$	D. $\frac{1}{2}$
	(e) $\log_e 1$	E. $\frac{9}{2}$
	(f) $\log_3 27^{3/2}$	F. 4

If the statement is in exponential form, write it in an equivalent logarithmic form. If the statement is in logarithmic form, write it in exponential form. See Example 1.

3. $3^4 = 81$

4. $2^5 = 32$

5. $\left(\frac{2}{3}\right)^{-3} = \frac{27}{8}$

6. $10^{-4} = 0.0001$

7. $\log_6 36 = 2$

8. $\log_5 5 = 1$

9. $\log_{\sqrt{3}} 81 = 8$

10. $\log_4 \frac{1}{64} = -3$

11. Explain why logarithms of negative numbers are not defined

12. ***Concept Check*** Why is $\log_a 1$ always equal to 0 for any valid base a?

Solve each logarithmic equation. See Example 2.

13. $x = \log_5 \frac{1}{625}$

14. $x = \log_3 \frac{1}{81}$

15. $\log_x \frac{1}{32} = 5$

16. $\log_x \frac{27}{64} = 3$

17. $x = \log_8 \sqrt[4]{8}$

18. $x = \log_7 \sqrt[5]{7}$

19. $x = 3^{\log_3 8}$

20. $x = 12^{\log_{12} 5}$

21. $x = 2^{\log_2 9}$

22. $x = 8^{\log_8 11}$

23. $\log_x 25 = -2$

24. $\log_x 16 = -2$

25. $\log_4 x = 3$

26. $\log_2 x = 3$

27. $x = \log_4 \sqrt[3]{16}$

28. $x = \log_5 \sqrt[4]{25}$

29. $\log_9 x = \frac{5}{2}$

30. $\log_4 x = \frac{7}{2}$

31. $\log_{1/2}(x + 3) = -4$

32. $\log_{1/3}(x + 6) = -2$

33. $\log_{(x+3)} 6 = 1$

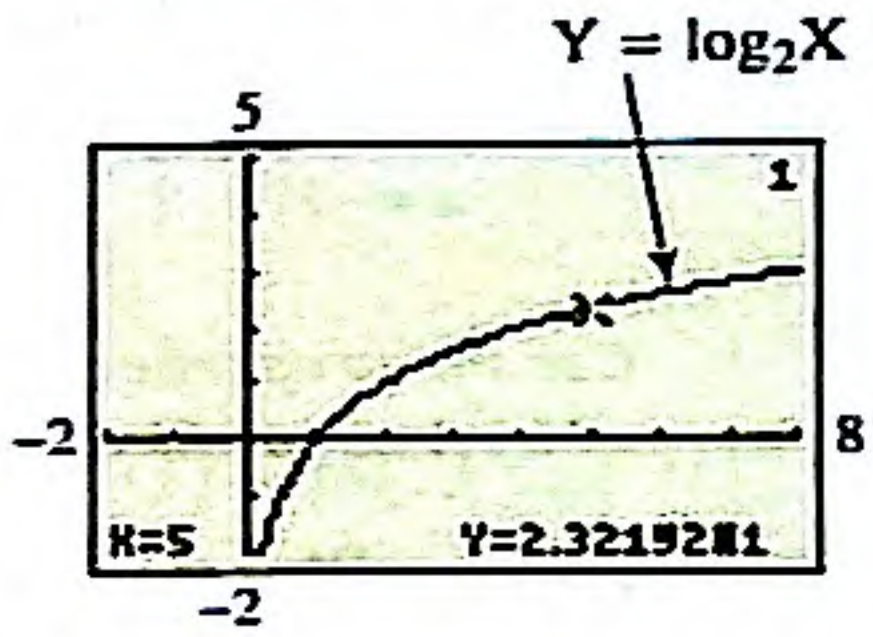
34. $\log_{(x-4)} 19 = 1$

35. $3x - 15 = \log_x 1 \quad (x > 0, x \neq 1)$

36. $4x - 24 = \log_x 1 \quad (x > 0, x \neq 1)$

37. Compare the summary of characteristics of the graph of $f(x) = \log_a x$ with the similar summary about the graph of $f(x) = a^x$ in Section 4.2. Make a list of characteristics that reinforce the idea that these are inverse functions.

38. ***Concept Check*** The calculator graph of $Y = \log_2 X$ shows the values of the ordered pair with $X = 5$. What does the value of Y represent?



Sketch the graph of $f(x) = \log_2 x$. Then refer to it and use the techniques of Chapter 2 to graph each function. Give the domain and range. See Example 4.

39. $f(x) = (\log_2 x) + 3$

40. $f(x) = \log_2(x + 3)$

41. $f(x) = |\log_2(x + 3)|$

Sketch the graph of $f(x) = \log_{1/2} x$. Then refer to it and use the techniques of Chapter 2 to graph each function. Give the domain and range. See Example 4.

42. $f(x) = (\log_{1/2} x) - 2$

43. $f(x) = \log_{1/2}(x - 2)$

44. $f(x) = |\log_{1/2}(x - 2)|$

Concept Check *In Exercises 45–50, match the function with its graph from choices A–F.*

45. $f(x) = \log_2 x$

46. $f(x) = \log_2 2x$

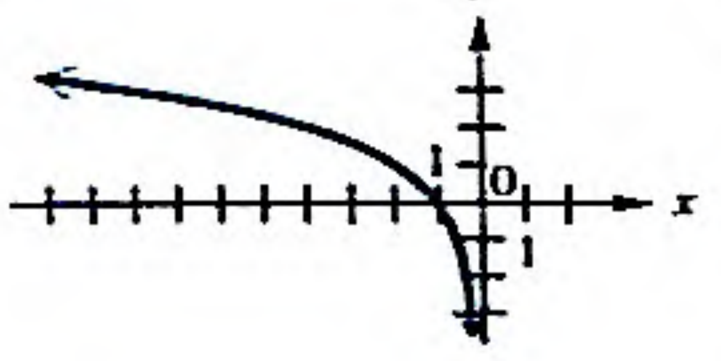
47. $f(x) = \log_2 \frac{1}{x}$

48. $f(x) = \log_2\left(\frac{1}{2}x\right)$

49. $f(x) = \log_2(x - 1)$

50. $f(x) = \log_2(-x)$

A.



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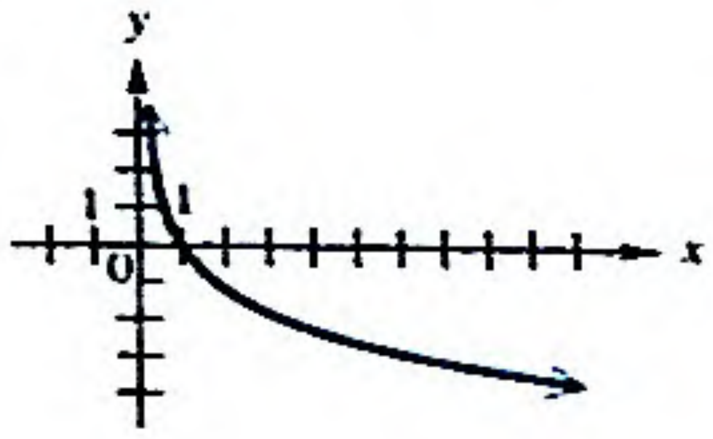

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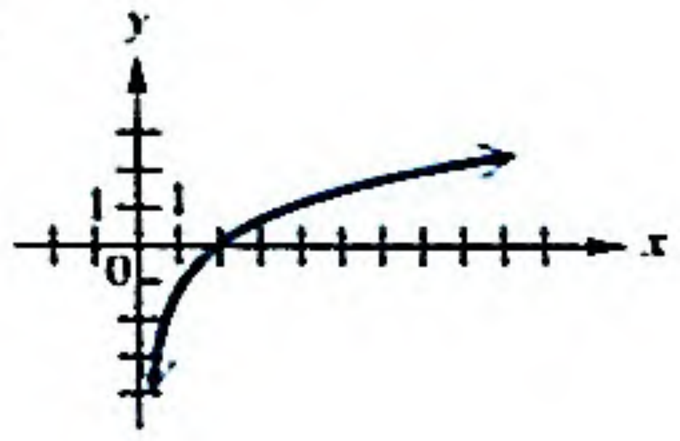

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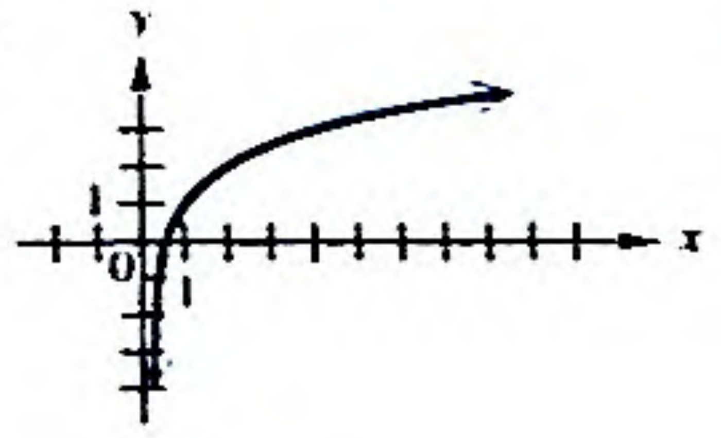
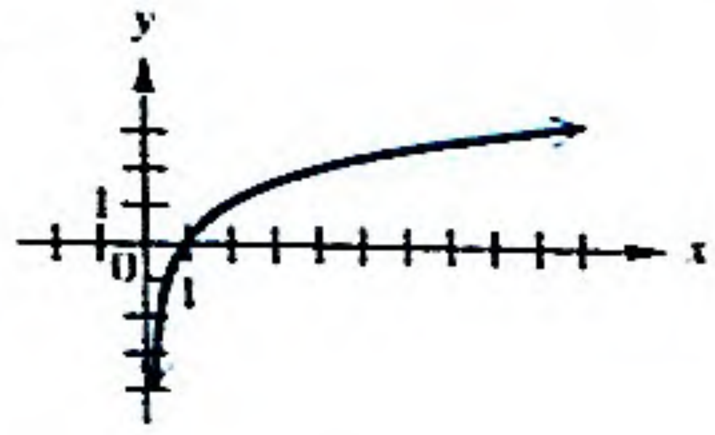
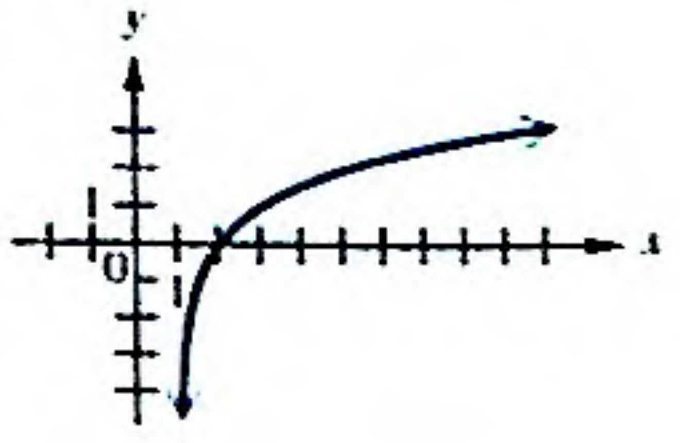


E.



F.



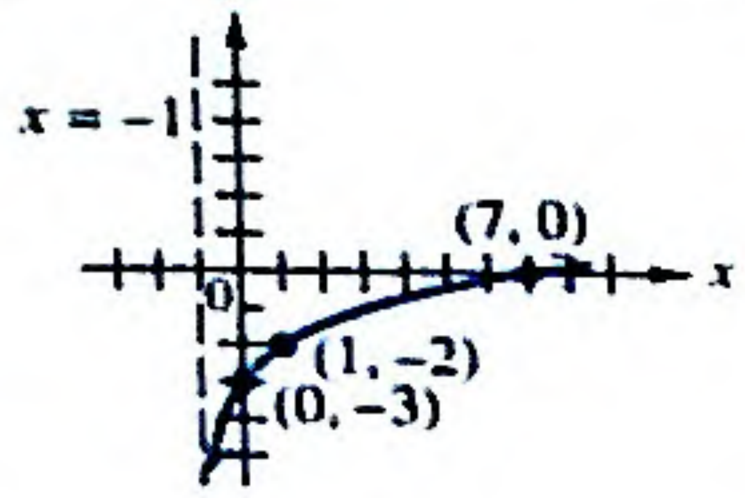
Graph each function. See Examples 3 and 4.

51. $f(x) = \log_5 x$ **52.** $f(x) = \log_{10} x$ **53.** $f(x) = \log_{1/2}(1 - x)$

54. $f(x) = \log_{1/3}(3 - x)$ **55.** $f(x) = \log_3(x - 1) + 2$ **56.** $f(x) = \log_2(x + 2) - 3$

Connecting Graphs with Equations *In Exercises 57–62, write an equation for the graph given. Each is a logarithmic function f with base 2 or 3, translated and/or reflected. See the Note following Example 4.*

57.



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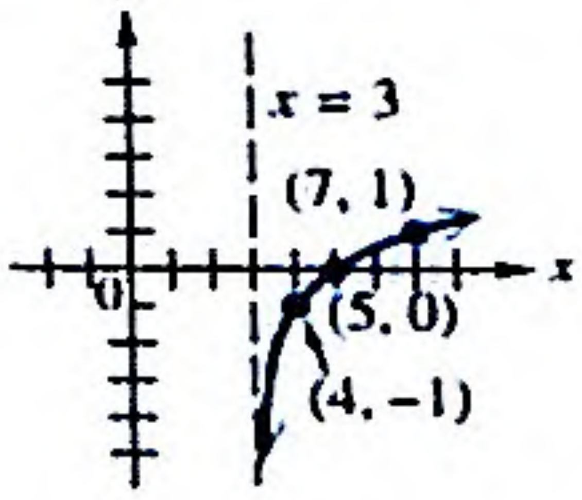

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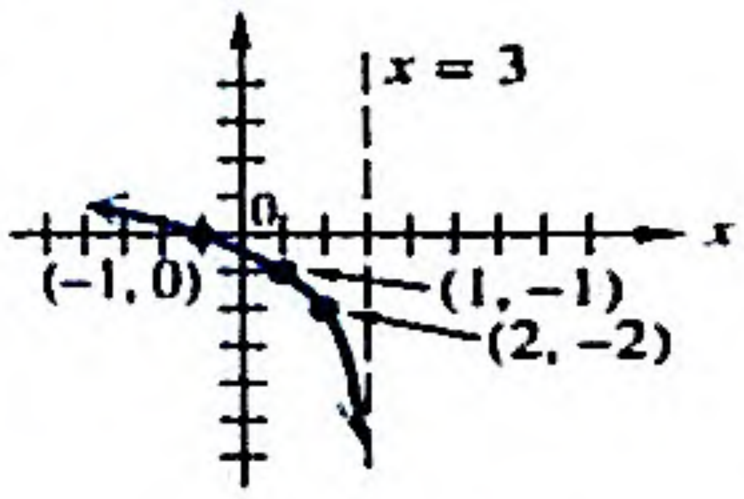

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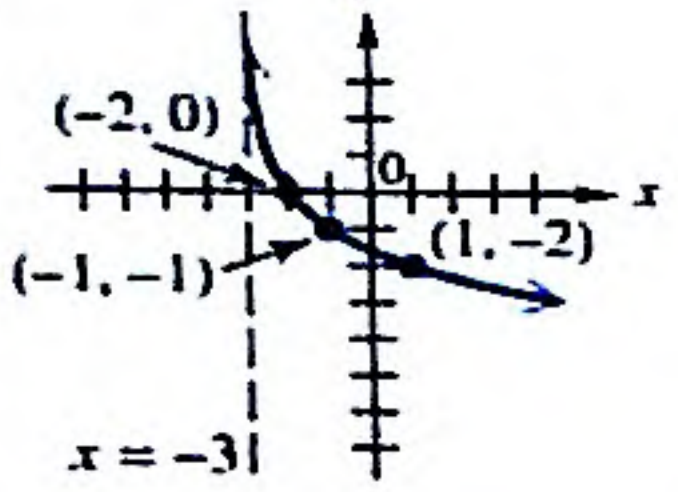

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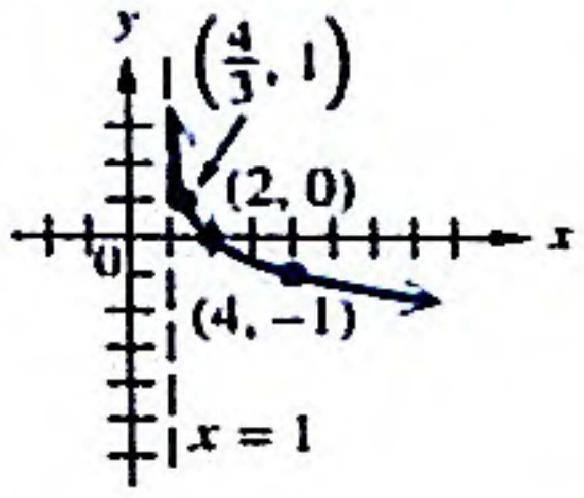

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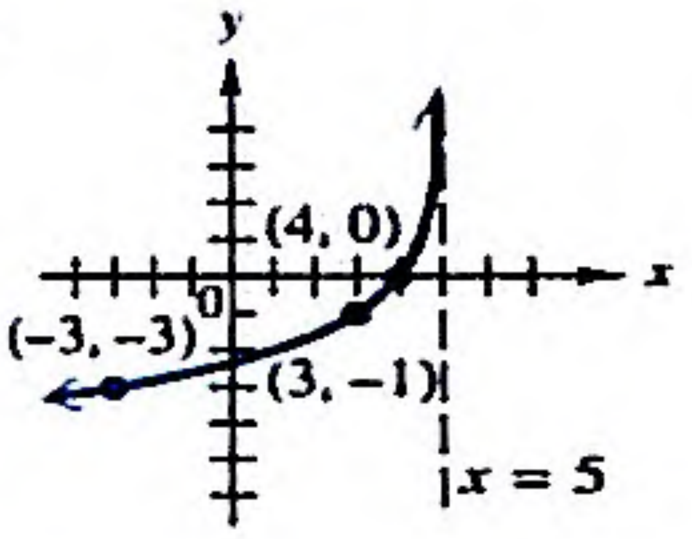


Use the properties of logarithms to rewrite each expression. Simplify the result if possible. Assume all variables represent positive real numbers. See Example 5.

63. $\log_2 \frac{6x}{y}$ **64.** $\log_3 \frac{4p}{q}$ **65.** $\log_5 \frac{5\sqrt{7}}{3}$

66. $\log_2 \frac{2\sqrt{3}}{5}$ **67.** $\log_4(2x + 5y)$ **68.** $\log_6(7m + 3q)$

69. $\log_m \sqrt{\frac{5r^3}{z^5}}$ **70.** $\log_p \sqrt[3]{\frac{m^5n^4}{t^2}}$ **71.** $\log_2 \frac{ab}{cd}$

72. $\log_2 \frac{xy}{tqr}$ **73.** $\log_3 \frac{\sqrt{x} \cdot \sqrt[3]{y}}{w^2\sqrt{z}}$ **74.** $\log_4 \frac{\sqrt[3]{a} \cdot \sqrt[4]{b}}{\sqrt{c} \cdot \sqrt[3]{d^2}}$

Write each expression as a single logarithm with coefficient 1. Assume all variables represent positive real numbers. See Example 6.

75. $\log_a x + \log_a y - \log_a m$ **76.** $\log_b k + \log_b m - \log_b a$

77. $\log_a m - \log_a n - \log_a t$ **78.** $\log_b p - \log_b q - \log_b r$

79. $\frac{1}{3} \log_b x^4y^5 - \frac{3}{4} \log_b x^2y$ **80.** $\frac{1}{2} \log_y p^3q^4 - \frac{2}{3} \log_y p^4q^3$

81. $2 \log_a(z + 1) + \log_a(3z + 2)$ **82.** $5 \log_a(z + 7) + \log_a(2z + 9)$

83. $-\frac{2}{3} \log_5 5m^2 + \frac{1}{2} \log_5 25m^2$ **84.** $-\frac{3}{4} \log_3 16p^4 - \frac{2}{3} \log_3 8p^3$

Given the approximations $\log_{10} 2 = 0.3010$ *and* $\log_{10} 3 = 0.4771$, *find each logarithm without using a calculator. See Example 7.*

85. $\log_{10} 6$ **86.** $\log_{10} 12$ **87.** $\log_{10} \frac{3}{2}$ **88.** $\log_{10} \frac{2}{9}$

89. $\log_{10} \frac{9}{4}$ **90.** $\log_{10} \frac{20}{27}$ **91.** $\log_{10} \sqrt{30}$ **92.** $\log_{10} 36^{1/3}$

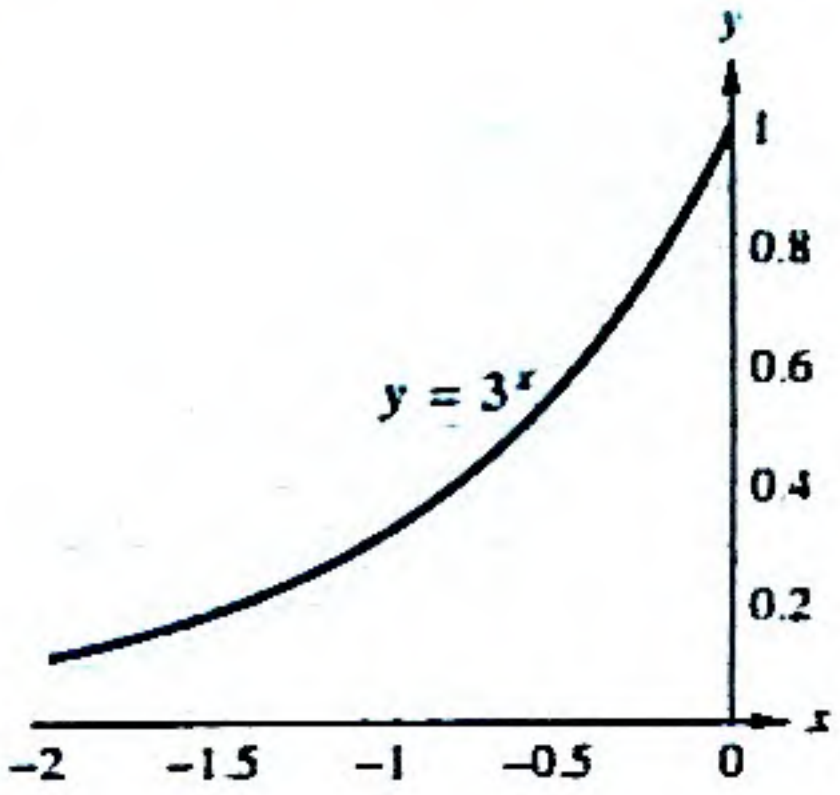
Solve each problem.

93. ***(Modeling) Interest Rates of Treasury Securities*** The table gives interest rates for various U.S. Treasury Securities on February 15, 2011.
   (a) Make a scatter diagram of the data.
   (b) Which type of function will model these data best: linear, exponential, or logarithmic?

Time	Yield
3-month	0.13%
6-month	0.17%
2-year	0.84%
5-year	2.35%
10-year	3.61%
30-year	4.66%

Source: www.federalreserve.gov

94. ***Concept Check*** Use the graph to estimate each logarithm.
   (a) $\log_3 0.3$ (b) $\log_3 0.8$



95. ***Concept Check*** Suppose $f(x) = \log_a x$ and $f(3) = 2$. Determine each function value.
   (a) $f\left(\frac{1}{9}\right)$ (b) $f(27)$ (c) $f(9)$ (d) $f\left(\frac{\sqrt{3}}{3}\right)$

96. Use properties of logarithms to evaluate each expression.
   (a) $100^{\log_{10} 3}$ (b) $\log_{10}(0.01)^3$ (c) $\log_{10}(0.0001)^5$ (d) $1000^{\log_{10} 5}$

97. Refer to the compound interest formula from **Section 4.2.** Show that the amount of time required for a deposit to double is

$$\frac{1}{\log_2\left(1 + \frac{r}{n}\right)^n}.$$

98. ***Concept Check*** If $(5, 4)$ is on the graph of the logarithmic function with base a, which of the following statements is true:

$$5 = \log_a 4 \quad \text{or} \quad 4 = \log_a 5?$$

Standard mathematical notation uses $\log x$ *as an abbreviation for* $\log_{10} x$. *Use the* log *key on your graphing calculator to graph each function.*

99. $f(x) = x \log_{10} x$

100. $f(x) = x^2 \log_{10} x$

Use a graphing calculator to find the solution set of each equation. Give solutions to the nearest hundredth.

101. $\log_{10} x = x - 2$

102. $2^{-x} = \log_{10} x$

103. Prove the quotient property of logarithms: $\log_a \frac{x}{y} = \log_a x - \log_a y$.

104. Prove the power property of logarithms: $\log_a x^r = r \log_a x$.

Summary Exercises on Inverse, Exponential, and Logarithmic Functions

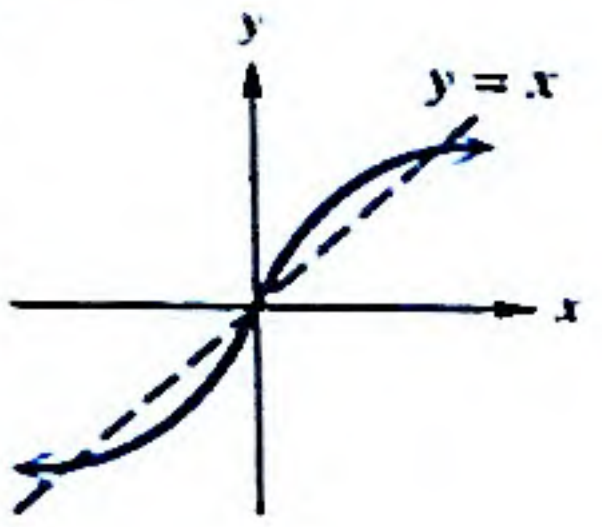
The following exercises are designed to help solidify your understanding of inverse, exponential, and logarithmic functions from Sections 4.1–4.3.

Determine whether the functions in each pair are inverses of each other.

1. $f(x) = 3x - 4, \quad g(x) = \frac{1}{3}x + \frac{4}{3}$
2. $f(x) = 8 - 5x, \quad g(x) = 8 + \frac{1}{5}x$
3. $f(x) = 1 + \log_2 x, \quad g(x) = 2^{x-1}$
4. $f(x) = 3^{x/5} - 2, \quad g(x) = 5 \log_3(x + 2)$

Determine whether each function is one-to-one. If it is, then sketch the graph of its inverse function.

5.



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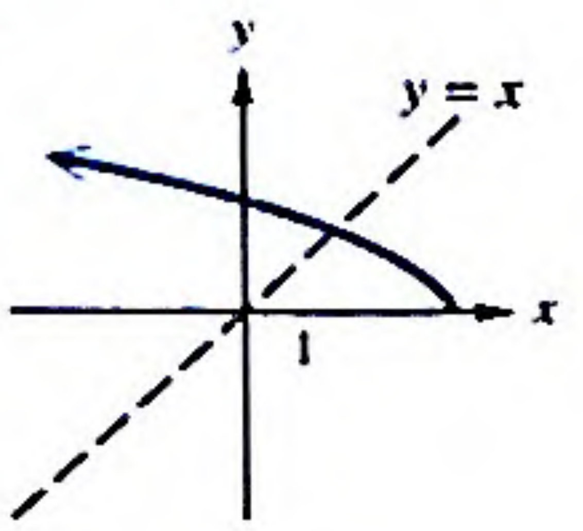

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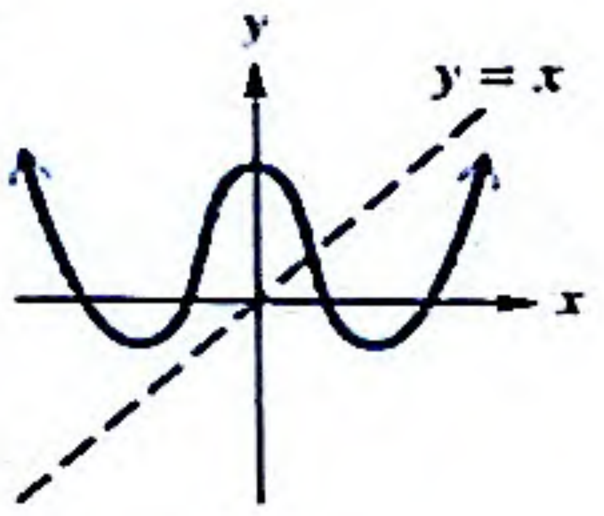

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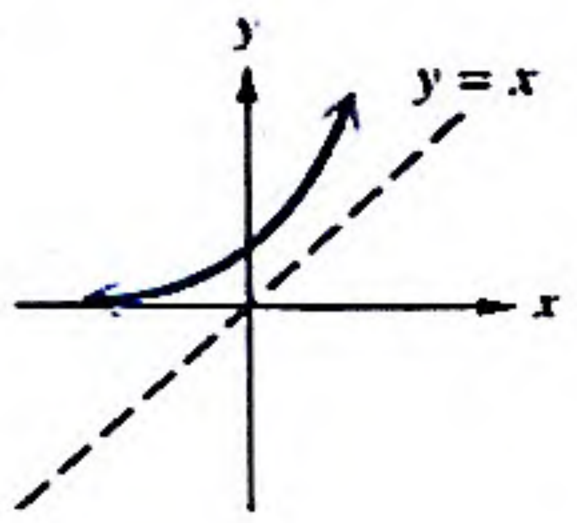
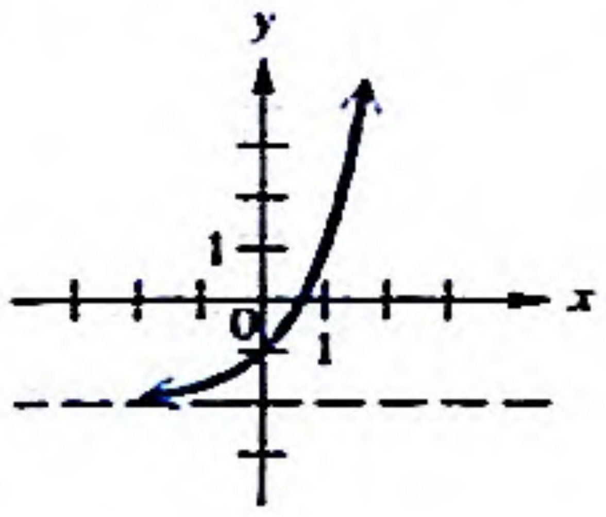


In Exercises 9–12, match the function with its graph from choices A–D.

9. $y = \log_3(x + 2)$
10. $y = 5 - 2^x$
11. $y = \log_2(5 - x)$
12. $y = 3^x - 2$

A.



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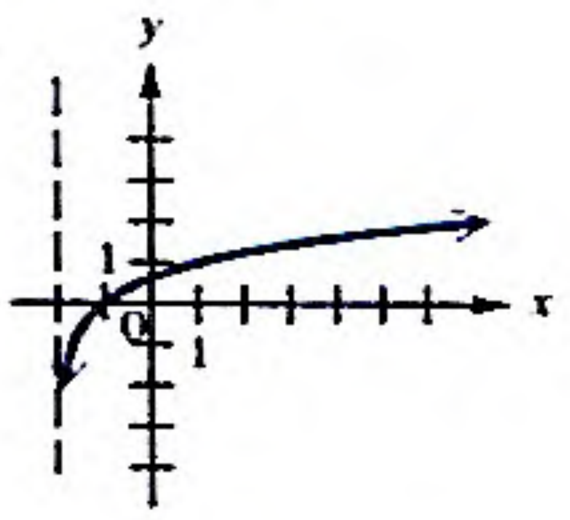

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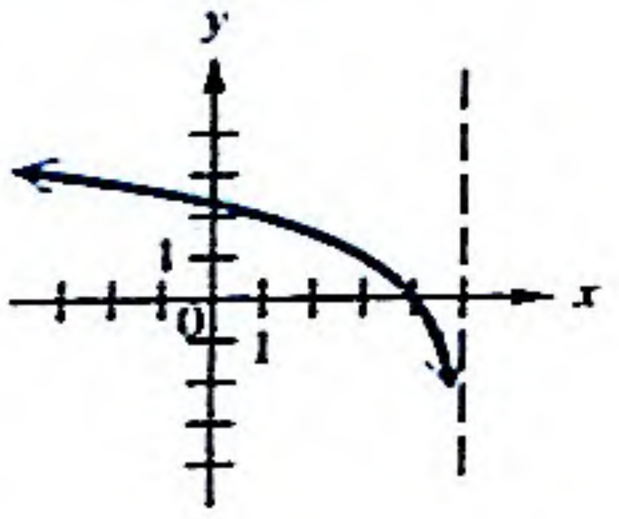

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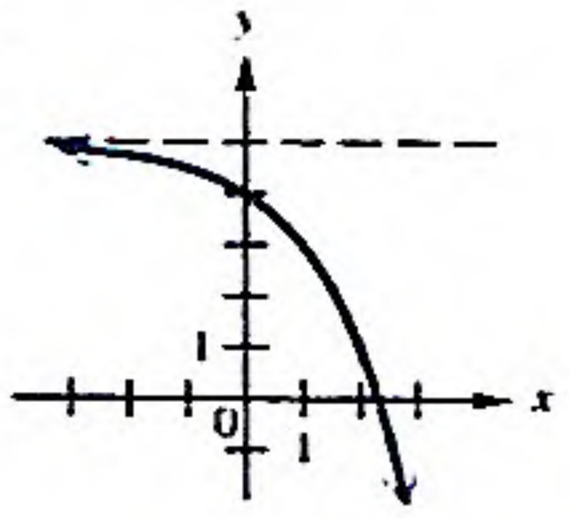


13. The functions in **Exercises 9–12** form two pairs of inverse functions. Determine which functions are inverses of each other.

14. Determine the inverse of the function $f(x) = \log_5 x$. (*Hint:* Replace $f(x)$ with y, and write in exponential form.)

For each function that is one-to-one, write an equation for its inverse function. Give the domain and range of both f and f^{-1}. If the function is not one-to-one, say so.

15. $f(x) = 3x - 6$

16. $f(x) = 2(x + 1)^3$

17. $f(x) = 3x^2$

18. $f(x) = \dfrac{2x - 1}{5 - 3x}$

19. $f(x) = \sqrt[3]{5 - x^4}$

20. $f(x) = \sqrt{x^2 - 9},\ x \geq 3$

Write an equivalent statement in logarithmic form.

21. $\left(\dfrac{1}{10}\right)^{-3} = 1000$

22. $a^b = c$

23. $(\sqrt{3})^4 = 9$

24. $4^{-3/2} = \dfrac{1}{8}$

25. $2^x = 32$

26. $27^{4/3} = 81$

Solve each equation.

27. $3x = 7^{\log_7 6}$

28. $x = \log_{10} 0.001$

29. $x = \log_6 \dfrac{1}{216}$

30. $\log_x 5 = \dfrac{1}{2}$

31. $\log_{10} 0.01 = x$

32. $\log_x 3 = -1$

33. $\log_x 1 = 0$

34. $x = \log_2 \sqrt{8}$

35. $\log_x \sqrt[3]{5} = \dfrac{1}{3}$

36. $\log_{1/3} x = -5$

37. $\log_{10}(\log_2 2^{10}) = x$

38. $x = \log_{4/5} \dfrac{25}{16}$

39. $2x - 1 = \log_6 6^x$

40. $x = \sqrt{\log_{1/2} \dfrac{1}{16}}$

41. $2^x = \log_2 16$

42. $\log_3 x = -2$

43. $\left(\dfrac{1}{3}\right)^{x+1} = 9^x$

44. $5^{2x-6} = 25^{x-3}$

4.4 Evaluating Logarithms and the Change-of-Base Theorem

- Common Logarithms
- Applications and Models with Common Logarithms
- Natural Logarithms
- Applications and Models with Natural Logarithms
- Logarithms with Other Bases

Common Logarithms Two of the most important bases for logarithms are 10 and e. Base 10 logarithms are called **common logarithms.** The common logarithm of x is written $\log x$, where the base is understood to be 10.

Common Logarithm

For all positive numbers x,

$$\log x = \log_{10} x.$$

A calculator with a log key can be used to find the base 10 logarithm of any positive number.

4.4 Exercises

***Concept Check** Answer each of the following.*

1. For the exponential function $f(x) = a^x$, where $a > 1$, is the function increasing or decreasing over its entire domain?

2. For the logarithmic function $g(x) = \log_a x$, where $a > 1$, is the function increasing or decreasing over its entire domain?

3. If $f(x) = 5^x$, what is the rule for $f^{-1}(x)$?

4. What is the name given to the exponent to which 4 must be raised to obtain 11?

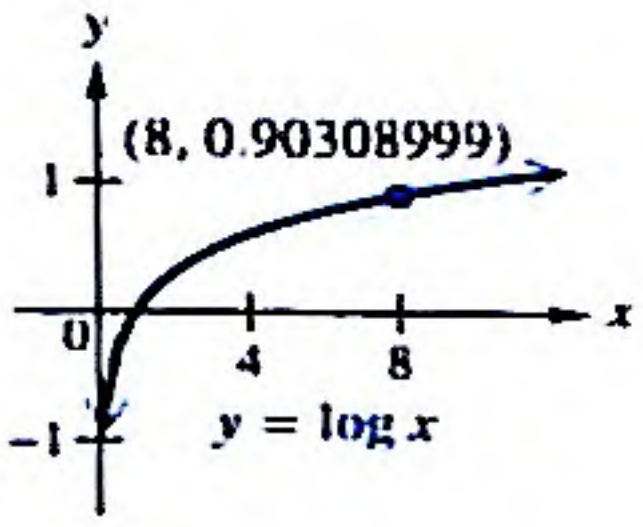
5. A base e logarithm is called a(n) ________ logarithm, and a base 10 logarithm is called a(n) ________ logarithm.

6. How is $\log_3 12$ written in terms of natural logarithms?

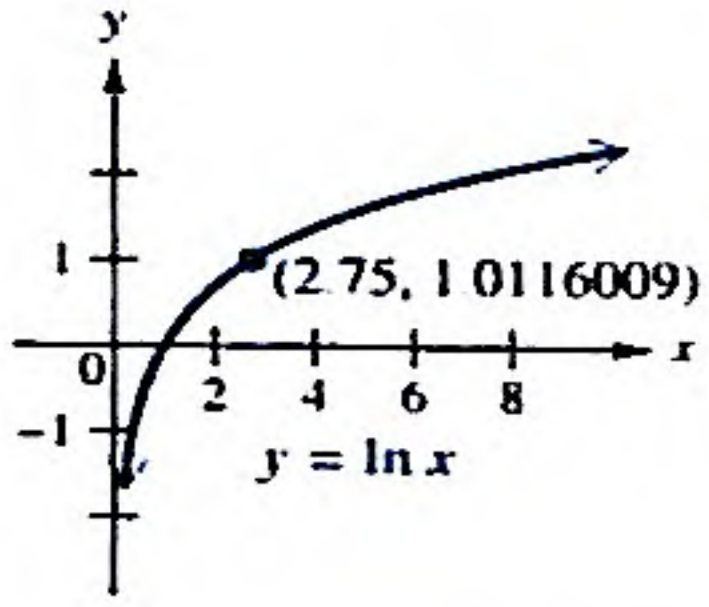
7. Why is $\log_2 0$ undefined?

8. Between what two consecutive integers must $\log_2 12$ lie?

9. The graph of $y = \log x$ shows a point on the graph. Write the logarithmic equation associated with that point.



10. The graph of $y = \ln x$ shows a point on the graph. Write the logarithmic equation associated with that point.



Find each value. If applicable, give an approximation to four decimal places. ***See Example 1.***

11. $\log 10^{12}$

12. $\log 10^7$

13. $\log 0.1$

14. $\log 0.01$

15. $\log 63$

16. $\log 94$

17. $\log 0.0022$

18. $\log 0.0055$

19. $\log(387 \times 23)$

20. $\log(296 \times 12)$

21. $\log\left(\frac{518}{342}\right)$

22. $\log\left(\frac{643}{287}\right)$

23. $\log 387 + \log 23$

24. $\log 296 + \log 12$

25. $\log 518 - \log 342$

26. $\log 643 - \log 287$

27. Explain why the result in Exercise 23 is the same as that in Exercise 19.

28. Explain why the result in Exercise 25 is the same as that in Exercise 21.

For each substance, find the pH from the given hydronium ion concentration. See Example 2(a).

29. grapefruit, 6.3×10^{-4} **30.** limes, 1.6×10^{-2}

31. crackers, 3.9×10^{-9} **32.** sodium hydroxide (lye), 3.2×10^{-14}

Find the $[H_3O^+]$ *for each substance with the given pH. See Example 2(b).*

33. soda pop, 2.7 **34.** wine, 3.4

35. beer, 4.8 **36.** drinking water, 6.5

In Exercises 37–42, suppose that water from a wetland area is sampled and found to have the given hydronium ion concentration. Determine whether the wetland is a rich fen, a poor fen, or a bog. See Example 3.

37. 2.49×10^{-5} **38.** 6.22×10^{-5} **39.** 2.49×10^{-2}

40. 3.14×10^{-2} **41.** 2.49×10^{-7} **42.** 5.86×10^{-7}

43. Use your calculator to find an approximation for each logarithm.
(a) log 398.4 (b) log 39.84 (c) log 3.984
(d) From your answers to parts (a)–(c), make a conjecture concerning the decimal values in the approximations of common logarithms of numbers greater than 1 that have the same digits.

44. Given that $\log 25 \approx 1.3979$, $\log 250 \approx 2.3979$, and $\log 2500 \approx 3.3979$, make a conjecture for an approximation of log 25,000. Then explain why this pattern continues.

Find each value. If applicable, give an approximation to four decimal places. See Example 5.

45. $\ln e^{1.6}$ **46.** $\ln e^{5.8}$ **47.** $\ln\left(\frac{1}{e^2}\right)$

48. $\ln\left(\frac{1}{e^4}\right)$ **49.** ln 28 **50.** ln 39

51. ln 0.00013 **52.** ln 0.0077 **53.** $\ln(27 \times 943)$

54. $\ln(33 \times 568)$ **55.** $\ln\left(\frac{98}{13}\right)$ **56.** $\ln\left(\frac{84}{17}\right)$

57. ln 27 + ln 943 **58.** ln 33 + ln 568

59. ln 98 − ln 13 **60.** ln 84 − ln 17

61. Explain why the result in Exercise 57 is the same as that in Exercise 53.

62. Explain why the result in Exercise 59 is the same as that in Exercise 55.

Solve each application of logarithms. See Examples 4, 6, 7, and 9.

63. ***Decibel Levels*** Find the decibel ratings of sounds having the following intensities.
(a) $100I_0$ (b) $1000I_0$ (c) $100{,}000I_0$ (d) $1{,}000{,}000I_0$
(e) If the intensity of a sound is doubled, by how much is the decibel rating increased?

64. ***Decibel Levels*** Find the decibel ratings of the following sounds, having intensities as given. Round each answer to the nearest whole number.
(a) whisper, $115I_0$ (b) busy street, $9{,}500{,}000I_0$
(c) heavy truck, 20 m away, $1{,}200{,}000{,}000I_0$
(d) rock music, $895{,}000{,}000{,}000I_0$
(e) jetliner at takeoff, $109{,}000{,}000{,}000{,}000I_0$

65. *Earthquake Intensity* The magnitude of an earthquake, measured on the Richter scale, is $\log_{10} \frac{I}{I_0}$, where I is the amplitude registered on a seismograph 100 km from the epicenter of the earthquake, and I_0 is the amplitude of an earthquake of a certain (small) size. Find the Richter scale ratings for earthquakes having the following amplitudes.

(a) $1000I_0$ (b) $1{,}000{,}000I_0$ (c) $100{,}000{,}000I_0$

66. *Earthquake Intensity* On December 26, 2004, an earthquake struck in the Indian Ocean with a magnitude of 9.1 on the Richter scale. The resulting tsunami killed an estimated 229,900 people in several countries. Express this reading in terms of I_0.

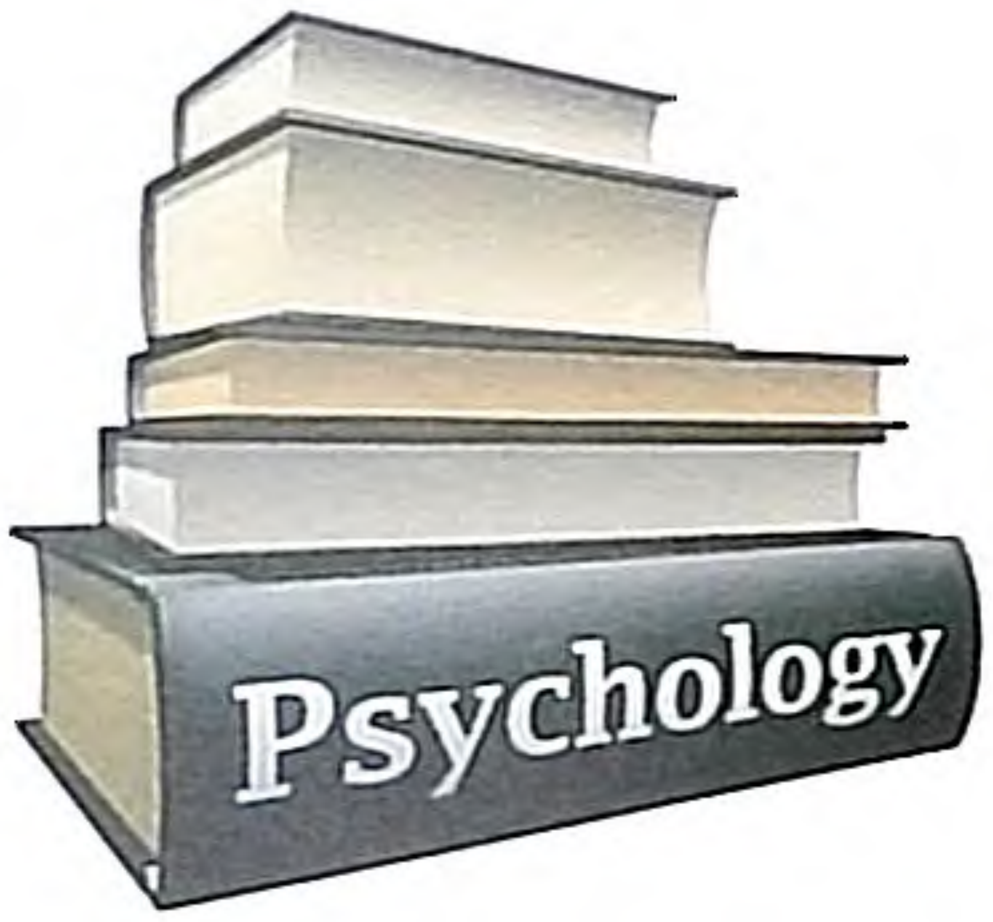
67. *Earthquake Intensity* On February 27, 2010, a massive earthquake struck Chile with a magnitude of 8.8 on the Richter scale. Express this reading in terms of I_0.

68. *Earthquake Intensity Comparison* Compare your answers to **Exercises 66 and 67.** How many times greater was the force of the 2004 earthquake than that of the 2010 earthquake?

69. *(Modeling) Bachelor's Degrees in Psychology* The table gives the number of bachelor's degrees in psychology (in thousands) earned at U.S. colleges and universities for selected years from 1980 through 2008. Suppose x represents the number of years since 1950. Thus, 1980 is represented by 30, 1990 is represented by 40, and so on.

Year	Degrees Earned (in thousands)
1980	42.1
1990	54.0
2000	74.2
2005	85.6
2007	90.0
2008	92.6

Source: U.S. National Center for Education Statistics.



The logarithmic function

$$f(x) = -228.1 + 78.19 \ln x$$

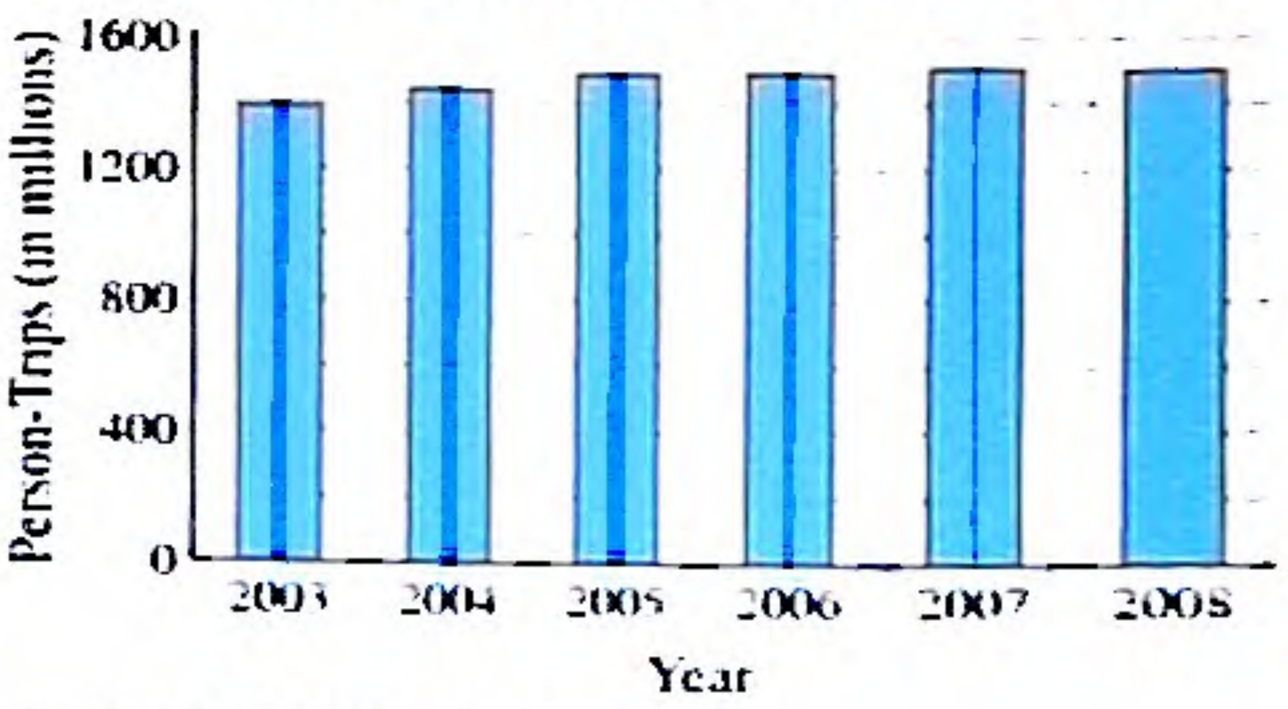
is the best-fitting logarithmic model for the data. Use this function to estimate the number of bachelor's degrees in psychology earned in the year 2012. What assumption must we make to estimate the number of degrees in years beyond 2012?

70. *(Modeling) Domestic Leisure Travel* The bar graph shows numbers of leisure trips within the United States (in millions of person-trips of 50 or more miles one-way) over the years 2003–2008. The function

$$f(t) = 1393 + 69.49 \ln t, \quad t \geq 1,$$

where t represents the number of years since 2002 and $f(t)$ is the number of person-trips, in millions, approximates the curve reasonably well.

U.S. Domestic Leisure Travel Volume



Source: U.S. Travel Association

(a) Use the function to approximate the number of person-trips in 2006. How does this approximation compare to the actual number of 1492 million?

(b) Explain why an exponential function would *not* provide a good model for these data.

71. *(Modeling) Diversity of Species* The number of species $S(n)$ in a sample is given by

$$S(n) = a \ln\left(1 + \frac{n}{a}\right),$$

where n is the number of individuals in the sample, and a is a constant that indicates the diversity of species in the community. If $a = 0.36$, find $S(n)$ for each value of n. (*Hint:* $S(n)$ must be a whole number.)

(a) 100 (b) 200 (c) 150 (d) 10

72. *(Modeling) Diversity of Species* In Exercise 71, find $S(n)$ if a changes to 0.88. Use the following values of n.

(a) 50 (b) 100 (c) 250

73. *(Modeling) Diversity of Species* Suppose a sample of a small community shows two species with 50 individuals each. Find the measure of diversity H.

74. *(Modeling) Diversity of Species* A virgin forest in northwestern Pennsylvania has 4 species of large trees with the following proportions of each: hemlock, 0.521; beech, 0.324; birch, 0.081; maple, 0.074. Find the measure of diversity H.

75. *(Modeling) Global Temperature Increase* In Example 7, we expressed the average global temperature increase T (in °F) as

$$T(k) = 1.03k \ln \frac{C}{C_0},$$

where C_0 is the preindustrial amount of carbon dioxide, C is the current carbon dioxide level, and k is a constant. Arrhenius determined that $10 \le k \le 16$ when C was double the value C_0. Use $T(k)$ to find the range of the rise in global temperature T (rounded to the nearest degree) that Arrhenius predicted. (*Source:* Clime, W., *The Economics of Global Warming*, Institute for International Economics, Washington, D.C.)

76. *(Modeling) Global Temperature Increase* (Refer to Exercise 75.) According to one study by the IPCC, future increases in average global temperatures (in °F) can be modeled by

$$T(C) = 6.489 \ln \frac{C}{280},$$

where C is the concentration of atmospheric carbon dioxide (in ppm). C can be modeled by the function

$$C(x) = 353(1.006)^{x-1990},$$

where x is the year. (*Source:* International Panel on Climate Change (IPCC).)

(a) Write T as a function of x.

(b) Using a graphing calculator, graph $C(x)$ and $T(x)$ on the interval [1990, 2275] using different coordinate axes. Describe the graph of each function. How are C and T related?

(c) Approximate the slope of the graph of T. What does this slope represent?

(d) Use graphing to estimate x and $C(x)$ when $T(x) = 10$°F.

77. *Age of Rocks* Use the formula of Example 6 to estimate the age of a rock sample having $\frac{A}{K} = 0.103$.

78. *(Modeling) Planets' Distances from the Sun and Periods of Revolution* The table contains the planets' average distances D from the sun and their periods P of revolution around the sun in years. The distances have been normalized so that Earth is one unit away from the sun. For example, since Jupiter's distance is 5.2, its distance from the sun is 5.2 times further than Earth's.

(a) Using a graphing calculator, make a scatter diagram by plotting the point ($\ln D$, $\ln P$) for each planet on the xy-coordinate axes. Do the data points appear to be linear?

(b) Determine a linear equation that models the data points. Graph your line and the data on the same coordinate axes.

(c) Use this linear model to predict the period of Pluto if its distance is 39.5. Compare your answer to the actual value of 248.5 yr.

Planet	D	P
Mercury	0.39	0.24
Venus	0.72	0.62
Earth	1	1
Mars	1.52	1.89
Jupiter	5.2	11.9
Saturn	9.54	29.5
Uranus	19.2	84.0
Neptune	30.1	164.8

Source: Ronan, C., *The Natural History of the Universe*, MacMillan Publishing Co., New York.

Use the change-of-base theorem to find an approximation to four decimal places for each logarithm. See Example 8.

79. $\log_2 5$ **80.** $\log_2 9$ **81.** $\log_8 0.59$ **82.** $\log_8 0.71$

83. $\log_{1/2} 3$ **84.** $\log_{1/3} 2$ **85.** $\log_\pi e$ **86.** $\log_\pi \sqrt{2}$

87. $\log_{\sqrt{13}} 12$ **88.** $\log_{\sqrt{19}} 5$ **89.** $\log_{0.32} 5$ **90.** $\log_{0.91} 8$

Let $u = \ln a$ and $v = \ln b$. Write each expression in terms of u and v without using the ln *function.*

91. $\ln\left(b^4\sqrt{a}\right)$ **92.** $\ln \frac{a^3}{b^2}$ **93.** $\ln\sqrt{\frac{a^3}{b^5}}$ **94.** $\ln\left(\sqrt[3]{a} \cdot b^4\right)$

Concept Check *In Exercises 95–98, use the various properties of exponential and logarithmic functions to evaluate the expressions in parts (a)–(c).*

95. Given $g(x) = e^x$, find **(a)** $g(\ln 4)$ **(b)** $g(\ln(5^2))$ **(c)** $g\left(\ln\left(\frac{1}{e}\right)\right)$.

96. Given $f(x) = 3^x$, find **(a)** $f(\log_3 2)$ **(b)** $f(\log_3(\ln 3))$ **(c)** $f(\log_3(2 \ln 3))$.

97. Given $f(x) = \ln x$, find **(a)** $f(e^6)$ **(b)** $f(e^{\ln 3})$ **(c)** $f(e^{2 \ln 3})$.

98. Given $f(x) = \log_2 x$, find **(a)** $f(2^7)$ **(b)** $f(2^{\log_2 2})$ **(c)** $f(2^{2 \log_2 2})$.

Work each problem.

99. ***Concept Check*** Which of the following is equivalent to $2 \ln(3x)$ for $x > 0$?

A. $\ln 9 + \ln x$ **B.** $\ln(6x)$ **C.** $\ln 6 + \ln x$ **D.** $\ln(9x^2)$

100. ***Concept Check*** Which of the following is equivalent to $\ln(4x) - \ln(2x)$ for $x > 0$?

A. $2 \ln x$ **B.** $\ln(2x)$ **C.** $\frac{\ln(4x)}{\ln(2x)}$ **D.** $\ln 2$

101. The function $f(x) = \ln|x|$ plays a prominent role in calculus. Find its domain, its range, and the symmetries of its graph.

102. Consider the function $f(x) = \log_3|x|$.

(a) What is the domain of this function?

(b) Use a graphing calculator to graph $f(x) = \log_3|x|$ in the window $[-4, 4]$ by $[-4, 4]$.

(c) How might one easily misinterpret the domain of the function by merely observing the calculator graph?

Use the properties of logarithms and the terminology of Chapter 2 to describe how the graph of the given function compares to the graph of $g(x) = \ln x$.

103. $f(x) = \ln(e^2x)$ **104.** $f(x) = \ln \frac{x}{e}$ **105.** $f(x) = \ln \frac{x}{e^2}$

Chapter 4 Quiz (Sections 4.1–4.4)

1. For the one-to-one function $f(x) = \sqrt[3]{3x - 6}$, find $f^{-1}(x)$.

2. Solve $4^{2x+1} = 8^{3x-6}$.

3. Graph $f(x) = -3^x$. Give the domain and range.

4. Graph $f(x) = \log_4(x + 2)$. Give the domain and range.

5. ***Future Value*** Suppose that \$15,000 is deposited in a bank certificate of deposit at an annual rate of 3.5% for 8 yr. Find the future value if interest is compounded as follows.

(a) annually **(b)** quarterly **(c)** monthly **(d)** daily (365 days)

6. Use a calculator to evaluate each logarithm to four decimal places.

(a) $\log 34.56$ **(b)** $\ln 34.56$

7. What is the meaning of the expression $\log_6 25$?

8. Solve.

(a) $x = 3^{\log_3 4}$ **(b)** $\log_x 25 = 2$ **(c)** $\log_4 x = -2$

9. Assuming all variables represent positive real numbers, use properties of logarithms to rewrite

$$\log_3 \frac{\sqrt{x} \cdot y}{pq^4}.$$

10. Given $\log_b 9 = 3.1699$ and $\log_b 5 = 2.3219$, find the value of $\log_b 225$.

11. Find the value of $\log_3 40$ to four decimal places.

12. If $f(x) = 4^x$, what is the value of $f(\log_4 12)$?

4.5 Exponential and Logarithmic Equations

- Exponential Equations
- Logarithmic Equations
- Applications and Models

Exponential Equations We solved exponential equations in earlier sections. General methods for solving these equations depend on the property below, which follows from the fact that logarithmic functions are one-to-one.

Property of Logarithms

If $x > 0$, $y > 0$, $a > 0$, and $a \neq 1$, then the following holds.

$$x = y \quad \text{is equivalent to} \quad \log_a x = \log_a y.$$

(b) Replace $f(t)$ with 25, and solve for t.

$$\begin{aligned} 25 &= 24.92 \ln t - 93.31 && f(t) = 25 \text{ in the given model.} \\ 118.31 &= 24.92 \ln t && \text{Add } 93.31. \\ \ln t &= \frac{118.31}{24.92} && \text{Divide by } 24.92. \text{ Rewrite.} \\ t &= e^{118.31/24.92} && \text{Write in exponential form.} \\ t &\approx 115.3 && \text{Use a calculator.} \end{aligned}$$

Add 115 to 1900 to get 2015. Based on this model, annual consumption will reach 25 quads in 2015.

✔ ***Now Try Exercise 105.***

4.5 Exercises

Concept Check *An exponential equation such as*

$$5^x = 9$$

can be solved for its exact solution using the meaning of logarithm and the change-of-base theorem. Since x is the exponent to which 5 must be raised in order to obtain 9, the exact solution is

$$\log_5 9, \quad or \quad \frac{\log 9}{\log 5}, \quad or \quad \frac{\ln 9}{\ln 5}.$$

For each equation, give the exact solution in three forms similar to the forms explained above.

1. $7^x = 19$ **2.** $3^x = 10$ **3.** $\left(\frac{1}{2}\right)^x = 12$ **4.** $\left(\frac{1}{3}\right)^x = 4$

Solve each exponential equation. In Exercises 5–28, express irrational solutions as decimals correct to the nearest thousandth. In Exercises 29–34, express solutions in exact form. See Examples 1–4.

5. $3^x = 7$ **6.** $5^x = 13$ **7.** $\left(\frac{1}{2}\right)^x = 5$

8. $\left(\frac{1}{3}\right)^x = 6$ **9.** $0.8^x = 4$ **10.** $0.6^x = 3$

11. $4^{x-1} = 3^{2x}$ **12.** $2^{x+3} = 5^{2x}$ **13.** $6^{x+1} = 4^{2x-1}$

14. $3^{x-4} = 7^{2x+5}$ **15.** $e^{x^2} = 100$ **16.** $e^{x^4} = 1000$

17. $e^{3x-7} \cdot e^{-2x} = 4e$ **18.** $e^{1-3x} \cdot e^{5x} = 2e$ **19.** $\left(\frac{1}{3}\right)^x = -3$

20. $\left(\frac{1}{9}\right)^x = -9$ **21.** $0.05(1.15)^x = 5$ **22.** $1.2(0.9)^x = 0.6$

23. $3(2)^{x-2} + 1 = 100$ **24.** $5(1.2)^{3x-2} + 1 = 7$ **25.** $2(1.05)^x + 3 = 10$

26. $3(1.4)^x - 4 = 60$ **27.** $5(1.015)^{x-1980} = 8$ **28.** $6(1.024)^{x-1900} = 9$

29. $e^{2x} - 6e^x + 8 = 0$ **30.** $e^{2x} - 8e^x + 15 = 0$ **31.** $2e^{2x} + e^x = 6$

32. $3e^{2x} + 2e^x = 1$ **33.** $5^{2x} + 3(5^x) = 28$ **34.** $3^{2x} - 12(3^x) = -35$

Solve each logarithmic equation. Express all solutions in exact form. See Examples 5–9.

35. $5 \ln x = 10$

36. $3 \ln x = 9$

37. $\ln(4x) = 1.5$

38. $\ln(2x) = 5$

39. $\log(2 - x) = 0.5$

40. $\log(3 - x) = 0.75$

41. $\log_6(2x + 4) = 2$

42. $\log_5(8 - 3x) = 3$

43. $\log_4(x^3 + 37) = 3$

44. $\log_7(x^3 + 65) = 0$

45. $\ln x + \ln x^2 = 3$

46. $\log x + \log x^2 = 3$

47. $\log_3[(x + 5)(x - 3)] = 2$

48. $\log_4[(3x + 8)(x - 6)] = 3$

49. $\log_2[(2x + 8)(x + 4)] = 5$

50. $\log_5[(3x + 5)(x + 1)] = 1$

51. $\log x + \log(x + 15) = 2$

52. $\log x + \log(2x + 1) = 1$

53. $\log(x + 25) = \log(x + 10) + \log 4$

54. $\log(3x + 5) - \log(2x + 4) = 0$

55. $\log(x - 10) - \log(x - 6) = \log 2$

56. $\log(x^2 + 10x - 39) - \log(x - 3) = \log 10$

57. $\ln(5 - x) + \ln(-3 - x) = \ln(1 - 8x)$

58. $\ln(10 - x) + \ln(-6 - x) = \ln(-34 - 15x)$

59. $\log_8(x + 2) + \log_8(x + 4) = \log_8 8$

60. $\log_2(5x - 6) - \log_2(x + 1) = \log_2 3$

61. $\log_2(x^2 - 100) - \log_2(x + 10) = 1$

62. $\log_2(x - 2) + \log_2(x - 1) = 1$

63. $\log x + \log(x - 21) = \log 100$

64. $\log x + \log(3x - 13) = \log 10$

65. $\log(9x + 5) = 3 + \log(x + 2)$

66. $\log(11x + 9) = 3 + \log(x + 3)$

67. $\ln(4x - 2) - \ln 4 = -\ln(x - 2)$

68. $\ln(5 + 4x) - \ln(3 + x) = \ln 3$

69. $\log_5(x + 2) + \log_5(x - 2) = 1$

70. $\log_2(x - 7) + \log_2 x = 3$

71. $\log_2(2x - 3) + \log_2(x + 1) = 1$

72. $\log_5(3x + 2) + \log_5(x - 1) = 1$

73. $\ln e^x - 2 \ln e = \ln e^4$

74. $\ln e^x - \ln e^3 = \ln e^3$

75. $\log_2(\log_2 x) = 1$

76. $\log x = \sqrt{\log x}$

77. $\log x^2 = (\log x)^2$

78. $\log_2 \sqrt{2x^2} = \frac{3}{2}$

79. ***Concept Check*** Suppose you overhear the following statement: "I must reject any negative proposed solution when I solve an equation involving logarithms." Is this correct? Why or why not?

80. ***Concept Check*** What values of x could not possibly be solutions of the following equation?

$$\log_a(4x - 7) + \log_a(x^2 + 4) = 0$$

Solve each equation for the indicated variable. Use logarithms with the appropriate bases. See Example 10.

81. $p = a + \frac{k}{\ln x}$, for x

82. $r = p - k \ln t$, for t

83. $T = T_0 + (T_1 - T_0)10^{-kt}$, for t

84. $A = \frac{Pr}{1 - (1 + r)^{-n}}$, for n

85. $I = \frac{E}{R}(1 - e^{-Rt/2})$, for t

86. $y = \frac{K}{1 + ae^{-bx}}$, for b

87. $y = A + B(1 - e^{-Cx})$, for x

88. $m = 6 - 2.5 \log\left(\frac{M}{M_0}\right)$, for M

89. $\log A = \log B - C \log x$, for A

90. $d = 10 \log\left(\frac{I}{I_0}\right)$, for I

91. $A = P\left(1 + \frac{r}{n}\right)^{tn}$, for t

92. $D = 160 + 10 \log x$, for x

For Exercises 93–98, refer to the formulas for compound interest.

$$A = P\left(1 + \frac{r}{n}\right)^{tn} \quad \text{and} \quad A = Pe^{rt} \quad \text{(Section 4.2)}$$

93. ***Compound Amount*** If $10,000 is invested in an account at 3% annual interest compounded quarterly, how much will be in the account in 5 yr if no money is withdrawn?

94. ***Compound Amount*** If $5000 is invested in an account at 4% annual interest compounded continuously, how much will be in the account in 8 yr if no money is withdrawn?

95. ***Investment Time*** Kurt Daniels wants to buy a $30,000 car. He has saved $27,000. Find the number of years (to the nearest tenth) it will take for his $27,000 to grow to $30,000 at 4% interest compounded quarterly.

96. ***Investment Time*** Find t to the nearest hundredth of a year if $1786 becomes $2063 at 2.6%, with interest compounded monthly.

97. ***Interest Rate*** Find the interest rate to the nearest hundredth of a percent that will produce $2500, if $2000 is left at interest compounded semiannually for 4.5 yr.

98. ***Interest Rate*** At what interest rate, to the nearest hundredth of a percent, will $16,000 grow to $20,000 if invested for 5.25 yr and interest is compounded quarterly?

(Modeling) Solve each application. See Example 11.

99. In the central Sierra Nevada (a mountain range in California), the percent of moisture that falls as snow rather than rain is approximated reasonably well by

$$f(x) = 86.3 \ln x - 680,$$

where x is the altitude in feet and $f(x)$ is the percent of moisture that falls as snow. Find the percent of moisture that falls as snow at each altitude.

(a) 3000 ft (b) 4000 ft (c) 7000 ft

100. C. Horne Creations finds that its total sales in dollars, $T(x)$, from the distribution of x thousand catalogues is approximated by

$$T(x) = 5000 \log(x + 1).$$

Find the total sales resulting from the distribution of each number of catalogues.

(a) 5000 (b) 24,000 (c) 49,000

101. ***Average Annual Public University Costs*** The table shows the cost of a year's tuition, room and board, and fees at 4-year public colleges for the years 2002–2010. Letting y represent the cost and x the number of years since 2002, we find that the function

$$f(x) = 9318(1.06)^x$$

models the data quite well. According to this function, when will the cost in 2002 be doubled?

Year	Average Annual Cost
2002	\$ 9,119
2003	\$ 9,951
2004	\$10,648
2005	\$11,322
2006	\$11,929
2007	\$12,698
2008	\$13,467
2009	\$14,265
2010	\$15,067

Source: The College Board, *Annual Survey of Colleges*.

102. ***Race Speed*** At the World Championship races held at Rome's Olympic Stadium in 1987, American sprinter Carl Lewis ran the 100-m race in 9.86 sec. His speed in meters per second after t seconds is closely modeled by the function

$$f(t) = 11.65(1 - e^{-t/1.27}).$$

(*Source:* Banks, Robert B., *Towing Icebergs, Falling Dominoes, and Other Adventures in Applied Mathematics,* Princeton University Press.)

(a) How fast was he running as he crossed the finish line?
(b) After how many seconds was he running at the rate of 10 m per sec?

103. ***Women in Labor Force*** The percent of women in the United States who were in the civilian labor force increased rapidly for several decades and then stabilized. If x represents the number of years since 1950, the function

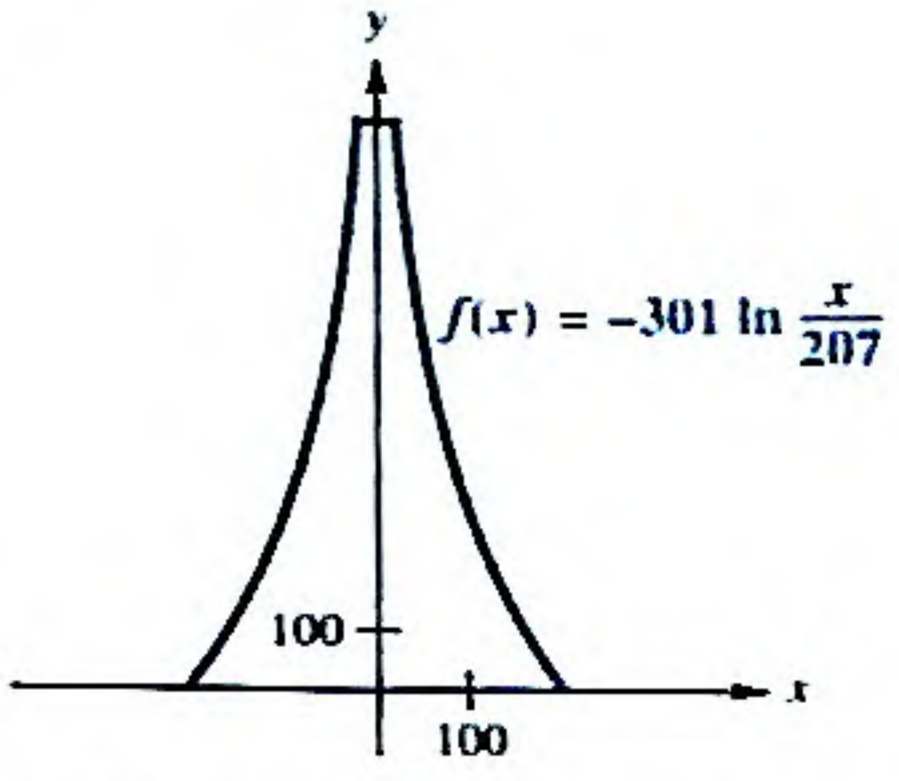
$$f(x) = \frac{67.21}{1 + 1.081e^{-x/24.71}}$$

models the percent fairly well. (*Source: Monthly Labor Review,* U.S. Bureau of Labor Statistics.)

(a) What percent of U.S. women were in the civilian labor force in 2008?
(b) In what year were 55% of U.S. women in the civilian labor force?

104. ***Height of the Eiffel Tower*** One side of the Eiffel Tower in Paris has a shape that can be approximated by the graph of the function

$$f(x) = -301 \ln \frac{x}{207}$$



See the figure. (*Source:* Banks, Robert B., *Towing Icebergs, Falling Dominoes, and Other Adventures in Applied Mathematics,* Princeton University Press.)

(a) Explain why the shape of the left side of the Eiffel Tower has the formula given by $f(-x)$.
(b) The short horizontal line at the top of the figure has length 15.7488 ft. Approximately how tall is the Eiffel Tower?
(c) Approximately how far from the center of the tower is the point on the right side that is 500 ft above the ground?

105. ***CO_2 Emissions Tax*** One action that government could take to reduce carbon emissions into the atmosphere is to levy a tax on fossil fuel. This tax would be based on the amount of carbon dioxide emitted into the air when the fuel is burned. The cost-benefit equation

$$\ln(1 - P) = -0.0034 - 0.0053T$$

models the approximate relationship between a tax of T dollars per ton of carbon and the corresponding percent reduction P (in decimal form) of emissions of carbon dioxide. (*Source:* Nordhause, W., "To Slow or Not to Slow: The Economics of the Greenhouse Effect," Yale University, New Haven, Connecticut.)

(a) Write P as a function of T.

(b) Graph P for $0 \leq T \leq 1000$. Discuss the benefit of continuing to raise taxes on carbon.

(c) Determine P when $T = \$60$, and interpret this result.

(d) What value of T will give a 50% reduction in carbon emissions?

106. ***Radiative Forcing*** (Refer to Example 7 in Section 4.4.) Using computer models, the International Panel on Climate Change (IPCC) in 1990 estimated k to be 6.3 in the radiative forcing equation

$$R = k \ln \frac{C}{C_0},$$

where C_0 is the preindustrial amount of carbon dioxide and C is the current level. (*Source:* Clime, W., *The Economics of Global Warming,* Institute for International Economics, Washington, D.C.)

(a) Use the equation $R = 6.3 \ln \frac{C}{C_0}$ to determine the radiative forcing R (in watts per square meter) expected by the IPCC if the carbon dioxide level in the atmosphere doubles from its preindustrial level.

(b) Determine the global temperature increase T that the IPCC predicted would occur if atmospheric carbon dioxide levels were to double. (*Hint:* $T(R) = 1.03R$.)

Find $f^{-1}(x)$, and give the domain and range.

107. $f(x) = e^{x+1} - 4$

108. $f(x) = 2 \ln 3x$

Use a graphing calculator to solve each equation. Give irrational solutions correct to the nearest hundredth.

109. $e^x + \ln x = 5$

110. $e^x - \ln(x + 1) = 3$

111. $2e^x + 1 = 3e^{-x}$

112. $e^x + 6e^{-x} = 5$

113. $\log x = x^2 - 8x + 14$

114. $\ln x = -\sqrt[3]{x + 3}$

115. Explain the error in the following "proof" that $2 < 1$.

$\frac{1}{9} < \frac{1}{3}$	True statement
$\left(\frac{1}{3}\right)^2 < \frac{1}{3}$	Rewrite the left side
$\log\left(\frac{1}{3}\right)^2 < \log \frac{1}{3}$	Take the logarithm on each side
$2 \log \frac{1}{3} < 1 \log \frac{1}{3}$	Property of logarithms; identity property
$2 < 1$	Divide each side by $\log \frac{1}{3}$

4.6 Applications and Models of Exponential Growth and Decay

- The Exponential Growth or Decay Function
- Growth Function Models
- Decay Function Models

The Exponential Growth or Decay Function In many situations that occur in ecology, biology, economics, and the social sciences, a quantity changes at a rate proportional to the amount present. In such cases the amount present at time t is a special function of t called an **exponential growth or decay function.**

LOOKING AHEAD TO CALCULUS
The exponential growth and decay function formulas are studied in calculus in conjunction with the topic known as differential equations.

Exponential Growth or Decay Function

Let y_0 be the amount or number present at time $t = 0$. Then, under certain conditions, the amount present at any time t is modeled by

$$y = y_0e^{kt}, \quad \text{where } k \text{ is a constant.}$$

When $k > 0$, the function describes growth. In Section 4.2, we saw examples of exponential growth: compound interest and atmospheric carbon dioxide, for example. When $k < 0$, the function describes decay; one example of exponential decay is radioactivity.

Growth Function Models The amount of time it takes for a quantity that grows exponentially to become twice its initial amount is its **doubling time.**

EXAMPLE 1 **Determining an Exponential Function to Model the Increase of Carbon Dioxide**

In Example 11, Section 4.2, we discussed the growth of atmospheric carbon dioxide over time. A function based on the data from the table was given in that example. Now we can see how to determine such a function from the data.

(a) Find an exponential function that gives the amount of carbon dioxide y in year x.

(b) Estimate the year when future levels of carbon dioxide will be double the preindustrial level of 280 ppm.

Year	Carbon Dioxide (ppm)
1990	353
2000	375
2075	590
2175	1090
2275	2000

Source: International Panel on Climate Change (IPCC).

SOLUTION

(a) Recall that the graph of the data points showed exponential growth, so the equation will take the form

$$y = y_0e^{kx}.$$

We must find the values of y_0 and k. The data begin with the year 1990, so to simplify our work we let 1990 correspond to $x = 0$, 1991 correspond to $x = 1$, and so on. Since y_0 is the initial amount, $y_0 = 353$ in 1990 when $x = 0$. Thus the equation is

$$y = 353e^{kx}. \quad \text{Let } y_0 = 353.$$

From the last pair of values in the table, we know that in 2275 the carbon dioxide level is expected to be 2000 ppm. The year 2275 corresponds to $2275 - 1990 = 285$. Substitute 2000 for y and 285 for x, and solve for k.

SOLUTION

(a) We must find values for C and k in the given formula. From the given information, when $t = 0$, $T_0 = 20$, and the temperature of the coffee is $f(0) = 100$. Also, when $t = 1$, $f(1) = 60$. Substitute the first pair of values into the function along with $T_0 = 20$.

$$f(t) = T_0 + Ce^{-kt} \qquad \text{Given formula}$$

$$100 = 20 + Ce^{-0k} \qquad \text{Let } t = 0,\ f(0) = 100, \text{ and } T_0 = 20.$$

$$100 = 20 + C \qquad e^0 = 1$$

$$80 = C \qquad \text{Subtract 20.}$$

The function that models the data can now be given.

$$f(t) = 20 + 80e^{-kt} \qquad \text{Let } T_0 = 20 \text{ and } C = 80.$$

Now use the remaining pair of values in this function to find k.

$$f(t) = 20 + 80e^{-kt} \qquad \text{Formula with } T_0 = 20 \text{ and } C = 80$$

$$60 = 20 + 80e^{-1k} \qquad \text{Let } t = 1 \text{ and } f(1) = 60.$$

$$40 = 80e^{-k} \qquad \text{Subtract 20.}$$

$$\frac{1}{2} = e^{-k} \qquad \text{Divide by 80.}$$

$$\ln \frac{1}{2} = \ln e^{-k} \qquad \text{Take the logarithm on each side.}$$

$$\ln \frac{1}{2} = -k \qquad \ln e^x = x, \text{ for all } x.$$

$$k = -\ln \frac{1}{2} \qquad \text{Multiply by } -1 \text{ and rewrite.}$$

$$k \approx 0.693 \qquad \text{Use a calculator.}$$

Thus, the model is $f(t) = 20 + 80e^{-0.693t}$.

(b) To find the temperature after $\frac{1}{2}$ hr, let $t = \frac{1}{2}$ in the model from part (a).

$$f(t) = 20 + 80e^{-0.693t} \qquad \text{Model from part (a)}$$

$$f\left(\frac{1}{2}\right) = 20 + 80e^{(-0.693)(1/2)} \qquad \text{Let } t = \tfrac{1}{2}.$$

$$f\left(\frac{1}{2}\right) \approx 76.6^\circ\text{C} \qquad \text{Use a calculator.}$$

(c) To find how long it will take for the coffee to cool to 50°C, let $f(t) = 50$.

$$50 = 20 + 80e^{-0.693t} \qquad \text{Let } f(t) = 50 \text{ in model from part (a).}$$

$$30 = 80e^{-0.693t} \qquad \text{Subtract 20.}$$

$$\frac{3}{8} = e^{-0.693t} \qquad \text{Divide by 80.}$$

$$\ln \frac{3}{8} = \ln e^{-0.693t} \qquad \text{Take the logarithm on each side.}$$

$$\ln \frac{3}{8} = -0.693t \qquad \ln e^x = x, \text{ for all } x.$$

$$t = \frac{\ln \frac{3}{8}}{-0.693} \qquad \text{Divide by } -0.693 \text{ and rewrite.}$$

$$t \approx 1.415 \text{ hr,} \quad \text{or} \quad \text{about 1 hr, 25 min}$$

✔ *Now Try Exercise 23.*

4.6 Exercises

Population Growth *A population is increasing according to the exponential function $y = 2e^{0.02x}$, where y is in millions and x is the number of years. Match each question in Column I with the correct procedure in Column II to answer the question.*

I	II
1. How long will it take for the population to triple?	**A.** Evaluate $y = 2e^{0.02(1/3)}$.
2. When will the population reach 3 million?	**B.** Solve $2e^{0.02x} = 6$.
3. How large will the population be in 3 yr?	**C.** Evaluate $y = 2e^{0.02(3)}$.
4. How large will the population be in 4 months?	**D.** Solve $2e^{0.02x} = 3$.

(Modeling) The exercises in this set are grouped according to discipline. *They involve exponential or logarithmic models.* ***See Examples 1–6.***

Physical Sciences (Exercises 5–24)

In Exercises 5–10, an initial amount of a radioactive substance y_0 is given, along with information about the amount remaining after a given time t in appropriate units. For an equation of the form $y = y_0e^{kt}$ that models the situation, give the exact value of k in terms of natural logarithms.

5. $y_0 = 60$ g; After 3 hr, 20 g remain.

6. $y_0 = 30$ g; After 6 hr, 10 g remain.

7. $y_0 = 10$ mg; The half-life is 100 days.

8. $y_0 = 20$ mg; The half-life is 200 days.

9. $y_0 = 2.56$ lb; After 2 yr, 0.64 lb remains.

10. $y_0 = 8.1$ kg; After 4 yr, 0.9 kg remains.

11. ***Decay of Lead*** A sample of 500 g of radioactive lead-210 decays to polonium-210 according to the function

$$A(t) = 500e^{-0.032t},$$

where t is time in years. Find the amount of radioactive lead remaining after

(a) 4 yr, (b) 8 yr, (c) 20 yr. (d) Find the half-life.

12. ***Decay of Plutonium*** Repeat Exercise 11 for 500 g of plutonium-241, which decays according to the function $A(t) = A_0e^{-0.053t}$, where t is time in years.

13. ***Decay of Radium*** Find the half-life of radium-226, which decays according to the function $A(t) = A_0e^{-0.00043t}$, where t is time in years.

14. ***Decay of Iodine*** How long will it take any quantity of iodine-131 to decay to 25% of its initial amount, knowing that it decays according to the exponential function $A(t) = A_0e^{-0.087t}$, where t is time in days?

15. ***Radioactive Decay*** If 12 g of a radioactive substance are present initially and 4 yr later only 6 g remain, how much of the substance will be present after 7 yr?

16. ***Magnitude of a Star*** The magnitude M of a star is modeled by

$$M = 6 - \frac{5}{2}\log\frac{I}{I_0},$$

where I_0 is the intensity of a just-visible star and I is the actual intensity of the star being measured. The dimmest stars are of magnitude 6, and the brightest are of magnitude 1. Determine the ratio of light intensities between a star of magnitude 1 and a star of magnitude 3.

17. *Carbon-14 Dating* Suppose an Egyptian mummy is discovered in which the amount of carbon-14 present is only about one-third the amount found in living human beings. About how long ago did the Egyptian die?

18. *Carbon-14 Dating* A sample from a refuse deposit near the Strait of Magellan had 60% of the carbon-14 of a contemporary sample. How old was the sample?

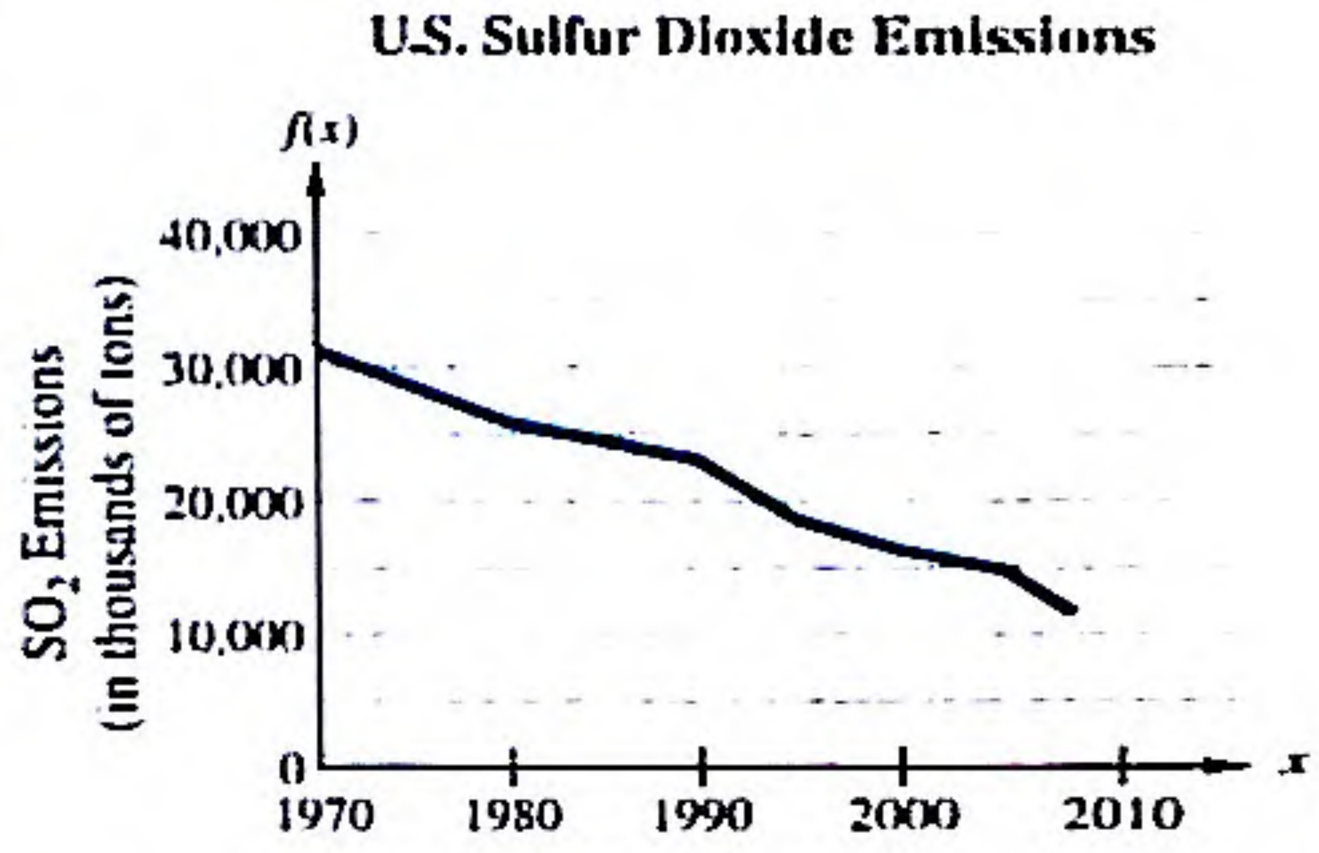
19. *Carbon-14 Dating* Paint from the Lascaux caves of France contains 15% of the normal amount of carbon-14. Estimate the age of the paintings.

20. *Dissolving a Chemical* The amount of a chemical that will dissolve in a solution increases exponentially as the (Celsius) temperature t is increased according to the model

$$A(t) = 10e^{0.0095t}.$$

At what temperature will 15 g dissolve?

21. *Sulfur Dioxide Levels* Sulfur dioxide (SO_2), a gas that is emitted by burning fossil fuels, is a source of air pollution and one of the major contributors to acid rain. Reductions in emissions of sulfur dioxide in the United States were mandated by the 1990 revision of the Clean Air Act. The graph displays approximate U.S. emissions of sulfur dioxide (in thousands of tons) over the period 1970–2008.



Source: U.S. Environmental Protection Agency

(a) Use the graph to find an exponential function

$$f(x) = A_0a^{x-1970}$$

that models U.S. emissions of sulfur dioxide over the years 1970–2008.

(b) Approximate the average annual percent decrease in sulfur dioxide emissions over this time period.

22. *Rock Sample Age* Estimate the age of a rock sample using the function

$$t = T\frac{\ln\left(1 + 8.33\left(\frac{A}{K}\right)\right)}{\ln 2}$$

Tests show that $\frac{A}{K}$ is 0.175 for the sample. Let $T = 1.26 \times 10^9$.

23. *Newton's Law of Cooling* Boiling water, at 100°C, is placed in a freezer at 0°C. The temperature of the water is 50°C after 24 min. Find the temperature of the water after 96 min. (*Hint:* Change minutes to hours.)

24. *Newton's Law of Cooling* A piece of metal is heated to 300°C and then placed in a cooling liquid at 50°C. After 4 min, the metal has cooled to 175°C. Find its temperature after 12 min. (*Hint:* Change minutes to hours.)

Finance (Exercises 25–30)

25. *Comparing Investments* Russ McClelland, who is self-employed, wants to invest $60,000 in a pension plan. One investment offers 5% compounded quarterly. Another offers 4.75% compounded continuously.

(a) Which investment will earn more interest in 5 yr?

(b) How much more will the better plan earn?

26. *Growth of an Account* If Russ (see Exercise 25) chooses the plan with continuous compounding, how long will it take for his $60,000 to grow to $80,000?

27. *Doubling Time* Find the doubling time of an investment earning 2.5% interest if interest is compounded continuously.

28. ***Doubling Time*** If interest is compounded continuously and the interest rate is tripled, what effect will this have on the time required for an investment to double?

29. ***Growth of an Account*** How long will it take an investment to triple if interest is compounded continuously at 5%?

30. ***Growth of an Account*** Use the Table feature of a graphing calculator to find how long it will take \$1500 invested at 5.75% compounded daily to triple in value. Zoom in on the solution by systematically decreasing the increment for x. Find the answer to the nearest day. (Find the answer to the nearest day by eventually letting the increment of x equal $\frac{1}{365}$. The decimal part of the solution can be multiplied by 365 to determine the number of days greater than the nearest year. For example, if the solution is determined to be 16.2027 yr, then multiply 0.2027 by 365 to get 73.9855. The solution is then, to the nearest day, 16 yr, 74 days.) Confirm the answer algebraically.

Social Sciences (Exercises 31–38)

31. ***Legislative Turnover*** The turnover of legislators is a problem of interest to political scientists. In a study during the 1970s, it was found that one model of legislative turnover in the U.S. House of Representatives was

$$M(t) = 434e^{-0.08t},$$

where $M(t)$ represents the number of continuously serving members at time t. Here, $t = 0$ represents 1965, $t = 1$ represents 1966, and so on. Use this model to approximate the number of continuously serving members in each year.

(a) 1969 (b) 1973 (c) 1979

32. ***Legislative Turnover*** Use the model in Exercise 31 to determine the year in which the number of continuously serving members was 338.

33. ***Population Growth*** In 2000 India's population reached 1 billion, and it is projected to be 1.4 billion in 2025. (*Source:* U.S. Census Bureau.)

(a) Find values for P_0 and a so that $f(x) = P_0a^{x-2000}$ models the population of India in year x.
(b) Predict India's population in 2020 to the nearest tenth of a billion.
(c) Use f to determine the year when India's population might reach 1.5 billion.

34. ***Population Decline*** A midwestern city finds its residents moving to the suburbs. Its population is declining according to the function defined by

$$P(t) = P_0e^{-0.04t},$$

where t is time measured in years and P_0 is the population at time $t = 0$. Assume that $P_0 = 1{,}000{,}000$.

(a) Find the population at time $t = 1$.
(b) Estimate the time it will take for the population to decline to 750,000.
(c) How long will it take for the population to decline to half the initial number?

35. ***Health Care Spending*** Out-of-pocket spending in the United States for health care increased between 2004 and 2008. The function

$$f(x) = 2572e^{0.0359x}$$

models average annual expenditures per household, in dollars. In this model, x represents the year, where $x = 0$ corresponds to 2004. (*Source:* U.S. Bureau of Labor Statistics.)

(a) Estimate out-of-pocket household spending on health care in 2008.
(b) Determine the year when spending reached \$2775 per household.

36. ***Recreational Expenditures*** Personal consumption expenditures for recreation in billions of dollars in the United States during the years 2004–2008 can be approximated by the function

$$A(t) = 769.5e^{0.05803t},$$

where $t = 0$ corresponds to the year 2004. Based on this model, how much were personal consumption expenditures in 2008? (*Source:* U.S. Bureau of Economic Analysis.)

37. ***Housing Costs*** Average annual per-household spending on housing over the years 1990–2008 is approximated by

$$H = 8790e^{0.0382t},$$

where t is the number of years since 1990. Find H to the nearest dollar for each year. (*Source:* U.S. Bureau of Labor Statistics.)

(a) 2000 **(b)** 2005 **(c)** 2008

38. ***Evolution of Language*** The number of years, n, since two independently evolving languages split off from a common ancestral language is approximated by

$$n \approx -7600 \log r,$$

where r is the proportion of words from the ancestral language common to both languages.

(a) Find n if $r = 0.9$. **(b)** Find n if $r = 0.3$.

(c) How many years have elapsed since the split if half of the words of the ancestral language are common to both languages?

Life Sciences *(Exercises 39–44)*

39. ***Spread of Disease*** During an epidemic, the number of people who have never had the disease and who are not immune (they are *susceptible*) decreases exponentially according to the function

$$f(t) = 15{,}000e^{-0.05t},$$

where t is time in days. Find the number of susceptible people at each time.

(a) at the beginning of the epidemic **(b)** after 10 days **(c)** after 3 weeks

40. ***Spread of Disease*** Refer to Exercise 39 and determine how long it will take for the initial number of people susceptible to decrease to half its amount.

41. ***Growth of Bacteria*** The growth of bacteria makes it necessary to time-date some food products so that they will be sold and consumed before the bacteria count is too high. Suppose for a certain product the number of bacteria present is given by

$$f(t) = 500e^{0.1t},$$

where t is time in days and the value of $f(t)$ is in millions. Find the number of bacteria present at each time.

(a) 2 days **(b)** 4 days **(c)** 1 week

42. ***Growth of Bacteria*** How long will it take the bacteria population in **Exercise 41** to double?

43. ***Medication Effectiveness*** Drug effectiveness decreases over time. If, each hour, a drug is only 90% as effective as the previous hour, at some point the patient will not be receiving enough medication and must receive another dose. If the initial dose was 200 mg and the drug was administered 3 hr ago, the expression $200(0.90)^3$, which equals 145.8, represents the amount of effective medication still in the system. (The exponent is equal to the number of hours since the drug was administered.)

The amount of medication still available in the system is given by the function

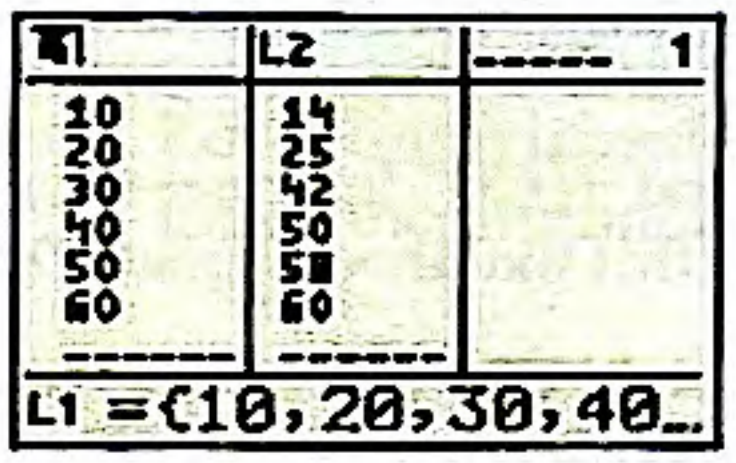
$$f(t) = 200(0.90)^t.$$

In this model, t is in hours and $f(t)$ is in milligrams. How long will it take for this initial dose to reach the dangerously low level of 50 mg?

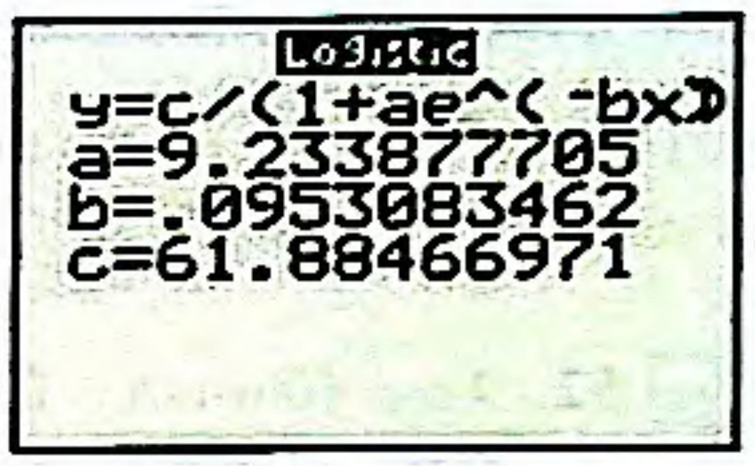
Population Size *Many environmental situations place effective limits on the growth of the number of an organism in an area. Many such limited-growth situations are described by the* ***logistic function***

$$G(x) = \frac{MG_0}{G_0 + (M - G_0)e^{-kMx}},$$

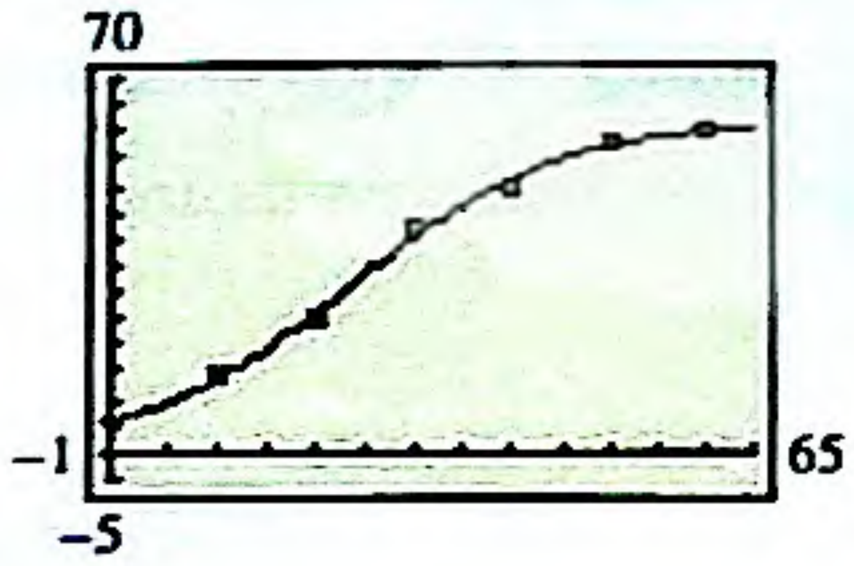
where G_0 is the initial number present, M is the maximum possible size of the population, and k is a positive constant. The screens illustrate a typical logistic function calculation and graph.



```
L1     L2     -----  1
10     14
20     25
30     42
40     50
50     58
60     60
------ ------
L1 ={10,20,30,40…
```



```
Logistic
y=c/(1+ae^(-bx))
a=9.233877705
b=.0953083462
c=61.88466971
```



44. Assume that $G_0 = 100$, $M = 2500$, $k = 0.0004$, and $x =$ time in decades (10-yr periods).
    - (a) Use a calculator to graph the function, using $0 \leq x \leq 8$, $0 \leq y \leq 2500$.
    - (b) Estimate the value of $G(2)$ from the graph. Then evaluate $G(2)$ algebraically to find the population after 20 yr.
    - (c) Find the x-coordinate of the intersection of the curve with the horizontal line $y = 1000$ to estimate the number of decades required for the population to reach 1000. Then solve $G(x) = 1000$ algebraically to obtain the exact value of x.

Economics *(Exercises 45–50)*

45. ***Consumer Price Index*** The U.S. Consumer Price Index for the years 1990–2009 is approximated by

$$A(t) = 100e^{0.026t},$$

where t represents the number of years after 1990. (Since $A(16)$ is about 152, the amount of goods that could be purchased for $100 in 1990 cost about $152 in 2006.) Use the function to determine the year in which costs will be 100% higher than in 1990. (*Source:* U.S. Bureau of Labor Statistics.)

46. ***Product Sales*** Sales of a product, under relatively stable market conditions but in the absence of promotional activities such as advertising, tend to decline at a constant yearly rate. This rate of sales decline varies considerably from product to product, but it seems to remain the same for any particular product. The sales decline can be expressed by the function

$$S(t) = S_0e^{-at},$$

where $S(t)$ is the rate of sales at time t measured in years, S_0 is the rate of sales at time $t = 0$, and a is the sales decay constant.
    - (a) Suppose the sales decay constant for a particular product is $a = 0.10$. Let $S_0 = 50{,}000$ and find $S(1)$ and $S(3)$.
    - (b) Find $S(2)$ and $S(10)$ if $S_0 = 80{,}000$ and $a = 0.05$.

47. ***Product Sales*** Use the sales decline function given in Exercise 46. If $a = 0.10$, $S_0 = 50{,}000$, and t is time measured in years, find the number of years it will take for sales to fall to half the initial sales.

48. *Cost of Bread* Assume the cost of a loaf of bread is \$4. With continuous compounding, find the time it would take for the cost to triple at an annual inflation rate of 4%.

49. *Electricity Consumption* Suppose that in a certain area the consumption of electricity has increased at a continuous rate of 6% per year. If it continued to increase at this rate, find the number of years before twice as much electricity would be needed.

50. *Electricity Consumption* Suppose a conservation campaign, together with higher rates, caused demand for electricity to increase at only 2% per year. (See **Exercise 49.**) Find the number of years before twice as much electricity would be needed.

(Modeling) Exercises 51 and 52 use another type of logistic function.

51. *Heart Disease* As age increases, so does the likelihood of coronary heart disease (CHD). The fraction of people x years old with some CHD is modeled by

$$f(x) = \frac{0.9}{1 + 271e^{-0.122x}}.$$

(*Source:* Hosmer, D., and S. Lemeshow, *Applied Logistic Regression*, John Wiley and Sons.)

(a) Evaluate $f(25)$ and $f(65)$. Interpret the results.
(b) At what age does this likelihood equal 50%?

52. *Tree Growth* The height of a certain tree in feet after x years is modeled by

$$f(x) = \frac{50}{1 + 47.5e^{-0.22x}}.$$

(a) Make a table for f starting at $x = 10$, and incrementing by 10. What appears to be the maximum height of the tree?
(b) Graph f and identify the horizontal asymptote. Explain its significance.
(c) After how long was the tree 30 ft tall?

Summary Exercises on Functions: Domains and Defining Equations

Finding the Domain of a Function: A Summary To find the domain of a function, given the equation that defines the function, remember that the value of x input into the equation must yield a real number for y when the function is evaluated. For the functions studied so far in this book, there are three cases to consider when determining domains.

Guidelines for Domain Restrictions

1. No input value can lead to 0 in a denominator, because division by 0 is undefined.
2. No input value can lead to an even root of a negative number, because this situation does not yield a real number.
3. No input value can lead to the logarithm of a negative number or 0, because this situation does not yield a real number.

Unless domains are otherwise specified, we can determine domains as follows.

- The domain of a **polynomial function** is the set of all real numbers.
- The domain of an **absolute value function** is the set of all real numbers for which the expression inside the absolute value bars (the argument) is defined.
- If a **function is defined by a rational expression,** the domain is the set of all real numbers for which the denominator is not zero.
- The domain of a **function defined by a radical with *even* root index** is the set of all real numbers that make the radicand greater than or equal to zero. **If the root index is *odd*,** the domain is the set of all real numbers for which the radicand is itself a real number.
- For an **exponential function** with constant base, the domain is the set of all real numbers for which the exponent is a real number.
- For a **logarithmic function,** the domain is the set of all real numbers that make the argument of the logarithm greater than zero.

Determining Whether an Equation Defines y as a Function of x ***For y to be a function of x, it is necessary that every input value of x in the domain leads to one and only one value of y.*** To determine whether an equation such as

$$x - y^3 = 0 \quad \text{or} \quad x - y^2 = 0$$

represents a function, solve the equation for y. In the first equation above, doing so leads to

$$y = \sqrt[3]{x}.$$

Notice that every value of x in the domain (that is, all real numbers) leads to one and only one value of y. So in the first equation, we can write y as a function of x. However, in the second equation above, solving for y leads to

$$y = \pm\sqrt{x}.$$

If we let $x = 4$, for example, we get two values of y: -2 and 2. Thus, in the second equation, we cannot write y as a function of x.

Find the domain of each function. Write answers using interval notation.

1. $f(x) = 3x - 6$

2. $f(x) = \sqrt{2x - 7}$

3. $f(x) = |x + 4|$

4. $f(x) = \dfrac{x + 2}{x - 6}$

5. $f(x) = \dfrac{-2}{x^2 + 7}$

6. $f(x) = \sqrt{x^2 - 9}$

7. $f(x) = \dfrac{x^2 + 7}{x^2 - 9}$

8. $f(x) = \sqrt[3]{x^3 + 7x - 4}$

9. $f(x) = \log_5(16 - x^2)$

10. $f(x) = \log \dfrac{x + 7}{x - 3}$

11. $f(x) = \sqrt{x^2 - 7x - 8}$

12. $f(x) = 2^{1/x}$

13. $f(x) = \dfrac{1}{2x^2 - x + 7}$

14. $f(x) = \dfrac{x^2 - 25}{x + 5}$

15. $f(x) = \sqrt{x^3 - 1}$

16. $f(x) = \ln|x^2 - 5|$

17. $f(x) = e^{x^2 + x + 4}$

18. $f(x) = \dfrac{x^3 - 1}{x^2 - 1}$

19. $f(x) = \sqrt{\dfrac{-1}{x^3 - 1}}$

20. $f(x) = \sqrt[3]{\dfrac{1}{x^3 - 8}}$

21. $f(x) = \ln(x^2 + 1)$

22. $f(x) = \sqrt{(x - 3)(x + 2)(x - 4)}$

23. $f(x) = \log\left(\dfrac{x + 2}{x - 3}\right)^2$

24. $f(x) = \sqrt[12]{(4 - x)^2(x + 3)}$

25. $f(x) = e^{|1/x|}$

26. $f(x) = \dfrac{1}{|x^2 - 7|}$

27. $f(x) = x^{100} - x^{50} + x^2 + 5$

28. $f(x) = \sqrt{-x^2 - 9}$

29. $f(x) = \sqrt[4]{16 - x^4}$

30. $f(x) = \sqrt[3]{16 - x^4}$

31. $f(x) = \sqrt{\dfrac{x^2 - 2x - 63}{x^2 + x - 12}}$

32. $f(x) = \sqrt[5]{5 - x}$

33. $f(x) = |\sqrt{5 - x}|$

34. $f(x) = \sqrt{\dfrac{-1}{x - 3}}$

35. $f(x) = \log\left|\dfrac{1}{4 - x}\right|$

36. $f(x) = 6^{x^2-9}$

37. $f(x) = 6^{\sqrt{x^2-25}}$

38. $f(x) = 6^{\sqrt[3]{x^2-25}}$

39. $f(x) = \ln\left(\dfrac{-3}{(x + 2)(x - 6)}\right)$

40. $f(x) = \dfrac{-2}{\log x}$

Determine which one of the choices (A, B, C, or D) is an equation in which y can be written as a function of x.

41. **A.** $3x + 2y = 6$ **B.** $x = \sqrt{|y|}$ **C.** $x = |y + 3|$ **D.** $x^2 + y^2 = 9$

42. **A.** $3x^2 + 2y^2 = 36$ **B.** $x^2 + y - 2 = 0$ **C.** $x - |y| = 0$ **D.** $x = y^2 - 4$

43. **A.** $x = \sqrt{y^2}$ **B.** $x = \log y^2$ **C.** $x^3 + y^3 = 5$ **D.** $x = \dfrac{1}{y^2 + 3}$

44. **A.** $\dfrac{x^2}{4} + \dfrac{y^2}{4} = 1$ **B.** $x = 5y^2 - 3$ **C.** $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$ **D.** $x = 10^y$

45. **A.** $x = \dfrac{2 - y}{y + 3}$ **B.** $x = \ln(y + 1)^2$ **C.** $\sqrt{x} = |y + 1|$ **D.** $\sqrt[4]{x} = y^2$

46. **A.** $e^{y^2} = x$ **B.** $e^{y+2} = x$ **C.** $e^{|y|} = x$ **D.** $10^{|y+2|} = x$

47. **A.** $x^2 = \dfrac{1}{y^2}$ **B.** $x + 2 = \dfrac{1}{y^2}$ **C.** $3x = \dfrac{1}{y^4}$ **D.** $2x = \dfrac{1}{y^3}$

48. **A.** $|x| = |y|$ **B.** $x = |y^2|$ **C.** $x = \dfrac{1}{y}$ **D.** $x^4 + y^4 = 81$

49. **A.** $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$ **B.** $\dfrac{y^2}{4} - \dfrac{x^2}{9} = 1$ **C.** $\dfrac{x}{4} - \dfrac{y}{9} = 0$ **D.** $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 0$

50. **A.** $y^2 - \sqrt{(x + 2)^2} = 0$ **B.** $y - \sqrt{(x + 2)^2} = 0$
C. $y^6 - \sqrt{(x + 1)^2} = 0$ **D.** $y^4 - \sqrt{x^2} = 0$

Chapter 4 Test Prep

Key Terms

4.1 one-to-one function
inverse function

4.2 exponential function
exponential equation
compound interest
future value
present value
compound amount
continuous compounding

4.3 logarithm
base
argument
logarithmic equation
logarithmic function

4.4 common logarithm
pH
natural logarithm

4.6 doubling time
half-life

New Symbols

$f^{-1}(x)$	the inverse of $f(x)$
e	a constant, approximately 2.718281828459045
$\log_a x$	the logarithm of x with the base a
$\log x$	common (base 10) logarithm of x
$\ln x$	natural (base e) logarithm of x

Quick Review

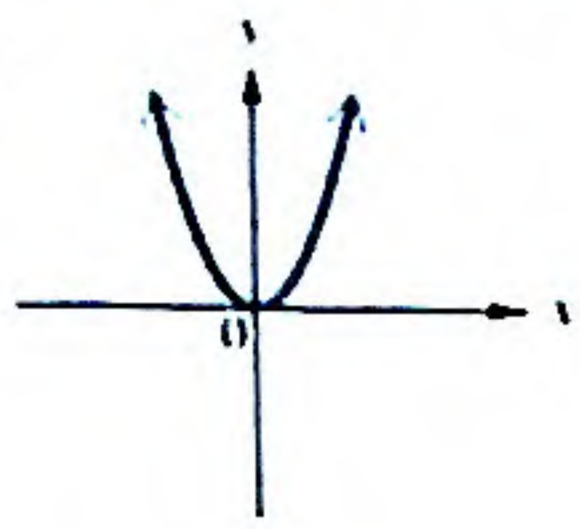
Concepts	Examples
4.1 Inverse Functions	
One-to-One Function In a one-to-one function, each x-value corresponds to only one y-value, and each y-value corresponds to only one x-value. A function f is one-to-one if, for elements a and b in the domain of f, $a \neq b$ implies $f(a) \neq f(b)$.	The function $y = x^2$ is not one-to-one, because $y = 16$, for example, corresponds to both $x = 4$ and $x = -4$.
Horizontal Line Test A function is one-to-one if every horizontal line intersects the graph of the function at most once.	The graph of $f(x) = 2x - 1$ is a straight line with slope 2. f is a one-to-one function by the horizontal line test.
Inverse Functions Let f be a one-to-one function. Then g is the inverse function of f if $(f \circ g)(x) = x$ for every x in the domain of g and $(g \circ f)(x) = x$ for every x in the domain of f To find $g(x)$, interchange x and y in $y = f(x)$, solve for y, and replace y with $g(x)$, which is $f^{-1}(x)$.	Find the inverse of $y = f(x) = 2x - 1$. $x = 2y - 1$ Interchange x and y $y = \frac{x+1}{2}$ Solve for y $f^{-1}(x) = \frac{x+1}{2}$ Replace y with $f^{-1}(x)$ $f^{-1}(x) = \frac{1}{2}x + \frac{1}{2}$ $\frac{x+1}{2} = \frac{x}{2} + \frac{1}{2} = \frac{1}{2}x + \frac{1}{2}$

(continued)

Chapter 4 Review Exercises

Decide whether each function as defined or graphed is one-to-one.

1.



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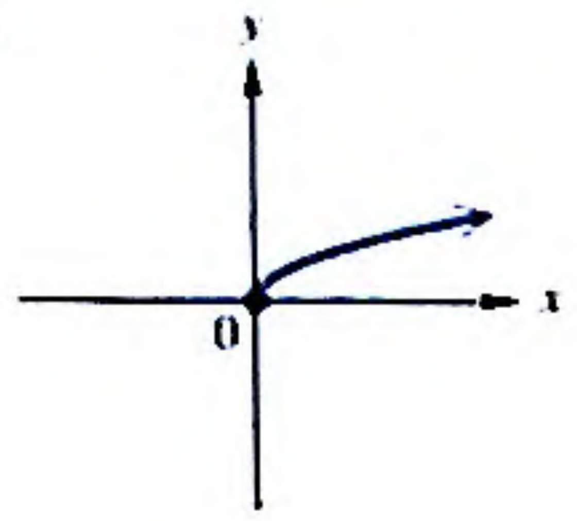


3. $y = 5x - 4$

4. $y = x^3 + 1$

5. $y = (x + 3)^2$

6. $y = \sqrt{3x^2 + 2}$

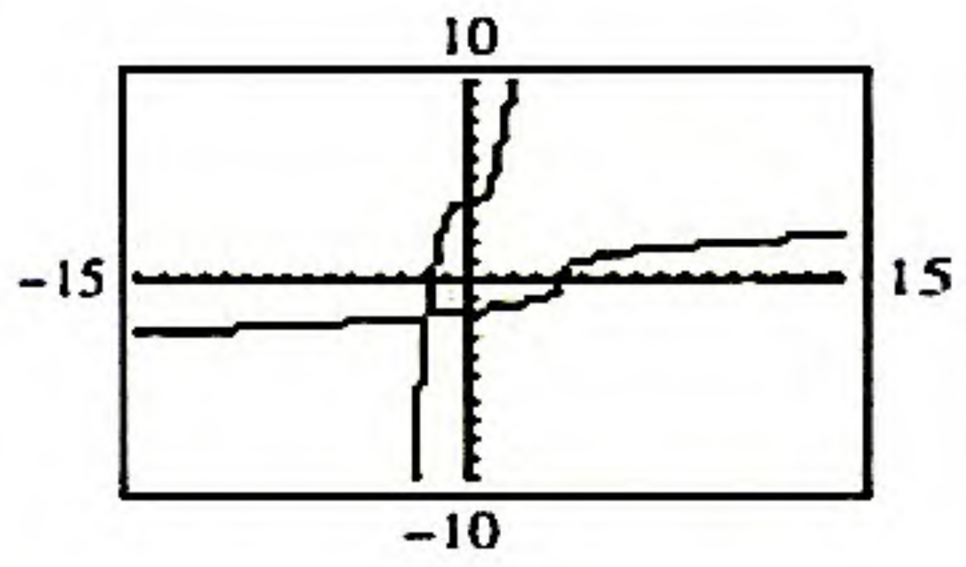
Write an equation, if possible, for each inverse function in the form $y = f^{-1}(x)$.

7. $f(x) = x^3 - 3$

8. $f(x) = \sqrt{25 - x^2}$

9. *Concept Check* Suppose $f(t)$ is the amount an investment will grow to t years after 2004. What does $f^{-1}(\$50{,}000)$ represent?

10. *Concept Check* The graphs of two functions are shown. Based on their graphs, are these functions inverses?



11. *Concept Check* To have an inverse, a function must be a(n) __________ function.

12. *Concept Check* Assuming that f has an inverse, is it true that every x-intercept of the graph of $y = f(x)$ is a y-intercept of the graph of $y = f^{-1}(x)$?

Match each equation with the figure that most closely resembles its graph.

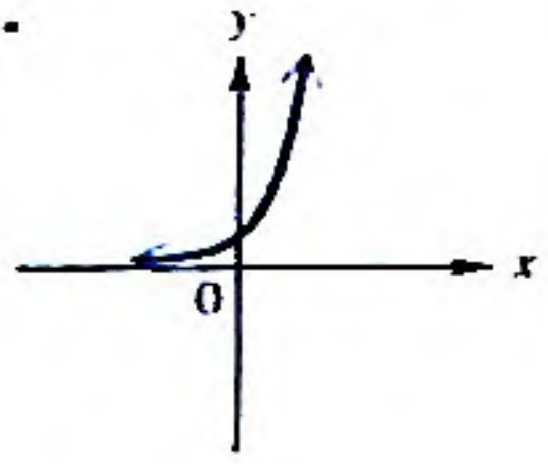
13. $y = \log_{0.3} x$

14. $y = e^x$

15. $y = \ln x$

16. $y = 0.3^x$

A.



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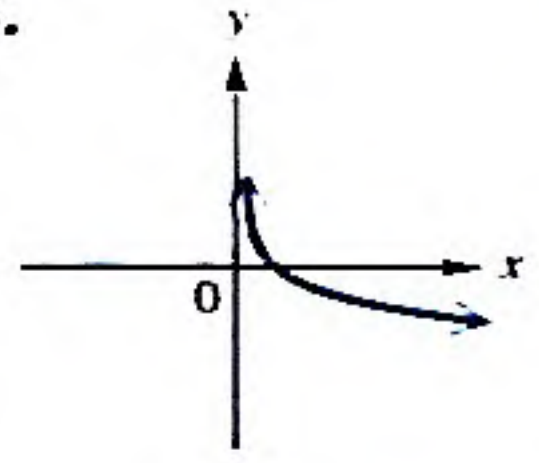
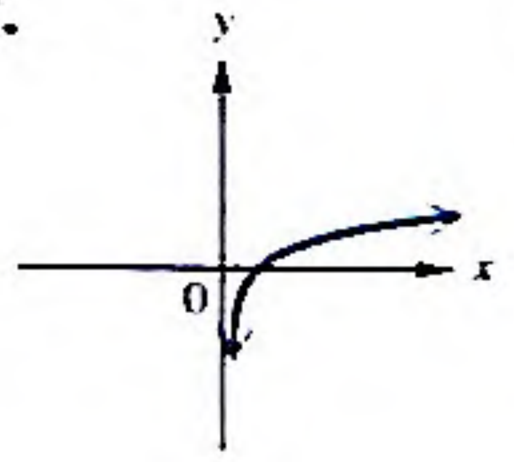
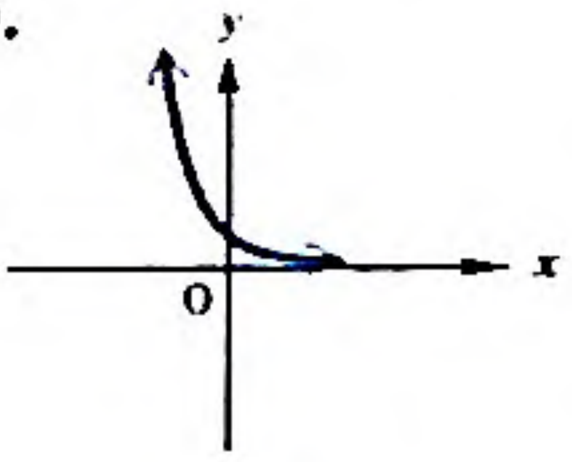


C.



D.



Write each equation in logarithmic form.

17. $2^5 = 32$

18. $100^{1/2} = 10$

19. $\left(\frac{3}{4}\right)^{-1} = \frac{4}{3}$

20. Graph $y = (1.5)^{x+2}$.

Write each equation in exponential form.

21. $\log 1000 = 3$

22. $\log_9 27 = \frac{3}{2}$

23. $\ln \sqrt{e} = \frac{1}{2}$

24. *Concept Check* What is the base of the logarithmic function whose graph contains the point $(81, 4)$?

25. *Concept Check* What is the base of the exponential function whose graph contains the point $\left(-4, \frac{1}{16}\right)$?

Use properties of logarithms to rewrite each expression. Simplify the result, if possible. Assume all variables represent positive numbers.

26. $\log_5\left(x^2y^4\sqrt[5]{m^3p}\right)$ 27. $\log_3 \frac{mn}{5r}$ 28. $\log_7(7k + 5r^2)$

Find each logarithm. Round to four decimal places.

29. $\log 0.0411$ 30. $\log 45.6$ 31. $\ln 144{,}000$

32. $\ln 470$ 33. $\log_{2/3} \frac{5}{8}$ 34. $\log_3 769$

Solve each exponential equation. Unless otherwise specified, express irrational solutions as decimals correct to the nearest thousandth.

35. $16^{x+4} = 8^{3x-2}$ 36. $4^x = 12$ 37. $3^{2x-5} = 13$

38. $2^{x+3} = 5^x$ 39. $6^{x+3} = 4^x$ 40. $e^{x-1} = 4$

41. $e^{2-x} = 12$ 42. $2e^{5x+2} = 8$ 43. $10e^{3x-7} = 5$

44. $5^{x+2} = 2^{2x-1}$ 45. $6^{x-3} = 3^{4x+1}$ 46. $e^{8x} \cdot e^{2x} = e^{20}$

47. $e^{6x} \cdot e^x = e^{21}$ 48. $100(1.02)^{x/4} = 200$ 49. $2e^{2x} - 5e^x - 3 = 0$ (Give exact form.)

50. $\left(\frac{1}{2}\right)^x + 2 = 0$ 51. $4(1.06)^x + 2 = 8$

52. ***Concept Check*** Which one or more of the following choices is the solution set of $5^x = 9$?

A. $\{\log_5 9\}$ B. $\{\log_9 5\}$ C. $\left\{\frac{\log 9}{\log 5}\right\}$ D. $\left\{\frac{\ln 9}{\ln 5}\right\}$

Solve each logarithmic equation. Express all solutions in exact form.

53. $3 \ln x = 13$ 54. $\ln(5x) = 16$

55. $\log(2x + 7) = 0.25$ 56. $\ln x + \ln x^3 = 12$

57. $\log_2(x^3 + 5) = 5$ 58. $\log_3(x^2 - 9) = 3$

59. $\log_4[(3x + 1)(x - 4)] = 2$ 60. $\ln e^{\ln x} - \ln(x - 4) = \ln 3$

61. $\log x + \log(13 - 3x) = 1$ 62. $\log_7(3x + 2) - \log_7(x - 2) = 1$

63. $\ln(6x) - \ln(x + 1) = \ln 4$ 64. $\log_{16}\sqrt{x + 1} = \frac{1}{4}$

65. $\ln[\ln(e^{-x})] = \ln 3$ 66. $S = a \ln\left(1 + \frac{n}{a}\right)$, for n

67. $d = 10 \log\left(\frac{I}{I_0}\right)$, for I_0 68. $D = 200 + 100 \log x$, for x

69. Use a graphing calculator to solve the equation $e^x = 4 - \ln x$. Give solution(s) to the nearest thousandth.

Solve each problem.

70. ***Earthquake Intensity*** On September 30, 2009, the region of Indonesia known as Sumatra was shaken by an earthquake that measured 7.5 on the Richter scale.

(a) Express this reading in terms of I_0. (See Section 4.4 Exercises.)

(b) On May 12, 2008, a quake measuring 7.9 on the Richter scale killed about 88,000 people in East Sichuan Province, China. Express the magnitude of a 7.9 reading in terms of I_0.

(c) How much greater was the force of the earthquake that measured 7.9?

71. ***Earthquake Intensity***
   (a) The San Francisco earthquake of 1906 had a Richter scale rating of 8.3. Express the magnitude of this earthquake as a multiple of I_0.
   (b) In 1989, the San Francisco region experienced an earthquake with a Richter scale rating of 7.1. Express the magnitude of this earthquake as a multiple of I_0.
   (c) Compare the magnitudes of the two San Francisco earthquakes discussed in parts (a) and (b).

72. ***(Modeling) Decibel Levels*** In Section 4.4 we found that the model for the decibel rating of the loudness of a sound is

$$d = 10 \log \frac{I}{I_0}.$$

A few years ago, there was a controversy about a proposed government limit on factory noise. One group wanted a maximum of 89 decibels, while another group wanted 86. This difference seemed very small to many people. Find the percent by which the 89-decibel intensity exceeds that for 86 decibels.

73. ***Interest Rate*** What annual interest rate, to the nearest tenth, will produce $5760 if $3500 is left at interest compounded annually for 10 yr?

74. ***Growth of an Account*** Find the number of years (to the nearest tenth) needed for $48,000 to become $58,344 at 5% interest compounded semiannually.

75. ***Growth of an Account*** Manuel deposits $10,000 for 12 yr in an account paying 4% compounded annually. He then puts this total amount on deposit in another account paying 5% compounded semiannually for another 9 yr. Find the total amount on deposit after the entire 21-yr period.

76. ***Growth of an Account*** Anne Kelly deposits $12,000 for 8 yr in an account paying 5% compounded annually. She then leaves the money alone with no further deposits at 6% compounded annually for an additional 6 yr. Find the total amount on deposit after the entire 14-yr period.

77. ***Cost from Inflation*** Suppose the inflation rate is 4%. Use the formula for continuous compounding to find the number of years, to the nearest tenth, for a $1 item to cost $2.

78. ***(Modeling) Drug Level in the Bloodstream*** After a medical drug is injected directly into the bloodstream, it is gradually eliminated from the body. Graph the following functions on the interval $[0, 10]$. Use $[0, 500]$ for the range of $A(t)$. Determine the function that best models the amount $A(t)$ (in milligrams) of a drug remaining in the body after t hours if 350 mg were initially injected.
   (a) $A(t) = t^2 - t + 350$
   (b) $A(t) = 350 \log(t + 1)$
   (c) $A(t) = 350(0.75)^t$
   (d) $A(t) = 100(0.95)^t$

79. ***(Modeling) Chicago Cubs' Payroll*** The table shows the total payroll (in millions of dollars) of the Chicago Cubs baseball team for the years 2007–2010.

Year	Total Payroll (millions of dollars)
2007	115.9
2008	118.3
2009	134.8
2010	146.6

Source: www.baseball.about.com, www.espn.go.com

Letting y represent the total payroll and x represent the number of years since 2007, we find that the function

$$f(x) = 113.2e^{0.0836x}$$

models the data quite well. According to this function, when would the total payroll double its 2010 value?

80. ***(Modeling) Transistors on Computer Chips*** Computing power of personal computers has increased dramatically as a result of the ability to place an increasing number of transistors on a single processor chip. The table lists the number of transistors on some popular computer chips made by Intel.

Year	Chip	Transistors
1985	386DX	275,000
1989	486DX	1,200,000
1994	Pentium	3,300,000
1999	Pentium 3	9,500,000
2000	Pentium 4	42,000,000
2006	Core 2 Duo	291,000,000
2008	Core 2 Quad	820,000,000

Source: Intel.

(a) Make a scatter diagram of the data. Let the x-axis represent the year, where $x = 0$ corresponds to 1985, and let the y-axis represent the number of transistors.

(b) Decide whether a linear, a logarithmic, or an exponential function describes the data best.

(c) Determine a function f that approximates these data. Plot f and the data on the same coordinate axes.

(d) Assuming that this trend continues, use f to predict the number of transistors on a chip in the year 2012.

81. ***Financial Planning*** The traditional IRA (individual retirement account) is a common tax-deferred saving plan in the United States. Earned income deposited into an IRA is not taxed in the current year, and no taxes are incurred on the interest paid in subsequent years. However, when you withdraw the money from the account after age $59\frac{1}{2}$, you pay taxes on the entire amount you withdraw.

Suppose you deposited \$5000 of earned income into an IRA, you can earn an annual interest rate of 4%, and you are in a 25% tax bracket. (*Note:* Interest rates and tax brackets are subject to change over time, but some assumptions must be made to evaluate the investment.) Also, suppose that you deposit the \$5000 at age 25 and withdraw it at age 60, and that interest is compounded continuously.

(a) How much money will remain after you pay the taxes at age 60?

(b) Suppose that instead of depositing the money into an IRA, you pay taxes on the money and the annual interest. How much money will you have at age 60? (*Note:* You effectively start with \$3750 (75% of \$5000), and the money earns 3% (75% of 4%) interest after taxes.)

(c) To the nearest dollar, how much additional money will you earn with the IRA?

(d) Suppose you pay taxes on the original \$5000 but are then able to earn 4% in a tax-free investment. Compare your balance at age 60 with the IRA balance.

82. Consider $f(x) = \log_4(2x^2 - x)$.

(a) Use the change-of-base theorem with base e to write $\log_4(2x^2 - x)$ in a suitable form to graph with a calculator.

(b) Graph the function using a graphing calculator. Use the window $[-2.5, 2.5]$ by $[-5, 2.5]$.

(c) What are the x-intercepts?

(d) Give the equations of the vertical asymptotes.

(e) Explain why there is no y-intercept.

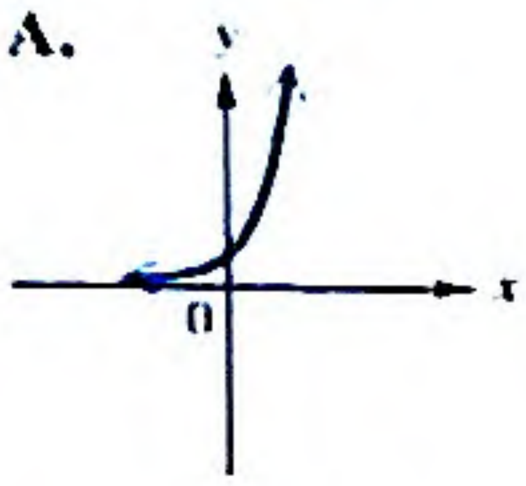
Chapter 4 Test

1. Consider the function $f(x) = \sqrt[3]{2x - 7}$.
   (a) What are the domain and range of f?
   (b) Explain why f^{-1} exists.
   (c) Find the rule for $f^{-1}(x)$.
   (d) What are the domain and range of f^{-1}?
   (e) Graph both f and f^{-1}. How are the two graphs related to the line $y = x$?

2. Match each equation with its graph.

   (a) $y = \log_{1/3} x$ (b) $y = e^x$ (c) $y = \ln x$ (d) $y = \left(\frac{1}{3}\right)^x$

A.



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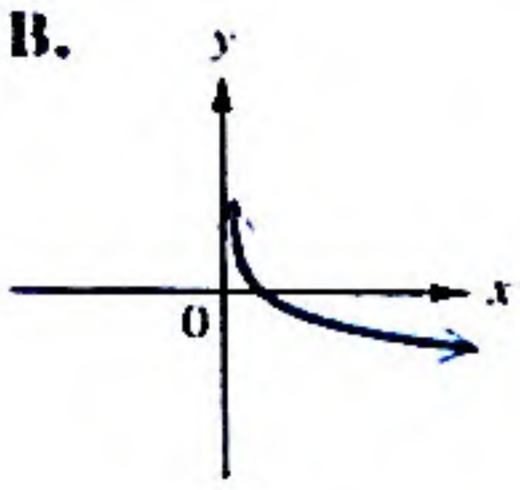
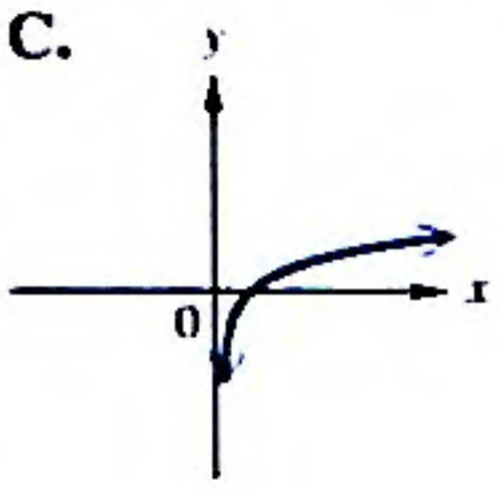
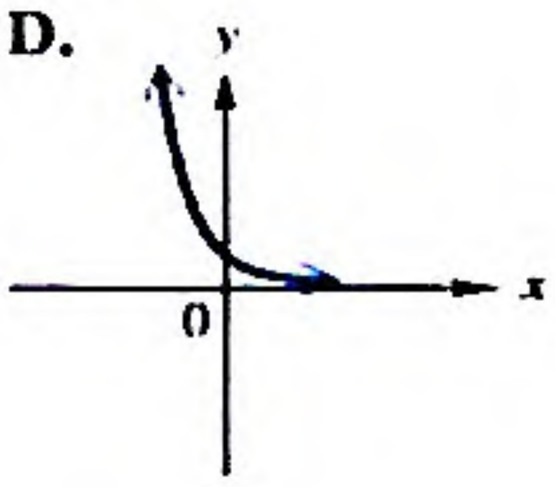


C.



D.



3. Solve $\left(\frac{1}{8}\right)^{2x-3} = 16^{x+1}$.

4. (a) Write $4^{3/2} = 8$ in logarithmic form.
   (b) Write $\log_8 4 = \frac{2}{3}$ in exponential form.

5. Graph $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \log_{1/2} x$ on the same axes. What is their relationship?

6. Use properties of logarithms to write

$$\log_7 \frac{x^2 \sqrt[4]{y}}{z^3}$$

as a sum, difference, or product of logarithms. Assume all variables represent positive numbers.

Use a calculator to find an approximation for each logarithm. Express answers to four decimal places.

7. $\log 2388$ 8. $\ln 2388$ 9. $\log_9 13$

10. Solve the following.

$$x^{2/3} = 25$$

Solve each exponential equation. If applicable, express solutions to the nearest thousandth.

11. $12^x = 1$ 12. $9^x = 4$ 13. $16^{2x+1} = 8^{3x}$

14. $2^{x+1} = 3^{x-4}$ 15. $e^{0.4x} = 4^{x-2}$

16. $2e^{2x} - 5e^x + 3 = 0$ (Give both exact and approximate values.)

Solve each logarithmic equation. Express all solutions in exact form.

17. $\log_x \frac{9}{16} = 2$ 18. $\log_2(x - 4)(x - 2) = 3$

19. $\log_2 x + \log_2(x + 2) = 3$ 20. $\ln x - 4 \ln 3 = \ln\left(\frac{1}{5}x\right)$

21. $\log_3(x + 1) - \log_3(x - 3) = 2$

22. One of your friends is taking another mathematics course and tells you, "I have no idea what an expression like $\log_5 27$ really means." Write an explanation of what it means, and tell how you can find an approximation for it with a calculator.

Solve each problem.

23. *(Modeling) Skydiver Fall Speed* A skydiver in free fall travels at a speed modeled by

$$v(t) = 176(1 - e^{-0.18t})$$

feet per second after t seconds. How long will it take for the skydiver to attain a speed of 147 ft per sec (100 mph)?

24. *Growth of an Account* How many years, to the nearest tenth, will be needed for \$5000 to increase to \$18,000 at 3.8% annual interest compounded (a) monthly (b) continuously?

25. *Tripling Time* For *any* amount of money invested at 3.8% annual interest compounded continuously, how long will it take to triple?

26. *(Modeling) Radioactive Decay* The amount of radioactive material, in grams, present after t days is modeled by

$$A(t) = 600e^{-0.05t}.$$

(a) Find the amount present after 12 days.
(b) Find the half-life of the material.

5

Systems and Matrices

A *system of equations* can describe the relationships between quantities combined in a mixture, such as a blend of several types of coffee beans, used to achieve a perfect balance of flavor, boldness, and acidity.

5.1 Systems of Linear Equations

5.2 Matrix Solution of Linear Systems

5.3 Determinant Solution of Linear Systems

5.4 Partial Fractions

Chapter 5 Quiz

5.5 Nonlinear Systems of Equations

Summary Exercises on Systems of Equations

5.6 Systems of Inequalities and Linear Programming

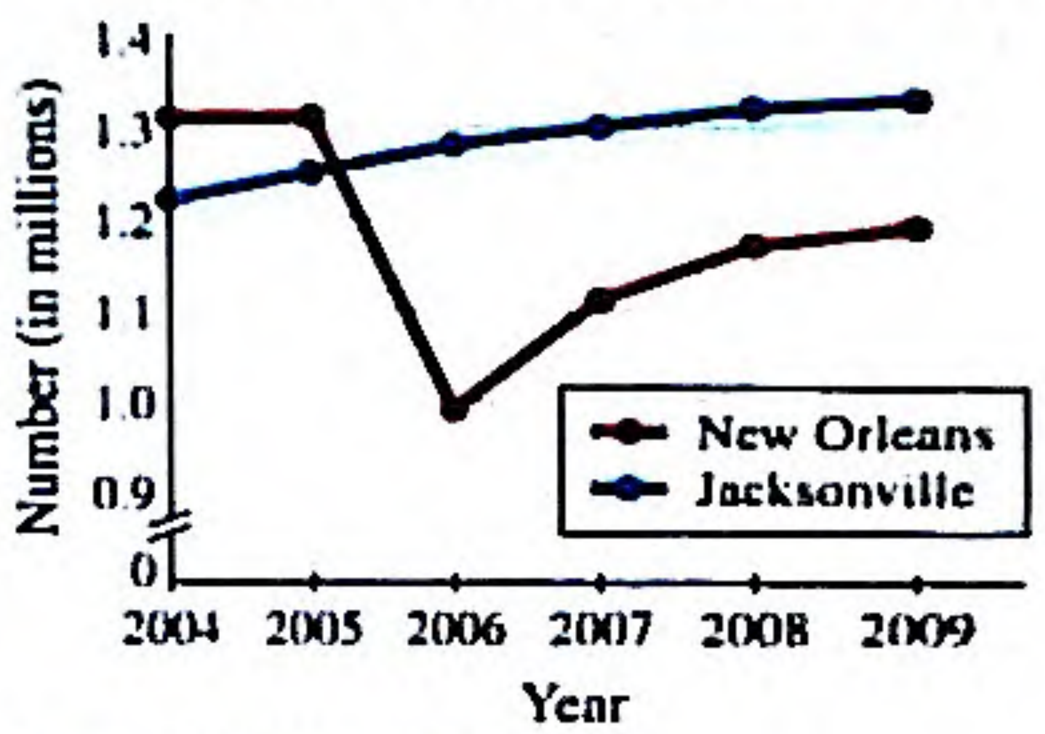
5.7 Properties of Matrices

5.8 Matrix Inverses

5.1 Exercises

Changes in Population *Many factors may contribute to population changes in metropolitan areas. The graph shows the populations of the New Orleans, Louisiana, and the Jacksonville, Florida, metropolitan areas over the years 2004–2009.*

Population of Metropolitan Areas



Source U.S. Census Bureau.

1. In what years was the population of the Jacksonville metropolitan area greater than that of the New Orleans metropolitan area?

2. At the time when the populations of the two metropolitan areas were equal, what was the approximate population of each area?

3. Express the solution of the system as an ordered pair.

4. Use the terms *increasing*, *decreasing*, and *constant* to describe the trends for the population of the New Orleans metropolitan area.

5. If equations of the form $y = f(t)$ were determined that modeled either of the two graphs, then the variable t would represent ______________ and the variable y would represent ______________.

6. Explain why each graph is that of a function.

Solve each system by substitution. See Example 1.

7. $\begin{array}{l} 4x + 3y = -13 \\ -x + y = 5 \end{array}$

8. $\begin{array}{l} 3x + 4y = 4 \\ x - y = 13 \end{array}$

9. $\begin{array}{l} x - 5y = 8 \\ x = 6y \end{array}$

10. $\begin{array}{l} 6x - y = 5 \\ y = 11x \end{array}$

11. $\begin{array}{l} 8x - 10y = -22 \\ 3x + y = 6 \end{array}$

12. $\begin{array}{l} 4x - 5y = -11 \\ 2x + y = 5 \end{array}$

13. $\begin{array}{l} 7x - y = -10 \\ 3y - x = 10 \end{array}$

14. $\begin{array}{l} 4x + 5y = 7 \\ 9y = 31 + 2x \end{array}$

15. $\begin{array}{l} -2x = 6y + 18 \\ -29 = 5y - 3x \end{array}$

16. $\begin{array}{l} 3x - 7y = 15 \\ 3x + 7y = 15 \end{array}$

17. $\begin{array}{l} 3y = 5x + 6 \\ x + y = 2 \end{array}$

18. $\begin{array}{l} 4y = 2x - 4 \\ x - y = 4 \end{array}$

Solve each system by elimination. In Exercises 27–30, first clear denominators. ***See Example 2.***

19. $\begin{array}{l} 3x - y = -4 \\ x + 3y = 12 \end{array}$

20. $\begin{array}{l} 4x + y = -23 \\ x - 2y = -17 \end{array}$

21. $\begin{array}{l} 2x - 3y = -7 \\ 5x + 4y = 17 \end{array}$

22. $\begin{array}{l} 4x + 3y = -1 \\ 2x + 5y = 3 \end{array}$

23. $\begin{array}{l} 5x + 7y = 6 \\ 10x - 3y = 46 \end{array}$

24. $\begin{array}{l} 12x - 5y = 9 \\ 3x - 8y = -18 \end{array}$

25. $\begin{array}{l} 6x + 7y + 2 = 0 \\ 7x - 6y - 26 = 0 \end{array}$

26. $\begin{array}{l} 5x + 4y + 2 = 0 \\ 4x - 5y - 23 = 0 \end{array}$

27. $\begin{array}{l} \frac{x}{2} + \frac{y}{3} = 4 \\ \frac{3x}{2} + \frac{3y}{2} = 15 \end{array}$

28. $\begin{array}{l} \frac{3x}{2} + \frac{y}{2} = -2 \\ \frac{x}{2} + \frac{y}{2} = 0 \end{array}$

29. $\begin{array}{l} \frac{2x - 1}{3} + \frac{y + 2}{4} = 4 \\ \frac{x + 3}{2} - \frac{x - y}{3} = 3 \end{array}$

30. $\begin{array}{l} \frac{x + 6}{5} + \frac{2y - x}{10} = 1 \\ \frac{x + 2}{4} + \frac{3y + 2}{5} = -3 \end{array}$

Solve each system. State whether it is inconsistent or has infinitely many solutions. If the system has infinitely many solutions, write the solution set with y arbitrary. See Examples 3 and 4.

31. $9x - 5y = 1$
$-18x + 10y = 1$

32. $3x + 2y = 5$
$6x + 4y = 8$

33. $4x - y = 9$
$-8x + 2y = -18$

34. $3x + 5y = -2$
$9x + 15y = -6$

35. $5x - 5y - 3 = 0$
$x - y - 12 = 0$

36. $2x - 3y - 7 = 0$
$-4x + 6y - 14 = 0$

37. $7x + 2y = 6$
$14x + 4y = 12$

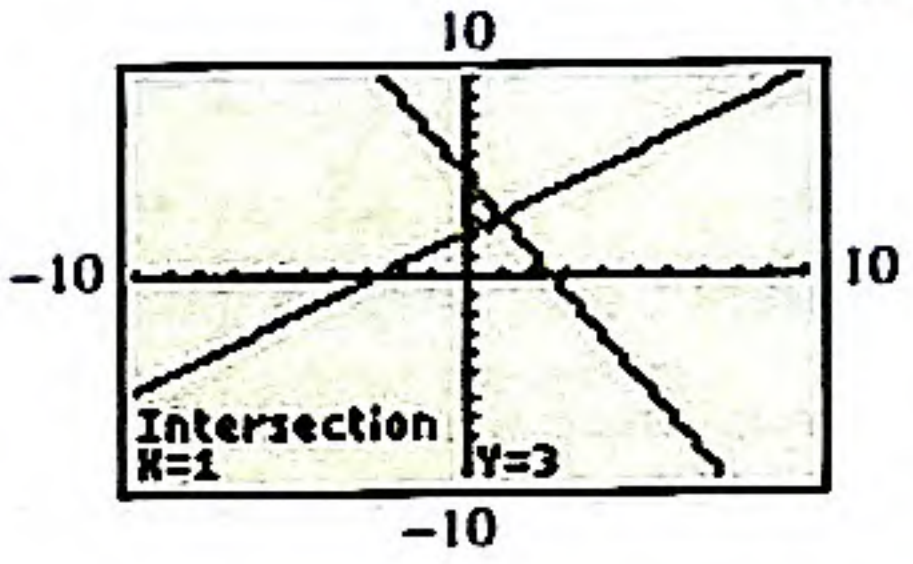
38. $2x - 8y = 4$
$x - 4y = 2$

39. $2x - 6y = 0$
$-7x + 21y = 10$

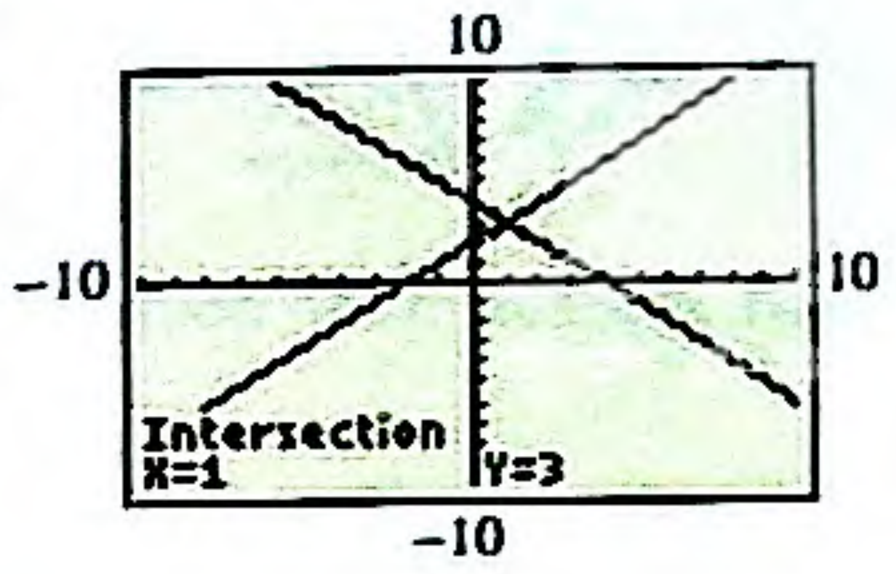
40. ***Concept Check*** Which screen gives the correct graphical solution of the system? (*Hint:* Solve for y first in each equation and use the slope-intercept forms to help you answer the question.)

$$\begin{aligned} 4x - 5y &= -11 \\ 2x + y &= 5 \end{aligned}$$

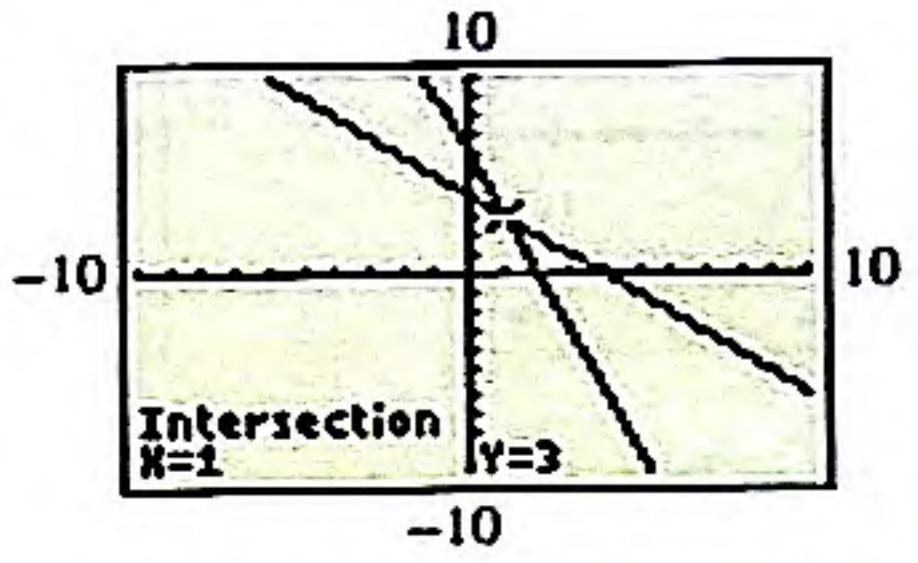
A.



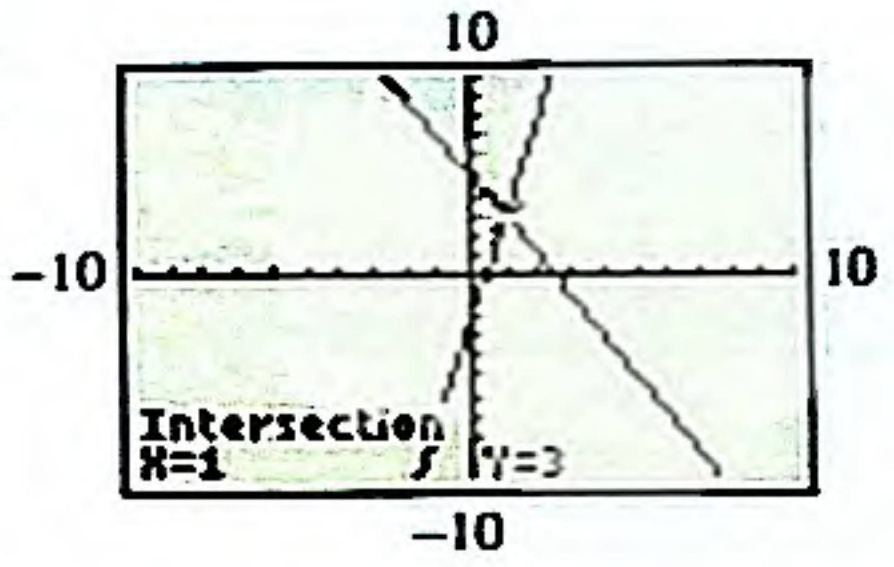
B.



C.

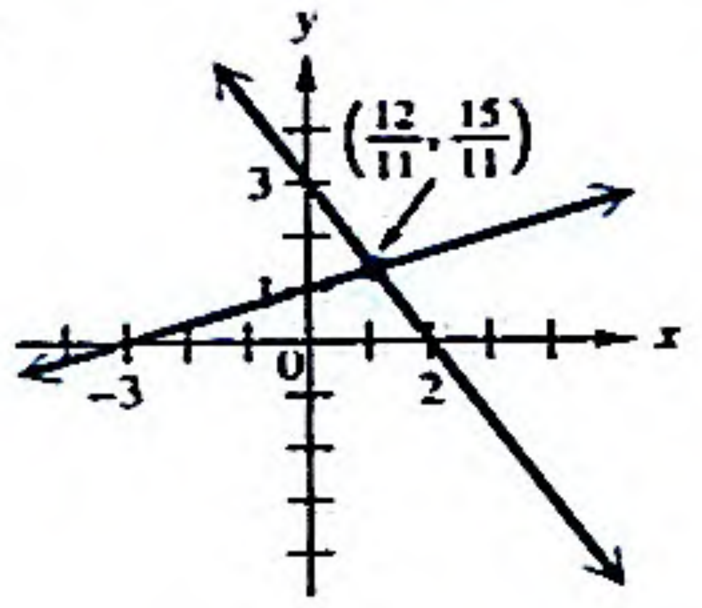


D.

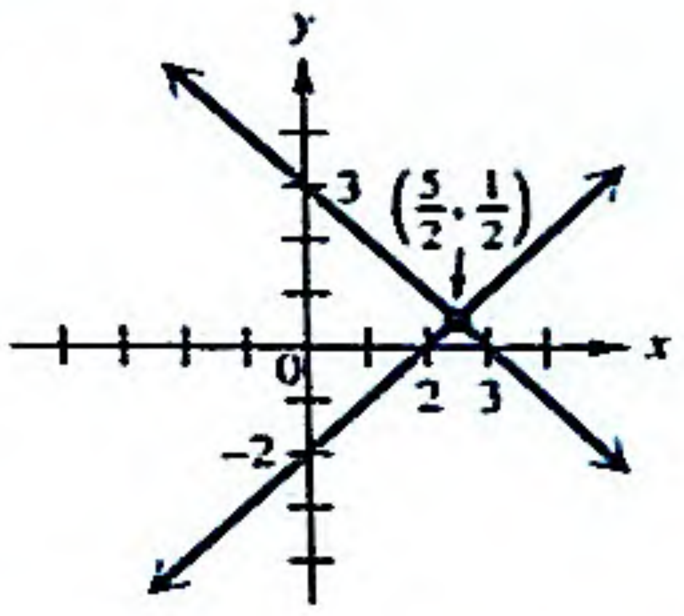


Connecting Graphs with Equations *Determine the system of equations illustrated in each graph. Write equations in standard form.*

41.



42.



Use a graphing calculator to solve each system. Express solutions with approximations to the nearest thousandth.

43. $\frac{11}{3}x + y = 0.5$
$0.6x - y = 3$

44. $\sqrt{3}x - y = 5$
$100x + y = 9$

45. $\sqrt{7}x + \sqrt{2}y - 3 = 0$
$\sqrt{6}x - y - \sqrt{3} = 0$

46. $0.2x + \sqrt{2}y = 1$
$\sqrt{5}x + 0.7y = 1$

Solve each system. See Example 6.

47. $x + y + z = 2$
$2x + y - z = 5$
$x - y + z = -2$

48. $2x + y + z = 9$
$-x - y + z = 1$
$3x - y + z = 9$

49. $x + 3y + 4z = 14$
$2x - 3y + 2z = 10$
$3x - y + z = 9$

50. $$\begin{aligned} 4x - y + 3z &= -2 \\ 3x + 5y - z &= 15 \\ -2x + y + 4z &= 14 \end{aligned}$$

51. $$\begin{aligned} x + 4y - z &= 6 \\ 2x - y + z &= 3 \\ 3x + 2y + 3z &= 16 \end{aligned}$$

52. $$\begin{aligned} 4x - 3y + z &= 9 \\ 3x + 2y - 2z &= 4 \\ x - y + 3z &= 5 \end{aligned}$$

53. $$\begin{aligned} x - 3y - 2z &= -3 \\ 3x + 2y - z &= 12 \\ -x - y + 4z &= 3 \end{aligned}$$

54. $$\begin{aligned} x + y + z &= 3 \\ 3x - 3y - 4z &= -1 \\ x + y + 3z &= 11 \end{aligned}$$

55. $$\begin{aligned} 2x + 6y - z &= 6 \\ 4x - 3y + 5z &= -5 \\ 6x + 9y - 2z &= 11 \end{aligned}$$

56. $$\begin{aligned} 8x - 3y + 6z &= -2 \\ 4x + 9y + 4z &= 18 \\ 12x - 3y + 8z &= -2 \end{aligned}$$

57. $$\begin{aligned} 2x - 3y + 2z - 3 &= 0 \\ 4x + 8y + z - 2 &= 0 \\ -x - 7y + 3z - 14 &= 0 \end{aligned}$$

58. $$\begin{aligned} -x + 2y - z - 1 &= 0 \\ -x - y - z + 2 &= 0 \\ x - y + 2z - 2 &= 0 \end{aligned}$$

Solve each system in terms of the arbitrary variable x. See Example 7.

59. $$\begin{aligned} x - 2y + 3z &= 6 \\ 2x - y + 2z &= 5 \end{aligned}$$

60. $$\begin{aligned} 3x - 2y + z &= 15 \\ x + 4y - z &= 11 \end{aligned}$$

61. $$\begin{aligned} 5x - 4y + z &= 9 \\ y + z &= 15 \end{aligned}$$

62. $$\begin{aligned} 3x - 5y - 4z &= -7 \\ y - z &= -13 \end{aligned}$$

63. $$\begin{aligned} 3x + 4y - z &= 13 \\ x + y + 2z &= 15 \end{aligned}$$

64. $$\begin{aligned} x - y + z &= -6 \\ 4x + y + z &= 7 \end{aligned}$$

Solve each system. State whether it is inconsistent or has infinitely many solutions. If the system has infinitely many solutions, write the solution set with z arbitrary. See Examples 3, 4, 6, and 7.

65. $$\begin{aligned} 3x + 5y - z &= -2 \\ 4x - y + 2z &= 1 \\ -6x - 10y + 2z &= 0 \end{aligned}$$

66. $$\begin{aligned} 3x + y + 3z &= 1 \\ x + 2y - z &= 2 \\ 2x - y + 4z &= 4 \end{aligned}$$

67. $$\begin{aligned} 5x - 4y + z &= 0 \\ x + y &= 0 \\ -10x + 8y - 2z &= 0 \end{aligned}$$

68. $$\begin{aligned} 2x + y - 3z &= 0 \\ 4x + 2y - 6z &= 0 \\ x - y + z &= 0 \end{aligned}$$

Solve each system. (Hint: In Exercises 69–72, let $\frac{1}{x} = t$ and $\frac{1}{y} = u$.)

69. $$\begin{aligned} \frac{2}{x} + \frac{1}{y} &= \frac{3}{2} \\ \frac{3}{x} - \frac{1}{y} &= 1 \end{aligned}$$

70. $$\begin{aligned} \frac{1}{x} + \frac{3}{y} &= \frac{16}{5} \\ \frac{5}{x} + \frac{4}{y} &= 5 \end{aligned}$$

71. $$\begin{aligned} \frac{2}{x} + \frac{1}{y} &= 11 \\ \frac{3}{x} - \frac{5}{y} &= 10 \end{aligned}$$

72. $$\begin{aligned} \frac{2}{x} + \frac{3}{y} &= 18 \\ \frac{4}{x} - \frac{5}{y} &= -8 \end{aligned}$$

73. $$\begin{aligned} \frac{2}{x} + \frac{3}{y} - \frac{2}{z} &= -1 \\ \frac{8}{x} - \frac{12}{y} + \frac{5}{z} &= 5 \\ \frac{6}{x} + \frac{3}{y} - \frac{1}{z} &= 1 \end{aligned}$$

74. $$\begin{aligned} -\frac{5}{x} + \frac{4}{y} + \frac{3}{z} &= 2 \\ \frac{10}{x} + \frac{3}{y} - \frac{6}{z} &= 7 \\ \frac{5}{x} + \frac{2}{y} - \frac{9}{z} &= 6 \end{aligned}$$

75. ***Concept Check*** For what value(s) of k will the following system of linear equations have no solution? infinitely many solutions?

$$\begin{aligned} x - 2y &= 3 \\ -2x + 4y &= k \end{aligned}$$

76. ***Concept Check*** Consider the linear equation in three variables $x + y + z = 4$. Find a pair of linear equations in three variables that, when considered together with the given equation, form a system having **(a)** exactly one solution, **(b)** no solution, **(c)** infinitely many solutions.

(Modeling) *Use a system of equations to solve each problem. See Example 8.*

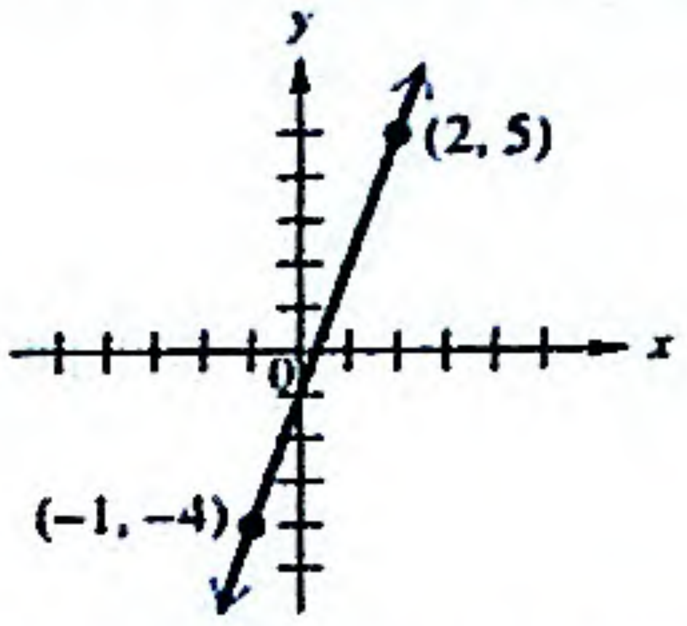
77. Find the equation of the line $y = ax + b$ that passes through the points $(-2, 1)$ and $(-1, -2)$.

78. Find the equation of the line $y = ax + b$ that passes through the points $(3, -4)$ and $(-1, 4)$.

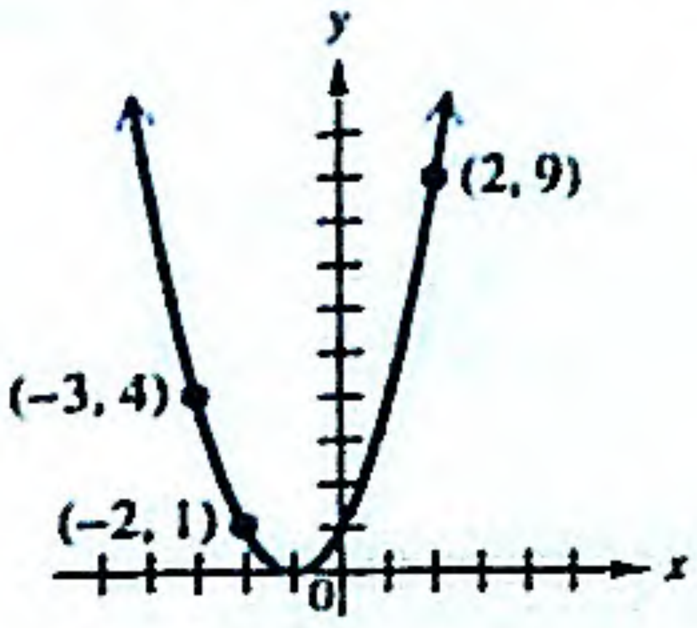
79. Find the equation of the parabola $y = ax^2 + bx + c$ that passes through the points $(2, 3)$, $(-1, 0)$, and $(-2, 2)$.

80. Find the equation of the parabola $y = ax^2 + bx + c$ that passes through the points $(-2, 4)$, $(2, 2)$, and $(4, 9)$.

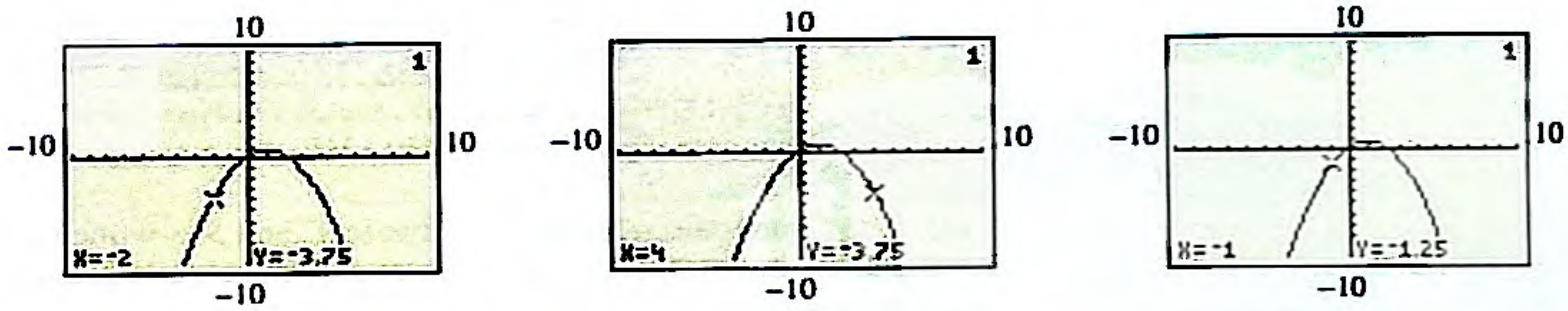
81. ***Connecting Graphs with Equations*** Use a system to find the equation of the line through the given points.



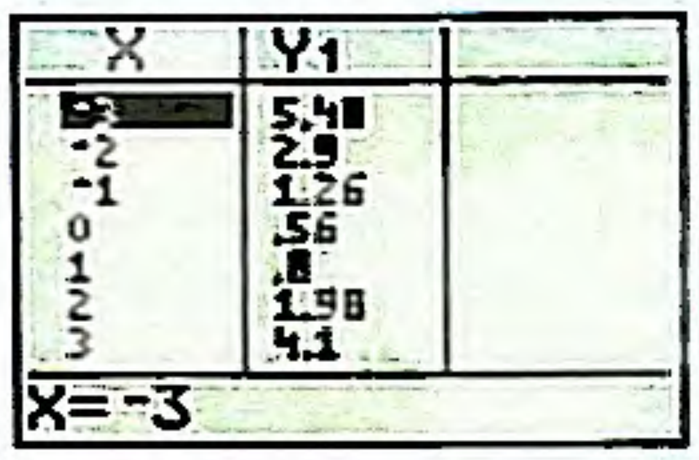
82. ***Connecting Graphs with Equations*** Use a system to find the equation of the parabola through the given points.



83. ***Connecting Graphs with Equations*** Find the equation of the parabola. Three views of the same curve are given.



84. ***(Modeling)*** The table was generated using a function defined by $Y_1 = aX^2 + bX + c$. Use any three points from the table to find the equation that defines the function.



X	Y_1
−3	5.48
−2	2.9
−1	1.26
0	.56
1	.8
2	1.98
3	4.1

X=−3

(Modeling) *Given three noncollinear points, there is one and only one circle that passes through them. Knowing that the equation of a circle may be written in the form*

$$x^2 + y^2 + ax + by + c = 0, \quad \text{(Section 2.2)}$$

find the equation of the circle passing through the given points.

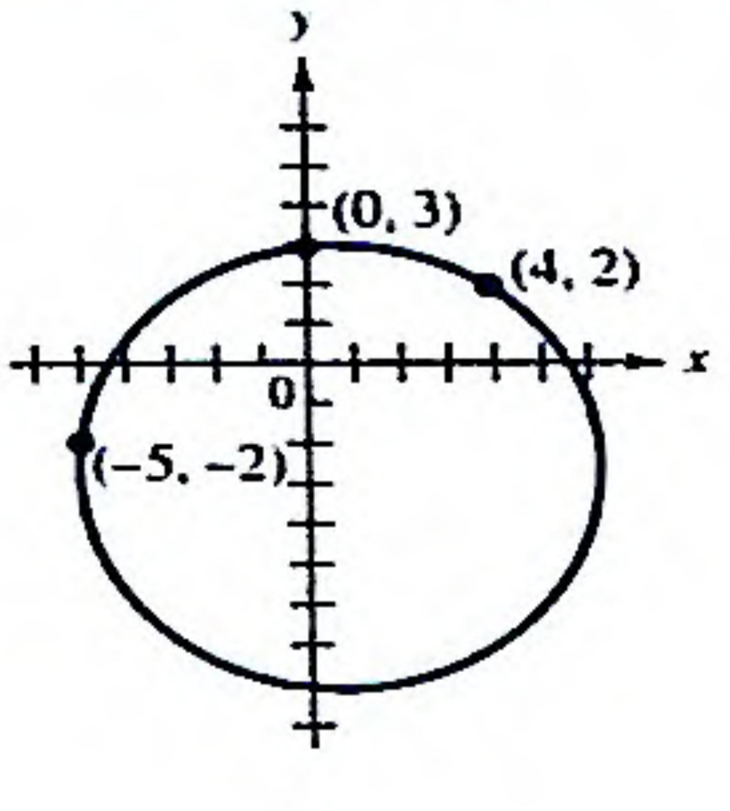
85. $(-1, 3)$, $(6, 2)$, and $(-2, -4)$

86. $(-1, 5)$, $(6, 6)$, and $(7, -1)$

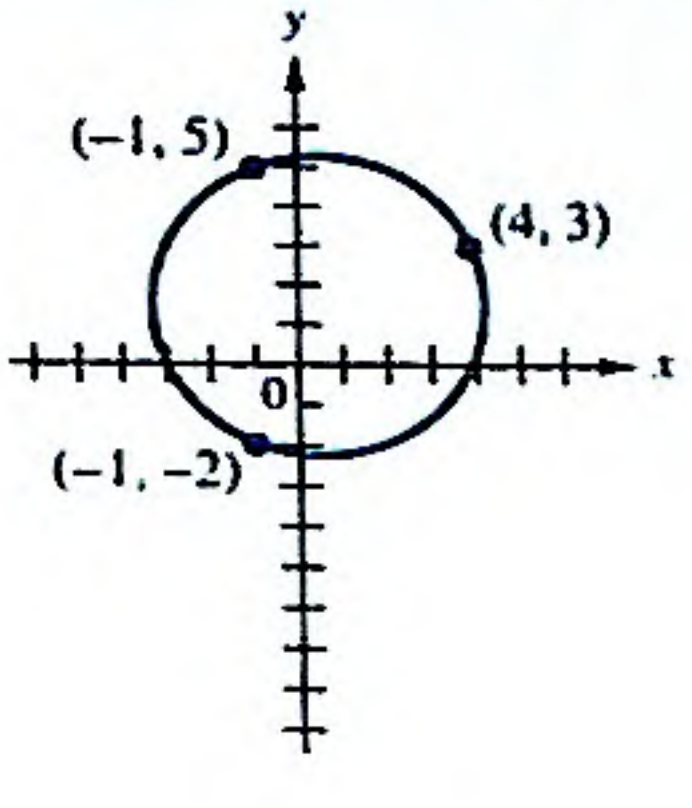
87. $(2, 1)$, $(-1, 0)$, and $(3, 3)$

88. $(-5, 0)$, $(2, -1)$, and $(4, 3)$

89. ***Connecting Graphs with Equations***



90. ***Connecting Graphs with Equations***



(Modeling) Use the method of Example 8 to work Exercises 91 and 92.

Year	CO_2
1960	317
1980	339
2009	385

Source: U.S. Department of Energy, Carbon Dioxide Information Analysis Center

91. *Atmospheric Carbon Dioxide* Carbon dioxide concentrations (in parts per million) have been measured directly from the atmosphere since 1960. This concentration has increased quadratically. The table lists readings for three years.

(a) If the quadratic relationship between the carbon dioxide concentration C and the year t is expressed as $C = at^2 + bt + c$, where $t = 0$ corresponds to 1960, use a system of linear equations to determine the constants a, b, and c, and give the equation.

(b) Predict when the amount of carbon dioxide in the atmosphere will be double its 1960 level.

92. *Aircraft Speed and Altitude* For certain aircraft there exists a quadratic relationship between an airplane's maximum speed S (in knots) and its ceiling C, or highest altitude possible (in thousands of feet). The table lists three airplanes that conform to this relationship.

Airplane	Max Speed (S)	Ceiling (C)
Hawkeye	320	33
Corsair	600	40
Tomcat	1283	50

Source: Sanders, D., *Statistics: A First Course*, Sixth Edition, McGraw Hill.

(a) If the quadratic relationship between C and S is written as $C = aS^2 + bS + c$, use a system of linear equations to determine the constants a, b and c, and give the equation.

(b) A new aircraft of this type has a ceiling of 45,000 ft. Predict its top speed.

Solve each problem. See Examples 5 and 9.

93. *Unknown Numbers* The sum of two numbers is 47, and the difference between the numbers is 1. Find the numbers.

94. *Costs of Goats and Sheep* At the Brendan Berger ranch, 6 goats and 5 sheep sell for $305, while 2 goats and 9 sheep sell for $285. Find the cost of a single goat and of a single sheep.

95. *Fan Cost Index* The Fan Cost Index (FCI) is a measure of how much it will cost a family of four to attend a professional sports event. In 2010, the FCI prices for Major League Baseball and the National Football League averaged $307.76. The FCI for baseball was $225.56 less than that for football. What were the FCIs for these sports? (*Source:* Team Marketing Report.)

96. *Money Denominations* A cashier has a total of 30 bills, made up of ones, fives, and twenties. The number of twenties is 9 more than the number of ones. The total value of the money is $351. How many of each denomination of bill are there?

97. *Mixing Water* A sparkling-water distributor wants to make up 300 gal of sparkling water to sell for $6.00 per gallon. She wishes to mix three grades of water selling for $9.00, $3.00, and $4.50 per gallon, respectively. She must use twice as much of the $4.50 water as of the $3.00 water. How many gallons of each should she use?

98. *Mixing Glue* A glue company needs to make some glue that it can sell for $120 per barrel. It wants to use 150 barrels of glue worth $100 per barrel, along with some glue worth $150 per barrel and some glue worth $190 per barrel. It must use the same number of barrels of $150 and $190 glue. How much of the $150 and $190 glue will be needed? How many barrels of $120 glue will be produced?

99. *Triangle Dimensions* The perimeter of a triangle is 59 in. The longest side is 11 in. longer than the medium side, and the medium side is 3 in. longer than the shortest side. Find the length of each side of the triangle.

100. *Triangle Dimensions* The sum of the measures of the angles of any triangle is 180°. In a certain triangle, the largest angle measures 55° less than twice the medium angle, and the smallest angle measures 25° less than the medium angle. Find the measures of all three angles.

101. *Investment Decisions* Patrick Summers wins $200,000 in the Louisiana state lottery. He invests part of the money in real estate with an annual return of 3% and another part in a money market account at 2.5% interest. He invests the rest, which amounts to $80,000 less than the sum of the other two parts, in certificates of deposit that pay 1.5%. If the total annual interest on the money is $4900, how much was invested at each rate?

	Amount Invested	Rate (as a decimal)	Annual Interest
Real Estate		0.03	
Money Market		0.025	
CDs		0.015	

102. *Investment Decisions* Jane Hooker invests $40,000 received as an inheritance in three parts. With one part she buys mutual funds that offer a return of 2% per year. The second part, which amounts to twice the first, is used to buy government bonds paying 2.5% per year. She puts the rest of the money into a savings account that pays 1.25% annual interest. During the first year, the total interest is $825. How much did she invest at each rate?

Relating Concepts

For individual or collaborative investigation *(Exercises 103–108)*

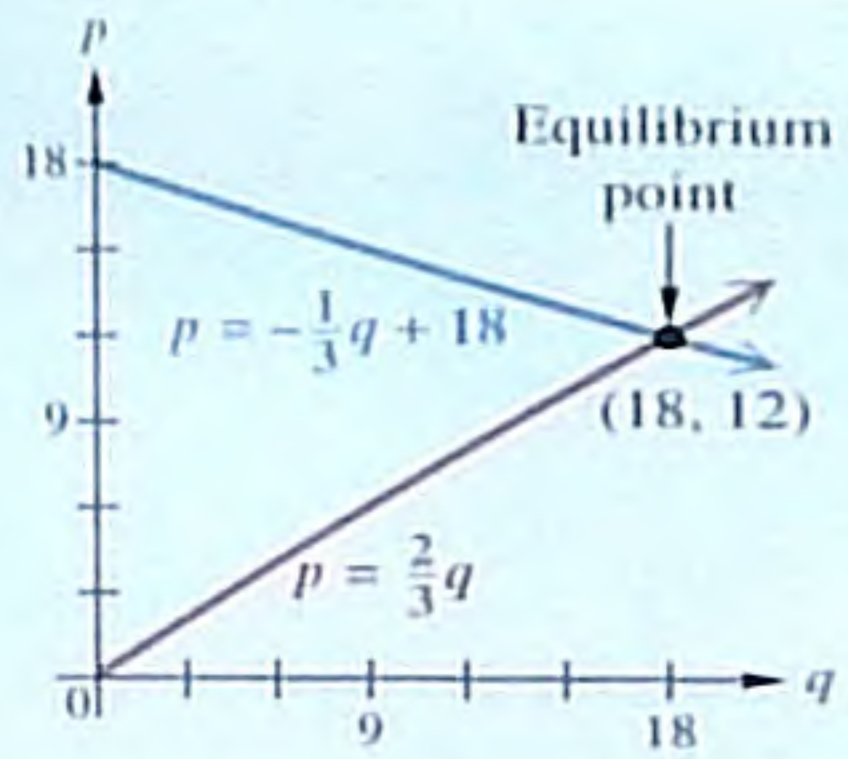
Supply and Demand In many applications of economics, as the price of an item goes up, demand for the item goes down and supply of the item goes up. The price where supply and demand are equal is the ***equilibrium price,*** *and the resulting supply or demand is the* ***equilibrium supply*** *or* ***equilibrium demand.***

Suppose the supply of a product is related to its price by the equation

$$p = \frac{2}{3}q,$$

where p is in dollars and q is supply in appropriate units. (Here, q stands for quantity.)

(continued)



Furthermore, suppose demand and price for the same product are related by

$$p = -\frac{1}{3}q + 18,$$

where p is price and q is demand. The system formed by these two equations has solution (18, 12), as seen in the graph. Use this information to ***work Exercises 103–108 in order.***

103. Suppose the demand and price for a certain model of electric can opener are related by $p = 16 - \frac{5}{4}q$, where p is price, in dollars, and q is demand, in appropriate units. Find the price when the demand is at each level.

(a) 0 units **(b)** 4 units **(c)** 8 units

104. Find the demand for the electric can opener at each price.

(a) \$6 **(b)** \$11 **(c)** \$16

105. Graph $p = 16 - \frac{5}{4}q$.

106. Suppose the price and supply of the can opener are related by $p = \frac{3}{4}q$, where q represents the supply and p the price. Find the supply at each price.

(a) \$0 **(b)** \$10 **(c)** \$20

107. Graph $p = \frac{3}{4}q$ on the same axes used for **Exercise 105.**

108. Use the result of **Exercise 107** to find the equilibrium price and the equilibrium demand.

109. Solve the system of equations (4), (5), and (6) from **Example 9.**

$$\begin{aligned} 25x + 40y + 20z &= 2200 \quad (4)\\ 4x + 2y + 3z &= 280 \quad (5)\\ 3x + 2y + z &= 180 \quad (6) \end{aligned}$$

110. Check your solution in **Exercise 109,** showing that it satisfies all three equations of the system.

111. ***Blending Coffee Beans*** Three varieties of coffee—Arabian Mocha Sanani, Organic Shade Grown Mexico, and Guatemala Antigua—are combined and roasted, yielding a 50-lb batch of coffee beans. Twice as many pounds of Guatemala Antigua, which retails for \$10.19 per lb, are needed as of Arabian Mocha Sanani, which retails for \$15.99 per lb. Organic Shade Grown Mexico retails for \$12.99 per lb. How many pounds of each coffee should be used in a blend that sells for \$12.37 per lb?

112. ***Blending Coffee Beans*** Rework Exercise 111 if Guatemala Antigua retails for \$12.49 per lb instead of \$10.19 per lb. Does your answer seem reasonable? Explain.

5.2 Matrix Solution of Linear Systems

- **The Gauss-Jordan Method**
- **Special Systems**

$$\begin{bmatrix} 2 & 3 & 7 \\ 5 & -1 & 10 \end{bmatrix} \quad \text{Matrix}$$

Since systems of linear equations occur in so many practical situations, computer methods have been developed for efficiently solving linear systems. Computer solutions of linear systems depend on the idea of a **matrix** (plural **matrices**), a rectangular array of numbers enclosed in brackets. Each number is an **element** of the matrix.

The Gauss-Jordan Method In this section, we develop a method for solving linear systems using matrices. We start with a system and write the coefficients of the variables and the constants as an **augmented matrix** of the system.

Linear system of equations | Augmented matrix

$$\begin{aligned} x + 3y + 2z &= 1 \\ 2x + y - z &= 2 \\ x + y + z &= 2 \end{aligned} \quad \text{can be written as} \quad \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 2 & 1 & -1 & 2 \\ 1 & 1 & 1 & 2 \end{array}\right] \begin{array}{l} \leftarrow \\ \leftarrow \text{ Rows} \\ \leftarrow \end{array} \\ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\ \text{Columns} \end{array}$$

The vertical line, which is optional, separates the coefficients from the constants. Because this matrix has 3 rows (horizontal) and 4 columns (vertical), we say its **dimension*** is 3×4 (read "three by four"). ***The number of rows is always given first.*** To refer to a number in the matrix, use its row and column numbers. For example, the number 3 is in the first row, second column.

We can treat the rows of this matrix just like the equations of the corresponding system of linear equations. Since the augmented matrix is nothing more than a shorthand form of the system, any transformation of the matrix that results in an equivalent system of equations can be performed.

Matrix Row Transformations

For any augmented matrix of a system of linear equations, the following row transformations will result in the matrix of an equivalent system.

1. Interchange any two rows.
2. Multiply or divide the elements of any row by a nonzero real number.
3. Replace any row of the matrix by the sum of the elements of that row and a multiple of the elements of another row.

These transformations are restatements in matrix form of the transformations of systems discussed in the previous section. From now on, when referring to the third transformation, we will abbreviate "a multiple of the elements of a row" as "a multiple of a row."

Before matrices can be used to solve a linear system, the system must be arranged in the proper form, with variable terms on the left side of the equation and constant terms on the right. The variable terms must be in the same order in each of the equations.

---

*Other terms used to describe the dimension of a matrix are *order* and *size*.

EXAMPLE 4 Solving a System with Infinitely Many Solutions

Use the Gauss-Jordan method to solve the system.

$$\begin{aligned} 2x - 5y + 3z &= 1 \\ x - 2y - 2z &= 8 \end{aligned}$$

SOLUTION Recall from the previous section that a system with two equations in three variables usually has an infinite number of solutions. We can use the Gauss-Jordan method to give the solution with z arbitrary.

$$\left[\begin{array}{ccc|c} 2 & -5 & 3 & 1 \\ 1 & -2 & -2 & 8 \end{array}\right] \quad \text{Write the augmented matrix.}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 2 & -5 & 3 & 1 \end{array}\right] \quad \text{Interchange rows to get 1 in the first row, first column position.}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 0 & -1 & 7 & -15 \end{array}\right] \quad -2R1 + R2$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & 8 \\ 0 & 1 & -7 & 15 \end{array}\right] \quad -1R2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -16 & 38 \\ 0 & 1 & -7 & 15 \end{array}\right] \quad 2R2 + R1$$

It is not possible to go further with the Gauss-Jordan method. The equations that correspond to the final matrix are

$$x - 16z = 38 \quad \text{and} \quad y - 7z = 15.$$

Solve these equations for x and y, respectively.

$$\begin{array}{r|r} x - 16z = 38 & y - 7z = 15 \\ x = 16z + 38 & y = 7z + 15 \quad \text{(Section 1.1)} \end{array}$$

The solution set, written with z arbitrary, is $\{(16z + 38, 7z + 15, z)\}$.

✓ *Now Try Exercise 37.*

Summary of Possible Cases

When matrix methods are used to solve a system of linear equations and the resulting matrix is written in diagonal form:

1. If the number of rows with nonzero elements to the left of the vertical line is equal to the number of variables in the system, then the system has a single solution. See **Examples 1 and 2.**
2. If one of the rows has the form $[0\ 0 \cdots 0 \mid a]$ with $a \neq 0$, then the system has no solution. See **Example 3.**
3. If there are fewer rows in the matrix containing nonzero elements than the number of variables, then the system has either no solution or infinitely many solutions. If there are infinitely many solutions, give the solutions in terms of one or more arbitrary variables. See **Example 4.**

5.2 Exercises

Use the given row transformation to change each matrix as indicated. See Example 1.

1. $\begin{bmatrix} 1 & 4 \\ 4 & 7 \end{bmatrix}$; −4 times row 1 added to row 2

2. $\begin{bmatrix} 1 & -4 \\ 7 & 0 \end{bmatrix}$; −7 times row 1 added to row 2

3. $\begin{bmatrix} 1 & 5 & 6 \\ -2 & 3 & -1 \\ 4 & 7 & 0 \end{bmatrix}$; 2 times row 1 added to row 2

4. $\begin{bmatrix} 1 & 5 & 6 \\ -4 & -1 & 2 \\ 3 & 7 & 1 \end{bmatrix}$; 4 times row 1 added to row 2

Concept Check Write the augmented matrix for each system and give its dimension. Do not solve.

5. $\begin{aligned} 2x + 3y &= 11 \\ x + 2y &= 8 \end{aligned}$

6. $\begin{aligned} 3x + 5y &= -13 \\ 2x + 3y &= -9 \end{aligned}$

7. $\begin{aligned} 2x + y + z - 3 &= 0 \\ 3x - 4y + 2z + 7 &= 0 \\ x + y + z - 2 &= 0 \end{aligned}$

8. $\begin{aligned} 4x - 2y + 3z - 4 &= 0 \\ 3x + 5y + z - 7 &= 0 \\ 5x - y + 4z - 7 &= 0 \end{aligned}$

Concept Check Write the system of equations associated with each augmented matrix. Do not solve.

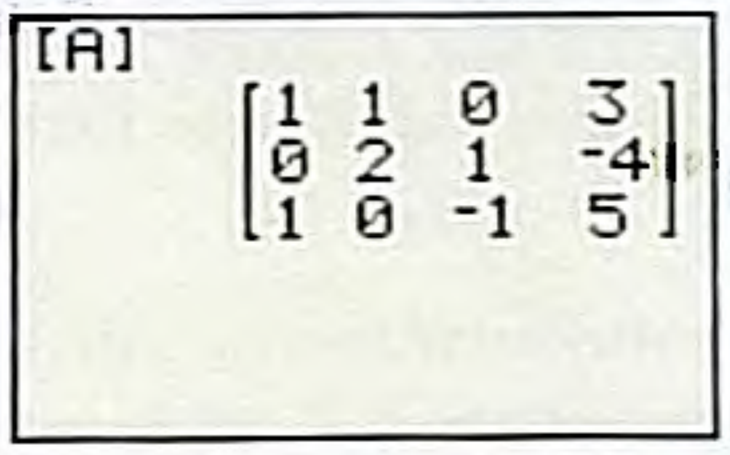
9. $\left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 0 & 2 & 4 & 22 \\ -1 & -2 & 3 & 15 \end{array}\right]$

10. $\left[\begin{array}{ccc|c} 2 & 1 & 3 & 12 \\ 4 & -3 & 0 & 10 \\ 5 & 0 & -4 & -11 \end{array}\right]$

11. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right]$

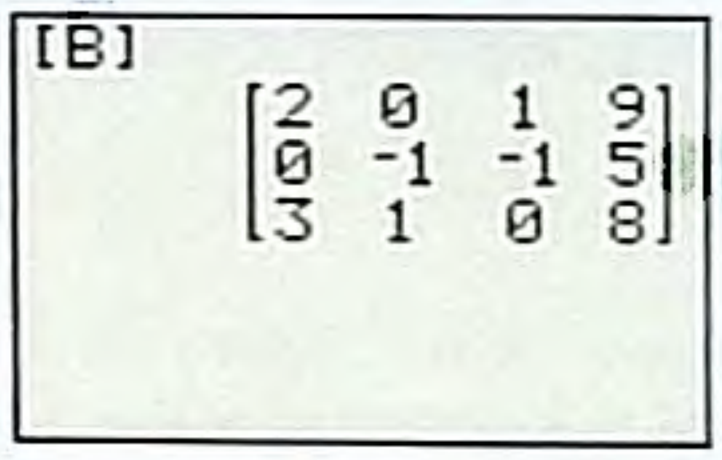
12. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

13.



```
[A]
        [1 1 0  3 ]
        [0 2 1  -4]
        [1 0 -1 5 ]
```

14.



```
[B]
        [2 0  1  9]
        [0 -1 -1 5]
        [3 1  0  8]
```

Use the Gauss-Jordan method to solve each system of equations. For systems in two variables with infinitely many solutions, give the solution with y arbitrary. For systems in three variables with infinitely many solutions, give the solution with z arbitrary. See Examples 1–4.

15. $\begin{aligned} x + y &= 5 \\ x - y &= -1 \end{aligned}$

16. $\begin{aligned} x + 2y &= 5 \\ 2x + y &= -2 \end{aligned}$

17. $\begin{aligned} 3x + 2y &= -9 \\ 2x - 5y &= -6 \end{aligned}$

18. $\begin{aligned} 2x - 5y &= 10 \\ 3x + y &= 15 \end{aligned}$

19. $\begin{aligned} 6x - 3y - 4 &= 0 \\ 3x + 6y - 7 &= 0 \end{aligned}$

20. $\begin{aligned} 2x - 3y - 10 &= 0 \\ 2x + 2y - 5 &= 0 \end{aligned}$

21. $\begin{aligned} 2x - y &= 6 \\ 4x - 2y &= 0 \end{aligned}$

22. $\begin{aligned} 3x - 2y &= 1 \\ 6x - 4y &= -1 \end{aligned}$

23. $\begin{aligned} \frac{3}{8}x - \frac{1}{2}y &= \frac{7}{8} \\ -6x + 8y &= -14 \end{aligned}$

24. $\begin{aligned} \frac{1}{2}x + \frac{3}{5}y &= \frac{1}{4} \\ 10x + 12y &= 5 \end{aligned}$

25. $\begin{aligned} x + y - 5z &= -18 \\ 3x - 3y + z &= 6 \\ x + 3y - 2z &= -13 \end{aligned}$

26. $\begin{aligned} -x + 2y + 6z &= 2 \\ 3x + 2y + 6z &= 6 \\ x + 4y - 3z &= 1 \end{aligned}$

27. $\begin{aligned} x + y - z &= 6 \\ 2x - y + z &= -9 \\ x - 2y + 3z &= 1 \end{aligned}$

28. $\begin{aligned} x + 3y - 6z &= 7 \\ 2x - y + z &= 1 \\ x + 2y + 2z &= -1 \end{aligned}$

29. $\begin{aligned} x - z &= -3 \\ y + z &= 9 \\ x + z &= 7 \end{aligned}$

30.
$$\begin{aligned} -x + y &= -1\\ y - z &= 6\\ x + z &= -1 \end{aligned}$$

31.
$$\begin{aligned} y &= -2x - 2z + 1\\ x &= -2y - z + 2\\ z &= x - y \end{aligned}$$

32.
$$\begin{aligned} x &= -y + 1\\ z &= 2x\\ y &= -2z - 2 \end{aligned}$$

33.
$$\begin{aligned} 2x - y + 3z &= 0\\ x + 2y - z &= 5\\ 2y + z &= 1 \end{aligned}$$

34.
$$\begin{aligned} 4x + 2y - 3z &= 6\\ x - 4y + z &= -4\\ -x + 2z &= 2 \end{aligned}$$

35.
$$\begin{aligned} 3x + 5y - z + 2 &= 0\\ 4x - y + 2z - 1 &= 0\\ -6x - 10y + 2z &= 0 \end{aligned}$$

36.
$$\begin{aligned} 3x + y + 3z - 1 &= 0\\ x + 2y - z - 2 &= 0\\ 2x - y + 4z - 4 &= 0 \end{aligned}$$

37.
$$\begin{aligned} x - 8y + z &= 4\\ 3x - y + 2z &= -1 \end{aligned}$$

38.
$$\begin{aligned} 5x - 3y + z &= 1\\ 2x + y - z &= 4 \end{aligned}$$

39.
$$\begin{aligned} x - y + 2z + w &= 4\\ y + z &= 3\\ z - w &= 2\\ x - y &= 0 \end{aligned}$$

40.
$$\begin{aligned} x + 2y + z - 3w &= 7\\ y + z &= 0\\ x - w &= 4\\ -x + y &= -3 \end{aligned}$$

41.
$$\begin{aligned} x + 3y - 2z - w &= 9\\ 4x + y + z + 2w &= 2\\ -3x - y + z - w &= -5\\ x - y - 3z - 2w &= 2 \end{aligned}$$

42.
$$\begin{aligned} 2x + y - z + 3w &= 0\\ 3x - 2y + z - 4w &= -24\\ x + y - z + w &= 2\\ x - y + 2z - 5w &= -16 \end{aligned}$$

Solve each system using a graphing calculator capable of performing row operations Give solutions with values correct to the nearest thousandth.

43.
$$\begin{aligned} 0.3x + 2.7y - \sqrt{2}z &= 3\\ \sqrt{7}x - 20y + 12z &= -2\\ 4x + \sqrt{3}y - 1.2z &= \frac{3}{4} \end{aligned}$$

44.
$$\begin{aligned} \sqrt{5}x - 1.2y + z &= -3\\ \frac{1}{2}x - 3y + 4z &= \frac{4}{3}\\ 4x + 7y - 9z &= \sqrt{2} \end{aligned}$$

Graph each system of three equations together on the same axes, and determine the number of solutions (exactly one, none, or infinitely many). If there is exactly one solution, estimate the solution. Then confirm your answer by solving the system with the Gauss-Jordan method.

45.
$$\begin{aligned} 2x + 3y &= 5\\ -3x + 5y &= 22\\ 2x + y &= -1 \end{aligned}$$

46.
$$\begin{aligned} 3x - 2y &= 3\\ -2x + 4y &= 14\\ x + y &= 11 \end{aligned}$$

For each equation, determine the constants A and B that make the equation an identity. (Hint: Combine terms on the right, and set coefficients of corresponding terms in the numerators equal.)

47. $\dfrac{1}{(x-1)(x+1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1}$

48. $\dfrac{x+4}{x^2} = \dfrac{A}{x} + \dfrac{B}{x^2}$

49. $\dfrac{x}{(x-a)(x+a)} = \dfrac{A}{x-a} + \dfrac{B}{x+a}$

50. $\dfrac{2x}{(x+2)(x-1)} = \dfrac{A}{x+2} + \dfrac{B}{x-1}$

Solve each problem using matrices

51. ***Daily Wages*** Dan Abbey is a building contractor. If he hires 7 day laborers and 2 concrete finishers, his payroll for the day is $1384. If he hires 1 day laborer and 5 concrete finishers, his daily cost is $952. Find the daily wage for each type of worker.

52. *Mixing Nuts* At the Everglades Nut Company, 5 lb of peanuts and 6 lb of cashews cost \$33.60, while 3 lb of peanuts and 7 lb of cashews cost \$32.40. Find the cost of a single pound of peanuts and a single pound of cashews.

53. *Unknown Numbers* Find three numbers whose sum is 20, if the first number is three times the difference between the second and the third, and the second number is two more than twice the third.

54. *Car Sales Quota* To meet a sales quota, a car salesperson must sell 24 new cars, consisting of small, medium, and large cars. She must sell 3 more small cars than medium cars, and the same number of medium cars as large cars. How many of each size must she sell?

55. *Mixing Acid Solutions* A chemist has two prepared acid solutions, one of which is 2% acid by volume, the other 7% acid. How many cubic centimeters of each should the chemist mix together to obtain 40 cm^3 of a 3.2% acid solution?

56. *Financing an Expansion* To get the necessary funds for a planned expansion, a small company took out three loans totaling \$25,000. The company was able to borrow some of the money at 4% interest. It borrowed \$2000 more than one-half the amount of the 4% loan at 6%, and the rest at 5%. The total annual interest was \$1220. How much did the company borrow at each rate?

57. *Financing an Expansion* In **Exercise 56**, suppose we drop the condition that the amount borrowed at 6% is \$2000 more than one-half the amount borrowed at 4%. How is the solution changed?

58. *Financing an Expansion* Suppose the company in **Exercise 56** can borrow only \$6000 at 5%. Is a solution possible that still meets the given conditions? Explain.

59. *Planning a Diet* In a special diet for a hospital patient, the total amount per meal of food groups A, B, and C must equal 400 g. The diet should include one-third as much of group A as of group B. The sum of the amounts of group A and group C should equal twice the amount of group B. How many grams of each food group should be included? (Give answers to the nearest tenth.)

60. *Planning a Diet* In **Exercise 59**, suppose that, in addition to the conditions given there, foods A and B cost \$0.02 per gram, food C costs \$0.03 per gram, and a meal must cost \$8. Is a solution possible? Explain.

(Modeling) Age Distribution in the United States As people live longer, a larger percentage of the U.S. population is 65 or over and a smaller percentage is in younger age brackets. Use matrices to solve the problems in Exercises 61 and 62. Let $x = 0$ represent 2010 and $x = 40$ represent 2050. Express percents in decimal form.

61. In 2010, 13.0% of the population was 65 or older. By 2050, this percentage is expected to be 20.2%. The percentage of the population aged 25–34 in 2010 was 13.5%. That age group is expected to include 12.8% of the population in 2050. (*Source:* U.S. Census Bureau.)

(a) Assuming these population changes are linear, use the data for the 65-or-older age group to write a linear equation. Then do the same for the 25–34 age group.

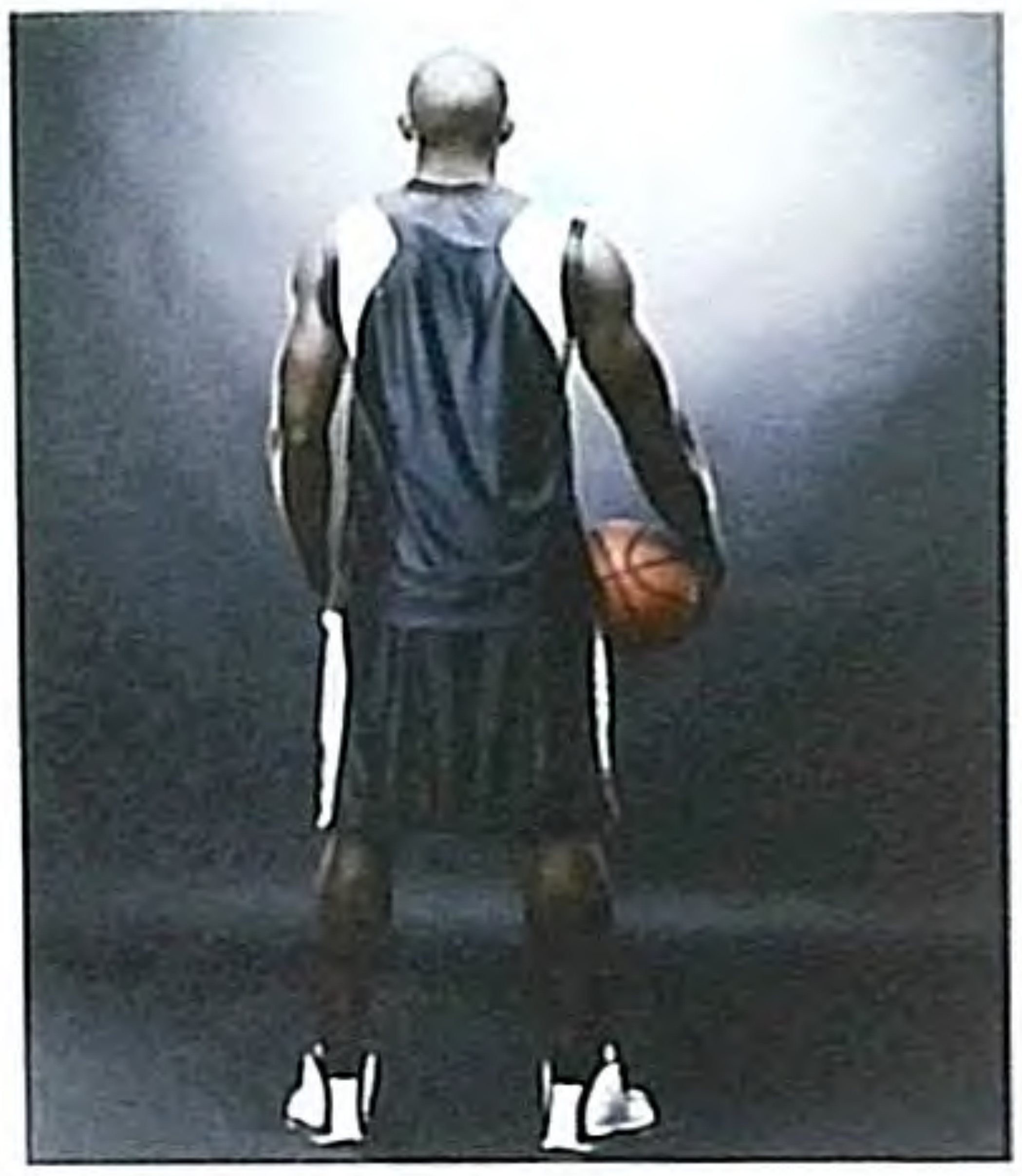
(b) Solve the system of linear equations from part (a). In what year will the two age groups include the same percentage of the population? What is that percentage?

(c) Does your answer to part (b) suggest that the *number* of people in the U.S. population aged 25–34 is decreasing? Explain.

62. In 2010, 13.3% of the U.S. population was aged 35–44. This percentage is expected to decrease to 12.5% in 2050. (*Source:* U.S. Census Bureau.)

(a) Write a linear equation representing this population change.

(b) Solve the system containing the equation from part (a) and the equation from **Exercise 61** for the 65-or-older age group. Give the year and percentage when these two age groups will include the same portion of the population.



63. *(Modeling) Athlete's Weight and Height* The relationship between a professional basketball player's height H (in inches) and weight W (in pounds) was modeled using two different samples of players. The resulting equations that modeled the two samples were

$$W = 7.46H - 374$$

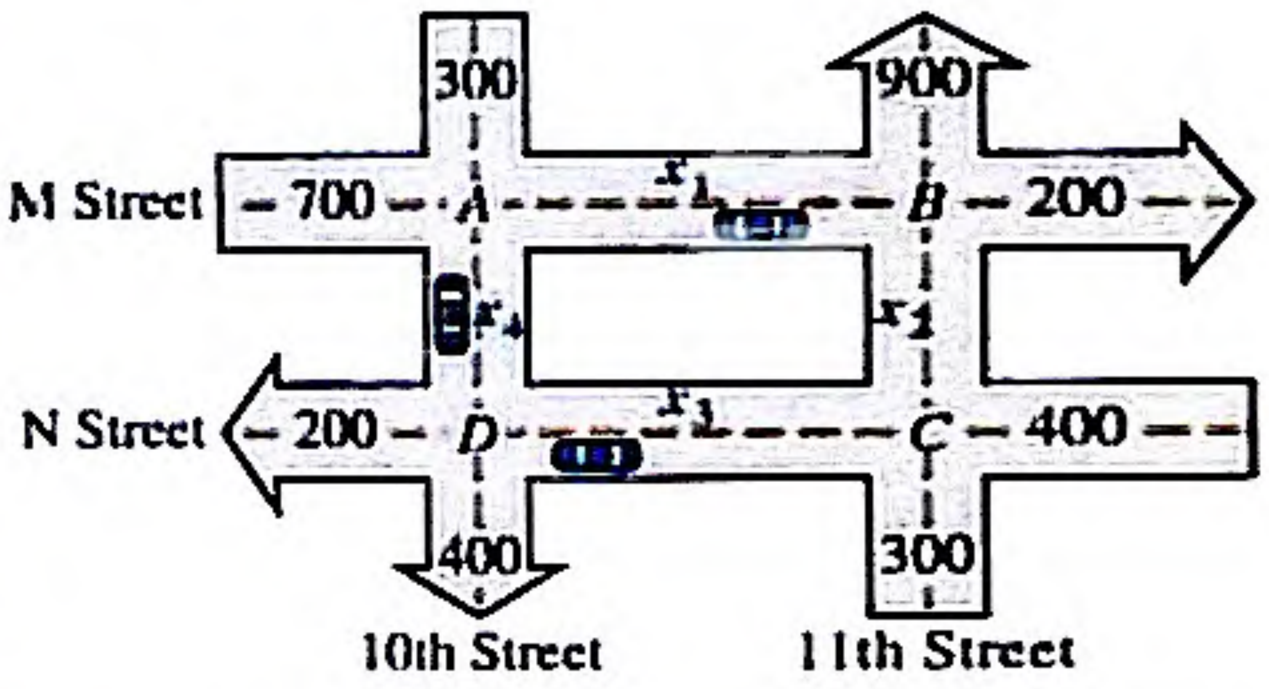
and

$$W = 7.93H - 405.$$

(a) Use each equation to predict the weight of a 6 ft 11 in. professional basketball player.

(b) According to each model, what change in weight is associated with a 1-in. increase in height?

(c) Determine the weight and height where the two models agree.



64. *(Modeling) Traffic Congestion* At rush hours, substantial traffic congestion is encountered at the traffic intersections shown in the figure. (All streets are one-way.) The city wishes to improve the signals at these corners to speed the flow of traffic. The traffic engineers first gather data. As the figure shows, 700 cars per hour come down M Street to intersection A, and 300 cars per hour come to intersection A on 10th Street. A total of x_1 of these cars leave A on M Street, while x_4 cars leave A on 10th Street. The number of cars entering A must equal the number leaving, which suggests the following equation.

$$x_1 + x_4 = 700 + 300$$
$$x_1 + x_4 = 1000$$

For intersection B, x_1 cars enter B on M Street, and x_2 cars enter B on 11th Street. As the figure shows, 900 cars leave B on 11th, while 200 leave on M, which leads to the following equation.

$$x_1 + x_2 = 900 + 200$$
$$x_1 + x_2 = 1100$$

At intersection C, 400 cars enter on N Street and 300 on 11th Street, while x_2 leave on 11th Street and x_3 leave on N Street.

$$x_2 + x_3 = 400 + 300$$
$$x_2 + x_3 = 700$$

Finally, intersection D has x_3 cars entering on N and x_4 entering on 10th. There are 400 cars leaving D on 10th and 200 leaving on N.

(a) Set up an equation for intersection D.

(b) Use the four equations to write an augmented matrix, and then transform it so that 1s are on the diagonal and 0s are below. This is **triangular form.**

(c) Since you got a row of all 0s, the system of equations does not have a unique solution. Write three equations, corresponding to the three nonzero rows of the matrix. Solve each of the equations for x_4.

(d) One of your equations should have been $x_4 = 1000 - x_1$. What is the greatest possible value of x_1 so that x_4 is not negative?

(e) Another equation should have been $x_4 = x_2 - 100$. Find the least possible value of x_2 so that x_4 is not negative.

(f) Find the greatest possible values of x_3 and x_4 so that neither variable is negative.

(g) Use the results of parts (a)–(f) to give a solution for the problem in which all the equations are satisfied and all variables are nonnegative. Is the solution unique?

Relating Concepts

For individual or collaborative investigation *(Exercises 65–68)*

(Modeling) Number of Fawns To model the spring fawn count F from the adult pronghorn population A, the precipitation P, and the severity of the winter W, environmentalists have used the equation

$$F = a + bA + cP + dW,$$

where the coefficients a, b, c, and d are constants that must be determined before using the equation. (Winter severity is scaled between 1 and 5, with 1 being mild and 5 being severe.) ***Work Exercises 65–68 in order.*** (*Source:* Brase, C. and C. Brase, *Understandable Statistics,* D.C. Heath and Company; Bureau of Land Management.)

65. Substitute the values for F, A, P, and W from the table for Years 1–4 into the equation

$$F = a + bA + cP + dW$$

and obtain four linear equations involving a, b, c, and d.

Year	Fawns	Adults	Precip. (in inches)	Winter Severity
1	239	871	11.5	3
2	234	847	12.2	2
3	192	685	10.6	5
4	343	969	14.2	1
5	?	960	12.6	3

66. Write an augmented matrix representing the system in **Exercise 65,** and solve for a, b, c, and d.

67. Write the equation for F using the values found in **Exercise 66** for the coefficients.

68. Use the information in the table to predict the spring fawn count in Year 5. (Compare this with the actual count of 320.)

(Modeling) Number of Calculations *When computers are programmed to solve large linear systems involved in applications like designing aircraft or electrical circuits, they frequently use an algorithm that is similar to the Gauss-Jordan method presented in this section. Solving a linear system with n equations and n variables requires the computer to perform a total of*

$$T(n) = \frac{2}{3}n^3 + \frac{3}{2}n^2 - \frac{7}{6}n$$

arithmetic calculations, including additions, subtractions, multiplications, and divisions. Use this model to solve the problems in Exercises 69–72. (*Source:* Burden, R. and J. Faires, *Numerical Analysis,* Sixth Edition, Brooks/Cole Publishing Company.)

69. Compute T for the following values of n.

$$n = 3, 6, 10, 29, 100, 200, 400, 1000, 5000, 10{,}000, 100{,}000$$

Write the results in a table.

70. In 1940, John Atanasoff, a physicist from Iowa State University, wanted to solve a 29×29 linear system of equations. How many arithmetic operations would this have required? Is this too many to do by hand? (Atanasoff's work led to the invention of the first fully electronic digital computer.) (*Source: The Gazette.*)

71. If the number of equations and variables is doubled, does the number of arithmetic operations double?

Atanasoff-Berry Computer

72. A Cray-T90 supercomputer can execute up to 60 billion arithmetic operations per second. How many hours would be required to solve a linear system with 100,000 variables?

5.3 Determinant Solution of Linear Systems

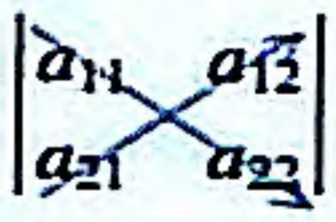
- Determinants
- Cofactors
- Evaluating $n \times n$ Determinants
- Determinant Theorems
- Cramer's Rule

Determinants Every $n \times n$ matrix A is associated with a real number called **the determinant** of A, written $|A|$. The determinant of a 2×2 matrix is defined as follows.

Determinant of a 2 × 2 Matrix

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$.

NOTE ***Matrices are enclosed with square brackets, while determinants are denoted with vertical bars.*** A matrix is an *array* of numbers, but its determinant is a *single* number.



The arrows in the diagram in the margin will remind you which products to find when evaluating a 2×2 determinant.

EXAMPLE 1 **Evaluating a 2 × 2 Determinant**

Let $A = \begin{bmatrix} -3 & 4 \\ 6 & 8 \end{bmatrix}$. Find $|A|$.

ALGEBRAIC SOLUTION

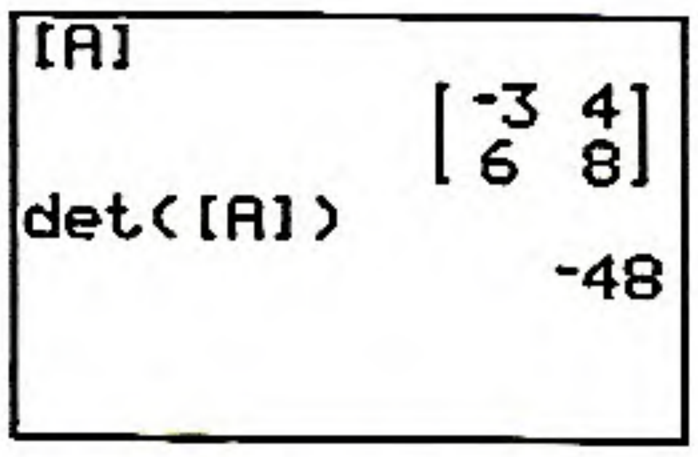
Use the definition with

$$a_{11} = -3, \quad a_{12} = 4, \quad a_{21} = 6, \quad a_{22} = 8.$$

$$\begin{aligned} |A| &= \underset{\substack{\uparrow \\ a_{11}}}{-3} \cdot \underset{\substack{\uparrow \\ a_{22}}}{8} - \underset{\substack{\uparrow \\ a_{21}}}{6} \cdot \underset{\substack{\uparrow \\ a_{12}}}{4} \\ &= -24 - 24 \\ &= -48 \end{aligned}$$

GRAPHING CALCULATOR SOLUTION

We can define a matrix and then use the capability of a graphing calculator to find the determinant of the matrix. In the screen in **Figure 8**, the symbol det([A]) represents the determinant of [A].



```
[A]
                [-3 4]
                [ 6 8]
det([A])
                   -48
```

Figure 8

✔ ***Now Try Exercise 1.***

LOOKING AHEAD TO CALCULUS

Determinants are used in calculus to find vector cross products, which are used to study the effect of forces in the plane or in space.

Determinant of a 3 × 3 Matrix

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then the determinant of A, symbolized $|A|$, is defined as follows.

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12})$$

The terms on the right side of the equation in the definition of $|A|$ can be rearranged to get the following.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}(a_{12}a_{33} - a_{32}a_{13}) + a_{31}(a_{12}a_{23} - a_{22}a_{13})$$

Each quantity in parentheses represents the determinant of a 2 × 2 matrix that is the part of the 3 × 3 matrix remaining when the row and column of the multiplier are eliminated, as shown below.

$$a_{11}(a_{22}a_{33} - a_{32}a_{23}) \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$a_{21}(a_{12}a_{33} - a_{32}a_{13}) \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$a_{31}(a_{12}a_{23} - a_{22}a_{13}) \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Cofactors The determinant of each 2 × 2 matrix above is the **minor** of the associated element in the 3 × 3 matrix. The symbol M_{ij} represents the minor that results when row i and column j are eliminated. The following list gives some of the minors from the matrix above.

Element	Minor	Element	Minor
a_{11}	$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$	a_{22}	$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$
a_{21}	$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$	a_{23}	$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$
a_{31}	$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$	a_{33}	$M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

5.3 Exercises

Find the value of each determinant. See Example 1.

1. $\begin{vmatrix} -5 & 9 \\ 4 & -1 \end{vmatrix}$

2. $\begin{vmatrix} -1 & 3 \\ -2 & 9 \end{vmatrix}$

3. $\begin{vmatrix} -1 & -2 \\ 5 & 3 \end{vmatrix}$

4. $\begin{vmatrix} 6 & -4 \\ 0 & -1 \end{vmatrix}$

5. $\begin{vmatrix} 9 & 3 \\ -3 & -1 \end{vmatrix}$

6. $\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix}$

7. $\begin{vmatrix} 3 & 4 \\ 5 & -2 \end{vmatrix}$

8. $\begin{vmatrix} -9 & 7 \\ 2 & 6 \end{vmatrix}$

9. $\begin{vmatrix} -7 & 0 \\ 3 & 0 \end{vmatrix}$

10. *Concept Check* Refer to Exercise 5. Make a conjecture about the value of the determinant of a matrix in which one row is a multiple of another row.

Find the cofactor of each element in the second row for each determinant. See Example 2.

11. $\begin{vmatrix} -2 & 0 & 1 \\ 1 & 2 & 0 \\ 4 & 2 & 1 \end{vmatrix}$

12. $\begin{vmatrix} 1 & -1 & 2 \\ 1 & 0 & 2 \\ 0 & -3 & 1 \end{vmatrix}$

13. $\begin{vmatrix} 1 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & 4 & 1 \end{vmatrix}$

14. $\begin{vmatrix} 2 & -1 & 4 \\ 3 & 0 & 1 \\ -2 & 1 & 4 \end{vmatrix}$

Find the value of each determinant. See Example 3.

15. $\begin{vmatrix} 4 & -7 & 8 \\ 2 & 1 & 3 \\ -6 & 3 & 0 \end{vmatrix}$

16. $\begin{vmatrix} 8 & -2 & -4 \\ 7 & 0 & 3 \\ 5 & -1 & 2 \end{vmatrix}$

17. $\begin{vmatrix} 1 & 2 & 0 \\ -1 & 2 & -1 \\ 0 & 1 & 4 \end{vmatrix}$

18. $\begin{vmatrix} 2 & 1 & -1 \\ 4 & 7 & -2 \\ 2 & 4 & 0 \end{vmatrix}$

19. $\begin{vmatrix} 10 & 2 & 1 \\ -1 & 4 & 3 \\ -3 & 8 & 10 \end{vmatrix}$

20. $\begin{vmatrix} 7 & -1 & 1 \\ 1 & -7 & 2 \\ -2 & 1 & 1 \end{vmatrix}$

21. $\begin{vmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 1 & 10 & -12 \end{vmatrix}$

22. $\begin{vmatrix} 2 & 3 & 0 \\ 1 & 9 & 0 \\ -1 & -2 & 0 \end{vmatrix}$

23. $\begin{vmatrix} 3 & 3 & -1 \\ 2 & 6 & 0 \\ -6 & -6 & 2 \end{vmatrix}$

24. $\begin{vmatrix} 5 & -3 & 2 \\ -5 & 3 & -2 \\ 1 & 0 & 1 \end{vmatrix}$

25. $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

26. $\begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$

27. $\begin{vmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$

28. $\begin{vmatrix} 0 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix}$

29. $\begin{vmatrix} \sqrt{2} & 4 & 0 \\ 1 & -\sqrt{5} & 7 \\ -5 & \sqrt{5} & 1 \end{vmatrix}$

30. $\begin{vmatrix} \sqrt{3} & 1 & 0 \\ \sqrt{7} & 4 & -1 \\ 5 & 0 & -\sqrt{7} \end{vmatrix}$

31. $\begin{vmatrix} 0.4 & -0.8 & 0.6 \\ 0.3 & 0.9 & 0.7 \\ 3.1 & 4.1 & -2.8 \end{vmatrix}$

32. $\begin{vmatrix} -0.3 & -0.1 & 0.9 \\ 2.5 & 4.9 & -3.2 \\ -0.1 & 0.4 & 0.8 \end{vmatrix}$

Relating Concepts

For individual or collaborative investigation *(Exercises 33–36)*

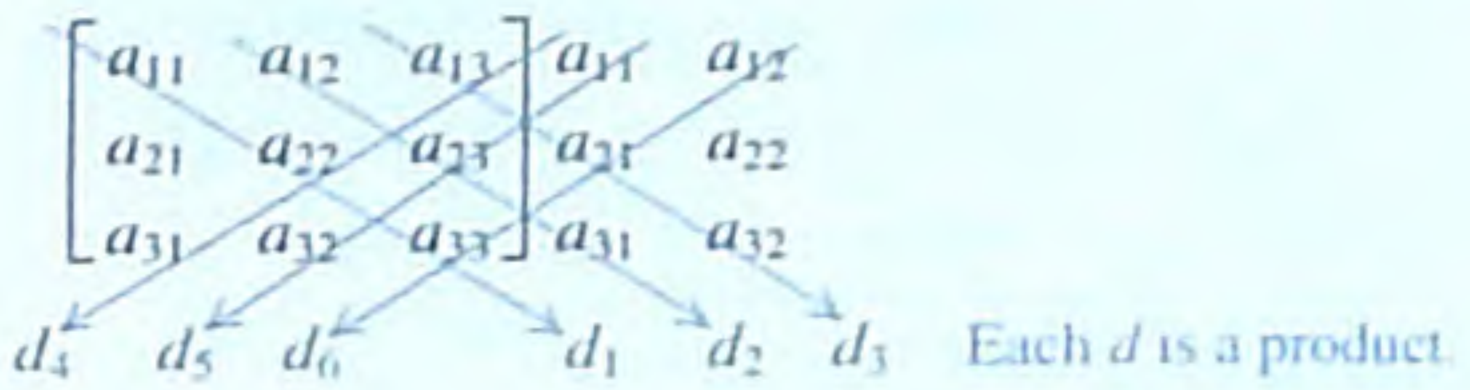
The determinant of a 3×3 *matrix A is defined as follows.*

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ then } |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12}).$$

Work these exercises in order.

33. The determinant of a 3×3 matrix can also be found using the method of "diagonals." To use this method, first rewrite columns 1 and 2 of matrix A to the right of matrix A. Next, identify the diagonals d_1 through d_6 and multiply their elements. Find the sum of the products from d_1, d_2, and d_3. Then subtract the sum of the products from d_4, d_5, and d_6 from that sum: $(d_1 + d_2 + d_3) - (d_4 + d_5 + d_6)$. Verify that this method produces the same results as the method given above.


a_{11} a_{12} a_{13} a_{11} a_{12}
a_{21} a_{22} a_{23} a_{21} a_{22}
a_{31} a_{32} a_{33} a_{31} a_{32}
d_4 d_5 d_6 d_1 d_2 d_3
Each d is a product.

34. Find the determinant $\begin{vmatrix} 1 & 3 & 2 \\ 0 & 2 & 6 \\ 7 & 1 & 5 \end{vmatrix}$ using the method of "diagonals."

35. Refer to Exercise 34. Find the determinant by expanding about column 1 and using the method of cofactors. Do these methods give the same determinant for 3×3 matrices?

36. *Concept Check* Does the method of finding a determinant using "diagonals" extend to 4×4 matrices?

Solve each equation for x.

37. $\begin{vmatrix} 5 & x \\ -3 & 2 \end{vmatrix} = 6$

38. $\begin{vmatrix} -0.5 & 2 \\ x & x \end{vmatrix} = 0$

39. $\begin{vmatrix} x & 3 \\ x & x \end{vmatrix} = 4$

40. $\begin{vmatrix} 2x & x \\ 11 & x \end{vmatrix} = 6$

41. $\begin{vmatrix} -2 & 0 & 1 \\ -1 & 3 & x \\ 5 & -2 & 0 \end{vmatrix} = 3$

42. $\begin{vmatrix} 4 & 3 & 0 \\ 2 & 0 & 1 \\ -3 & x & -1 \end{vmatrix} = 5$

43. $\begin{vmatrix} 5 & 3x & -3 \\ 0 & 2 & -1 \\ 4 & -1 & x \end{vmatrix} = -7$

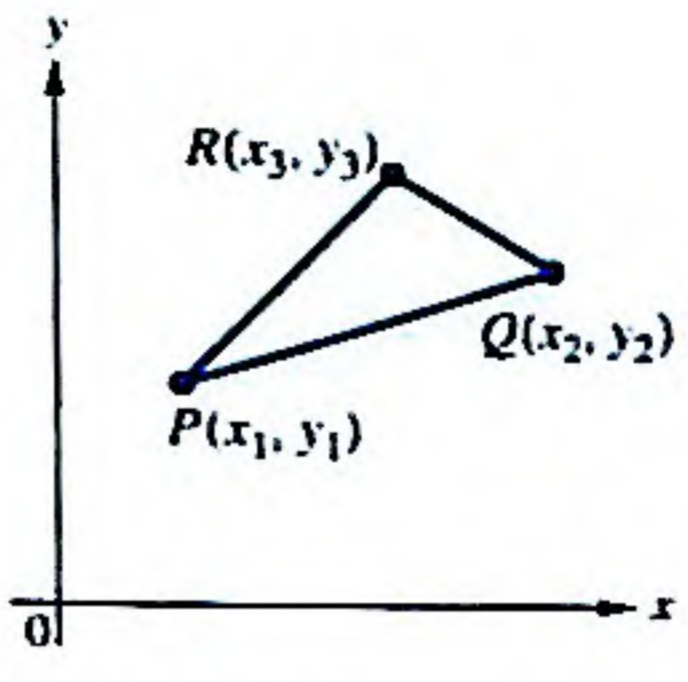
44. $\begin{vmatrix} 2x & 1 & -1 \\ 0 & 4 & x \\ 3 & 0 & 2 \end{vmatrix} = x$

45. $\begin{vmatrix} x & 0 & -1 \\ 2 & -3 & x \\ x & 0 & 7 \end{vmatrix} = 12$

46. ***Concept Check*** **Write the sign array representing $(-1)^{i+j}$ for each element of a 4×4 matrix.**

Area of a Triangle *A triangle with vertices at (x_1, y_1), (x_2, y_2), and (x_3, y_3), as shown in the figure, has area equal to the absolute value of D, where*

$$D = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$



Find the area of each triangle having vertices at P, Q, and R.

47. $P(0, 0)$, $Q(0, 2)$, $R(1, 4)$

48. $P(0, 1)$, $Q(2, 0)$, $R(1, 5)$

49. $P(2, 5)$, $Q(-1, 3)$, $R(4, 0)$

50. $P(2, -2)$, $Q(0, 0)$, $R(-3, -4)$

51. *Area of a Triangle* Find the area of a triangular lot whose vertices have coordinates in feet of

(101.3, 52.7), (117.2, 253.9), and (313.1, 301.6)

(*Source:* Al-Khafaji, A. and J. Tooley, *Numerical Methods in Engineering Practice*, Holt, Rinehart, and Winston.)

52. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Find $|A|$ by expansion about row 3 of the matrix. Show that your result is really equal to $|A|$ as given in the definition of the determinant of a 3×3 matrix at the beginning of this section.

Use the determinant theorems and the fact that $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 10 \end{vmatrix} = 3$ *to find the value of each determinant.*

53. $\begin{vmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 9 & 10 \end{vmatrix}$

54. $\begin{vmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \\ 10 & 9 & 7 \end{vmatrix}$

55. $\begin{vmatrix} 5 & 10 & 15 \\ 4 & 5 & 6 \\ 7 & 9 & 10 \end{vmatrix}$

56. $\begin{vmatrix} 1 & 20 & 3 \\ 4 & 50 & 6 \\ 7 & 90 & 10 \end{vmatrix}$

57. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 11 & 13 \end{vmatrix}$

58. $\begin{vmatrix} 1 & 2 & 0 \\ 4 & 5 & -6 \\ 7 & 9 & -11 \end{vmatrix}$

Use the determinant theorems to find the value of each determinant. See ***Example 4.***

59. $\begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 3 & 0 & 0 \end{vmatrix}$

60. $\begin{vmatrix} -1 & 2 & 4 \\ 4 & -8 & -16 \\ 0 & 0 & 0 \end{vmatrix}$

61. $\begin{vmatrix} 6 & 8 & -12 \\ -1 & 0 & 2 \\ 4 & 0 & -8 \end{vmatrix}$

62. $\begin{vmatrix} 4 & 8 & 0 \\ -1 & -2 & 1 \\ 2 & 4 & 3 \end{vmatrix}$

63. $\begin{vmatrix} -4 & 1 & 4 \\ 2 & 0 & 1 \\ 0 & 2 & 4 \end{vmatrix}$

64. $\begin{vmatrix} 6 & 3 & 2 \\ 1 & 0 & 2 \\ 5 & 7 & 3 \end{vmatrix}$

65. $\begin{vmatrix} 0 & 1 & -3 \\ 7 & 5 & 2 \\ 1 & -2 & 6 \end{vmatrix}$

66. $\begin{vmatrix} 7 & 9 & -3 \\ 7 & -6 & 2 \\ 8 & 1 & 0 \end{vmatrix}$

67. $\begin{vmatrix} 1 & 6 & 7 \\ 0 & 6 & 7 \\ 0 & 0 & 9 \end{vmatrix}$

68. $\begin{vmatrix} 7 & 0 & 0 \\ 1 & 6 & 0 \\ 4 & 2 & 4 \end{vmatrix}$

69. $\begin{vmatrix} 2 & -1 & 3 \\ 6 & 4 & 10 \\ 4 & 5 & 7 \end{vmatrix}$

70. $\begin{vmatrix} 9 & 1 & 7 \\ 12 & 5 & 2 \\ 11 & 4 & 3 \end{vmatrix}$

71. $\begin{vmatrix} -1 & 0 & 2 & 3 \\ 5 & 4 & -3 & 7 \\ 8 & 2 & 9 & -5 \\ 4 & 4 & -1 & 10 \end{vmatrix}$

72. $\begin{vmatrix} 5 & 1 & 4 & 2 \\ 4 & -3 & 7 & -4 \\ 5 & 8 & -3 & 6 \\ 9 & 9 & 0 & 8 \end{vmatrix}$

73. $\begin{vmatrix} 4 & 0 & 0 & 2 \\ -1 & 0 & 3 & 0 \\ 2 & 4 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix}$

74. $\begin{vmatrix} -2 & 0 & 4 & 2 \\ 3 & 6 & 0 & 4 \\ 0 & 0 & 0 & 3 \\ 9 & 0 & 2 & -1 \end{vmatrix}$

75. $\begin{vmatrix} 3 & -6 & 5 & -1 \\ 0 & 2 & -1 & 3 \\ -6 & 4 & 2 & 0 \\ -7 & 3 & 1 & 1 \end{vmatrix}$

76. $\begin{vmatrix} 4 & 5 & -1 & -1 \\ 2 & -3 & 1 & 0 \\ -5 & 1 & 3 & 9 \\ 0 & -2 & 1 & 5 \end{vmatrix}$

77. *Concept Check* For the system below, match each determinant in (a)–(d) with its equivalent from choices A–D.

$$\begin{aligned} 4x + 3y - 2z &= 1 \\ 7x - 4y + 3z &= 2 \\ -2x + y - 8z &= 0 \end{aligned}$$

(a) D (b) D_x (c) D_y (d) D_z

A. $\begin{vmatrix} 1 & 3 & -2 \\ 2 & -4 & 3 \\ 0 & 1 & -8 \end{vmatrix}$

B. $\begin{vmatrix} 4 & 3 & 1 \\ 7 & -4 & 2 \\ -2 & 1 & 0 \end{vmatrix}$

C. $\begin{vmatrix} 4 & 1 & -2 \\ 7 & 2 & 3 \\ -2 & 0 & -8 \end{vmatrix}$

D. $\begin{vmatrix} 4 & 3 & -2 \\ 7 & -4 & 3 \\ -2 & 1 & -8 \end{vmatrix}$

78. *Concept Check* For the following system, $D = -43$, $D_x = -43$, $D_y = 0$, and $D_z = 43$. What is the solution set of the system?

$$\begin{aligned} x + 3y - 6z &= 7 \\ 2x - y + z &= 1 \\ x + 2y + 2z &= -1 \end{aligned}$$

Use Cramer's rule to solve each system of equations. If $D = 0$, use another method to determine the solution set. See Examples 5–7.

79. $\begin{aligned} x + y &= 4 \\ 2x - y &= 2 \end{aligned}$

80. $\begin{aligned} 3x + 2y &= -4 \\ 2x - y &= -5 \end{aligned}$

81. $\begin{aligned} 4x + 3y &= -7 \\ 2x + 3y &= -11 \end{aligned}$

82. $\begin{aligned} 4x - y &= 0 \\ 2x + 3y &= 14 \end{aligned}$

83. $\begin{aligned} 5x + 4y &= 10 \\ 3x - 7y &= 6 \end{aligned}$

84. $\begin{aligned} 3x + 2y &= -4 \\ 5x - y &= 2 \end{aligned}$

85. $\begin{aligned} 1.5x + 3y &= 5 \\ 2x + 4y &= 3 \end{aligned}$

86. $\begin{aligned} 12x + 8y &= 3 \\ 1.5x + y &= 0.9 \end{aligned}$

87. $\begin{aligned} 3x + 2y &= 4 \\ 6x + 4y &= 8 \end{aligned}$

88. $\begin{aligned} 4x + 3y &= 9 \\ 12x + 9y &= 27 \end{aligned}$

89. $\begin{aligned} \frac{1}{2}x + \frac{1}{3}y &= 2 \\ \frac{3}{2}x - \frac{1}{2}y &= -12 \end{aligned}$

90. $\begin{aligned} -\frac{3}{4}x + \frac{2}{3}y &= 16 \\ \frac{5}{2}x + \frac{1}{2}y &= -37 \end{aligned}$

91. $\begin{aligned} 2x - y + 4z &= -2 \\ 3x + 2y - z &= -3 \\ x + 4y + 2z &= 17 \end{aligned}$

92. $\begin{aligned} x + y + z &= 4 \\ 2x - y + 3z &= 4 \\ 4x + 2y - z &= -15 \end{aligned}$

93. $\begin{aligned} 4x - 3y + z + 1 &= 0 \\ 5x + 7y + 2z + 2 &= 0 \\ 3x - 5y - z - 1 &= 0 \end{aligned}$

94. $\begin{aligned} 2x - 3y + z - 8 &= 0 \\ -x - 5y + z + 4 &= 0 \\ 3x - 5y + 2z - 12 &= 0 \end{aligned}$

95. $\begin{aligned} x + 2y + 3z &= 4 \\ 4x + 3y + 2z &= 1 \\ -x - 2y - 3z &= 0 \end{aligned}$

96. $\begin{aligned} 2x - y + 3z &= 1 \\ -2x + y - 3z &= 2 \\ 5x - y + z &= 2 \end{aligned}$

97. $\begin{aligned} -2x - 2y + 3z &= 4 \\ 5x + 7y - z &= 2 \\ 2x + 2y - 3z &= -4 \end{aligned}$

98. $\begin{aligned} 3x - 2y + 4z &= 1 \\ 4x + y - 5z &= 2 \\ -6x + 4y - 8z &= -2 \end{aligned}$

99. $\begin{aligned} 5x - y &= -4 \\ 3x + 2z &= 4 \\ 4y + 3z &= 22 \end{aligned}$

100. $\begin{aligned} 3x + 5y &= -7 \\ 2x + 7z &= 2 \\ 4y + 3z &= -8 \end{aligned}$

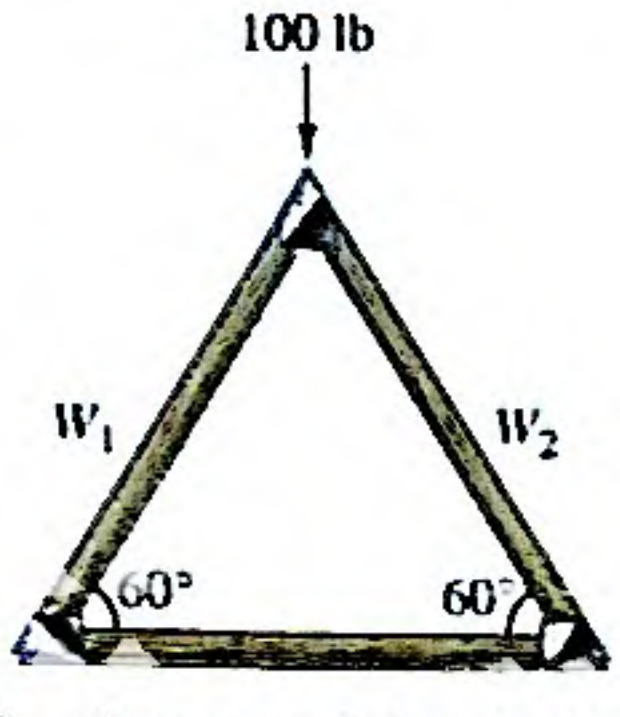
101. $\begin{aligned} x + 2y &= 10 \\ 3x + 4z &= 7 \\ -y - z &= 1 \end{aligned}$

102. $\begin{aligned} 5x - 2y &= 3 \\ 4y + z &= 8 \\ x + 2z &= 4 \end{aligned}$

(Modeling) Solve the systems in Exercises 103 and 104.

103. ***Roof Trusses*** The simplest type of roof truss is a triangle. The truss shown in the figure is used to frame roofs of small buildings. If a 100-pound force is applied at the peak of the truss, then the forces or weights W_1 and W_2 exerted parallel to each rafter of the truss are determined by the following linear system of equations.

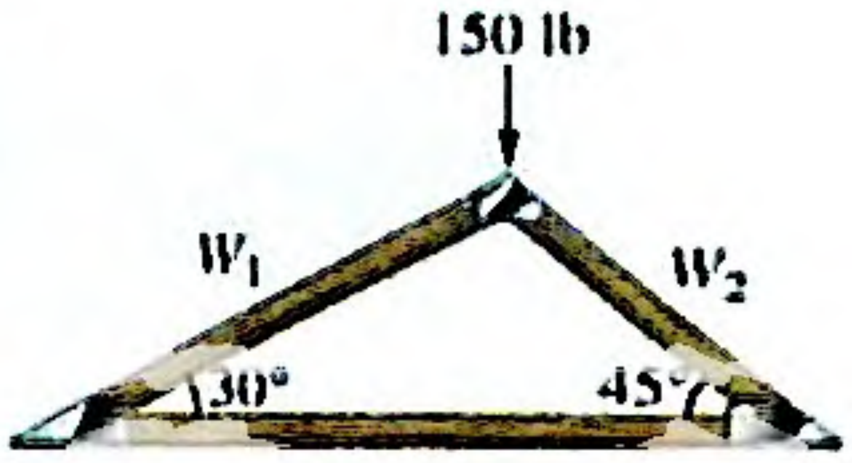
$$\begin{aligned} \frac{\sqrt{3}}{2}(W_1 + W_2) &= 100 \\ W_1 - W_2 &= 0 \end{aligned}$$



Solve the system to find W_1 and W_2. (*Source:* Hibbeler, R., *Structural Analysis*, Fourth Edition. Reprinted by permission of Pearson Education, Inc., Upper Saddle River, NJ.)

104. *Roof Trusses* **(Refer to Exercise 103.)** Use the following system of equations to determine the forces or weights W_1 and W_2 exerted on each rafter for the truss shown in the figure.

$$W_1 + \sqrt{2}W_2 = 300$$
$$\sqrt{3}W_1 - \sqrt{2}W_2 = 0$$



Solve each system for x and y using Cramer's rule. Assume a and b are nonzero constants.

105. $bx + y = a^2$
$ax + y = b^2$

106. $ax + by = \dfrac{b}{a}$
$x + y = \dfrac{1}{b}$

107. $b^2x + a^2y = b^2$
$ax + by = a$

108. $x + by = b$
$ax + y = a$

109. Use Cramer's Rule to find the solution set if $a, b, c, d, e,$ and f are consecutive integers.

$$ax + by = c$$
$$dx + ey = f$$

110. In the following system, $a, b, c, \ldots, l$ are consecutive integers. Express the solution set in terms of z.

$$ax + by + cz = d$$
$$ex + fy + gz = h$$
$$ix + jy + kz = l$$

5.4 Partial Fractions

- Decomposition of Rational Expressions
- Distinct Linear Factors
- Repeated Linear Factors
- Distinct Linear and Quadratic Factors
- Repeated Quadratic Factors

Decomposition of Rational Expressions The sums of rational expressions are found by combining two or more rational expressions into one rational expression. Here, the reverse process is considered:

> *Given one rational expression, express it as the sum of two or more rational expressions.*

A special type of sum involving rational expressions is a **partial fraction decomposition**—each term in the sum is a **partial fraction.** The technique of decomposing a rational expression into partial fractions is useful in calculus and other areas of mathematics.

Add rational expressions

$$\frac{2}{x+1} + \frac{3}{x} = \frac{5x+3}{x(x+1)}$$

Partial fraction decomposition

LOOKING AHEAD TO CALCULUS
In calculus, partial fraction decomposition provides a powerful technique for determining integrals of rational functions.

Partial Fraction Decomposition of $\frac{f(x)}{g(x)}$

To form a partial fraction decomposition of a rational expression, follow these steps.

Step 1 If $\frac{f(x)}{g(x)}$ is not a proper fraction (a fraction with the numerator of lesser degree than the denominator), divide $f(x)$ by $g(x)$. For example,

$$\frac{x^4 - 3x^3 + x^2 + 5x}{x^2 + 3} = x^2 - 3x - 2 + \frac{14x + 6}{x^2 + 3}.$$

Then apply the following steps to the remainder, which is a proper fraction.

Step 2 Factor the denominator $g(x)$ completely into factors of the form $(ax + b)^m$ or $(cx^2 + dx + e)^n$, where $cx^2 + dx + e$ is irreducible and m and n are positive integers.

Step 3 (a) For each distinct linear factor $(ax + b)$, the decomposition must include the term $\frac{A}{ax + b}$.

(b) For each repeated linear factor $(ax + b)^m$, the decomposition must include the terms

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_m}{(ax + b)^m}.$$

Step 4 (a) For each distinct quadratic factor $(cx^2 + dx + e)$, the decomposition must include the term $\frac{Bx + C}{cx^2 + dx + e}$.

(b) For each repeated quadratic factor $(cx^2 + dx + e)^n$, the decomposition must include the terms

$$\frac{B_1x + C_1}{cx^2 + dx + e} + \frac{B_2x + C_2}{(cx^2 + dx + e)^2} + \cdots + \frac{B_nx + C_n}{(cx^2 + dx + e)^n}.$$

Step 5 Use algebraic techniques to solve for the constants in the numerators of the decomposition.

To find the constants in Step 5, the goal is to get a system of equations with as many equations as there are unknowns in the numerators.

Distinct Linear Factors

EXAMPLE 1 **Finding a Partial Fraction Decomposition**

Find the partial fraction decomposition of $\frac{2x^4 - 8x^2 + 5x - 2}{x^3 - 4x}$.

SOLUTION The given fraction is not a proper fraction—the numerator has greater degree than the denominator. Perform the division.

$$\begin{array}{r} 2x \\ x^3 - 4x\overline{)2x^4 - 8x^2 + 5x - 2} \\ \underline{2x^4 - 8x^2 } \\ 5x - 2 \end{array} \quad \text{(Section R.3)}$$

The quotient is $$\frac{2x^4 - 8x^2 + 5x - 2}{x^3 - 4x} = 2x + \frac{5x - 2}{x^3 - 4x}.$$

For fractions with denominators that have quadratic factors, another method is often more convenient. The system of equations is formed by equating coefficients of like terms on each side of the partial fraction decomposition. For instance, in **Example 3**, equation (1) was

$$x^2 + 3x - 1 = A(x^2 + 2) + (Bx + C)(x + 1). \quad (1)$$

Multiply on the right and collect like terms.

$$x^2 + 3x - 1 = Ax^2 + 2A + Bx^2 + Bx + Cx + C$$

$$1x^2 + 3x - 1 = (A + B)x^2 + (B + C)x + (C + 2A)$$

Now, equate the coefficients of like powers of x to obtain three equations.

$$1 = A + B$$

$$3 = B + C$$

$$-1 = C + 2A$$

Solving this system for A, B, and C gives the partial fraction decomposition.

Repeated Quadratic Factors

EXAMPLE 4 Finding a Partial Fraction Decomposition

Find the partial fraction decomposition of $\dfrac{2x}{(x^2 + 1)^2(x - 1)}$.

SOLUTION This expression has both a linear factor and a repeated quadratic factor. Use Steps 3(a) and 4(b) from the beginning of this section.

$$\frac{2x}{(x^2 + 1)^2(x - 1)} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2} + \frac{E}{x - 1}$$

Multiply each side by $(x^2 + 1)^2(x - 1)$.

$$2x = (Ax + B)(x^2 + 1)(x - 1) + (Cx + D)(x - 1) + E(x^2 + 1)^2 \quad (1)$$

If $x = 1$, then equation (1) reduces to $2 = 4E$, or $E = \frac{1}{2}$. Substitute $\frac{1}{2}$ for E in equation (1), and expand and combine like terms on the right.

$$2x = \left(A + \frac{1}{2}\right)x^4 + (-A + B)x^3 + (A - B + C + 1)x^2$$

$$+ (-A + B + D - C)x + \left(-B - D + \frac{1}{2}\right) \quad (2)$$

To get additional equations involving the unknowns, equate the coefficients of like powers of x on the two sides of equation (2). Setting corresponding coefficients of x^4 equal, $0 = A + \frac{1}{2}$, or $A = -\frac{1}{2}$. From the corresponding coefficients of x^3, $0 = -A + B$. Since $A = -\frac{1}{2}$, $B = -\frac{1}{2}$.

Using the coefficients of x^2, $0 = A - B + C + 1$. Since $A = -\frac{1}{2}$ and $B = -\frac{1}{2}$, $C = -1$. Finally, from the coefficients of x, $2 = -A + B + D - C$. Substituting for A, B, and C gives $D = 1$. With $A = -\frac{1}{2}$, $B = -\frac{1}{2}$, $C = -1$, $D = 1$, and $E = \frac{1}{2}$, the given fraction has the following partial fraction decomposition.

$$\frac{2x}{(x^2 + 1)^2(x - 1)} = \frac{-\frac{1}{2}x - \frac{1}{2}}{x^2 + 1} + \frac{-x + 1}{(x^2 + 1)^2} + \frac{\frac{1}{2}}{x - 1},$$

or $$\frac{2x}{(x^2 + 1)^2(x - 1)} = \frac{-(x + 1)}{2(x^2 + 1)} + \frac{-x + 1}{(x^2 + 1)^2} + \frac{1}{2(x - 1)}$$

Simplify complex fractions (Section R.5)

✓ *Now Try Exercise 25.*

In summary, to solve for the constants in the numerators of a partial fraction decomposition, use either of the following methods or a combination of the two.

Techniques for Decomposition into Partial Fractions

Method 1 **For Linear Factors**

Step 1 Multiply each side of the resulting rational equation by the common denominator.

Step 2 Substitute the zero of each factor in the resulting equation. For repeated linear factors, substitute as many other numbers as necessary to find all the constants in the numerators. The number of substitutions required will equal the number of constants $A, B, \ldots$.

Method 2 **For Quadratic Factors**

Step 1 Multiply each side of the resulting rational equation by the common denominator.

Step 2 Collect like terms on the right side of the equation.

Step 3 Equate the coefficients of like terms to get a system of equations.

Step 4 Solve the system to find the constants in the numerators.

5.4 Exercises

Find the partial fraction decomposition for each rational expression. See Examples 1–4.

1. $\dfrac{5}{3x(2x+1)}$

2. $\dfrac{3x-1}{x(x+1)}$

3. $\dfrac{4x+2}{(x+2)(2x-1)}$

4. $\dfrac{x+2}{(x+1)(x-1)}$

5. $\dfrac{x}{x^2+4x-5}$

6. $\dfrac{5x-3}{x^2-2x-3}$

7. $\dfrac{4}{x(1-x)}$

8. $\dfrac{9}{x(x-3)}$

9. $\dfrac{4x^2-x-15}{x(x+1)(x-1)}$

10. $\dfrac{3}{(x+1)(x+3)}$

11. $\dfrac{2x+1}{(x+2)^3}$

12. $\dfrac{2x}{(x+1)(x+2)^2}$

13. $\dfrac{x^2}{x^2+2x+1}$

14. $\dfrac{2}{x^2(x+3)}$

15. $\dfrac{2x^5+3x^4-3x^3-2x^2+x}{2x^2+5x+2}$

16. $\dfrac{6x^5+7x^4-x^2+2x}{3x^2+2x-1}$

17. $\dfrac{x^3+4}{9x^3-4x}$

18. $\dfrac{x^3+2}{x^3-3x^2+2x}$

19. $\dfrac{-3}{x^2(x^2+5)}$

20. $\dfrac{1}{x^2(x^2-2)}$

21. $\dfrac{3x-2}{(x+4)(3x^2+1)}$

22. $\dfrac{2x+1}{(x+1)(x^2+2)}$

23. $\dfrac{1}{x(2x+1)(3x^2+4)}$

24. $\dfrac{3}{x(x+1)(x^2+1)}$

25. $\dfrac{3x-1}{x(2x^2+1)^2}$

26. $\dfrac{x^4+1}{x(x^2+1)^2}$

27. $\dfrac{-x^4-8x^2+3x-10}{(x+2)(x^2+4)^2}$

28. $\dfrac{3x^4+x^3+5x^2-x+4}{(x-1)(x^2+1)^2}$

29. $\dfrac{5x^5+10x^4-15x^3+4x^2+13x-9}{x^3+2x^2-3x}$

30. $\dfrac{3x^6+3x^4+3x}{x^4+x^2}$

31. $\dfrac{x^2}{x^4-1}$

32. $\dfrac{-2x^2-24}{x^4-16}$

33. $\dfrac{4x^2-3x-4}{x^3+x^2-2x}$

34. $\dfrac{2x+4}{x^3-2x^2}$

Chapter 5 Quiz (Sections 5.1–5.4)

Solve each system, using the method indicated, if possible.

1. (Substitution)
$$\begin{aligned}2x+y&=-4\\-x+2y&=2\end{aligned}$$

2. (Substitution)
$$\begin{aligned}5x+10y&=10\\x+2y&=2\end{aligned}$$

3. (Elimination)
$$\begin{aligned}x-y&=6\\x-y&=4\end{aligned}$$

4. (Elimination)
$$\begin{aligned}2x-3y&=18\\5x+2y&=7\end{aligned}$$

5. (Gauss-Jordan)
$$\begin{aligned}3x+5y&=-5\\-2x+3y&=16\end{aligned}$$

6. (Cramer's rule)
$$\begin{aligned}5x+2y&=-3\\4x-3y&=-30\end{aligned}$$

7. (Elimination)
$$\begin{aligned}x+y+z&=1\\-x+y+z&=5\\y+2z&=5\end{aligned}$$

8. (Gauss-Jordan)
$$\begin{aligned}2x+4y+4z&=4\\x+3y+z&=4\\-x+3y+2z&=-1\end{aligned}$$

9. (Cramer's rule)
$$\begin{aligned}7x+y-z&=4\\2x-3y+z&=2\\-6x+9y-3z&=-6\end{aligned}$$

Solve each problem.

10. ***Spending on Food*** In 2008, the amount spent by a typical American household on food was about $6450. For every $10 spent on food away from home, about $14 was spent on food at home. Find the amount of household spending on food in each category. (*Source:* U.S. Bureau of Labor Statistics.)

11. ***Investments*** A sum of $5000 is invested in three accounts that pay 3%, 4%, and 6% interest rates. The amount of money invested in the account paying 6% equals the total amount of money invested in the other two accounts, and the total annual interest from all three investments is $240. Find the amount invested at each rate

12. Let $A=\begin{bmatrix}-5 & 4\\ 2 & -1\end{bmatrix}$. Find $|A|$.

13. Evaluate $\begin{vmatrix}-1 & 2 & 4\\ -3 & -2 & -3\\ 2 & -1 & 5\end{vmatrix}$. Use determinant theorems if desired.

Find the partial fraction decomposition for each rational expression

14. $\dfrac{10x+13}{x^2-x-20}$

15. $\dfrac{2x^2-15x-32}{(x-1)(x^2+6x+8)}$

5.5 Nonlinear Systems of Equations

- **Solving Nonlinear Systems with Real Solutions**
- **Solving Nonlinear Systems with Nonreal Complex Solutions**
- **Applying Nonlinear Systems**

Solving Nonlinear Systems with Real Solutions

A system of equations in which at least one equation is *not* linear is a **nonlinear system.**

$$\begin{aligned} x^2 - y &= 4 \qquad & x^2 + y^2 &= 16 \\ x + y &= -2 \qquad & |x| + y &= 4 \end{aligned} \qquad \text{Nonlinear systems}$$

The substitution method works well for solving many such systems, particularly when one of the equations is linear, as in the next example.

EXAMPLE 1 Solving a Nonlinear System by Substitution

Solve the system.

$$\begin{aligned} x^2 - y &= 4 \quad (1) \\ x + y &= -2 \quad (2) \end{aligned}$$

ALGEBRAIC SOLUTION

When one of the equations in a nonlinear system is linear, it is usually best to begin by solving the linear equation for one of the variables.

$$y = -2 - x \qquad \text{Solve equation (2) for } y.$$

Substitute this result for y in equation (1).

$$\begin{aligned} x^2 - y &= 4 && (1) \\ x^2 - (-2 - x) &= 4 && \text{Let } y = -2 - x. \text{ (Section 5.1)} \\ x^2 + 2 + x &= 4 && \text{Distributive property (Section R.2)} \\ x^2 + x - 2 &= 0 && \text{Standard form (Section 1.4)} \\ (x + 2)(x - 1) &= 0 && \text{Factor. (Section R.4)} \\ x + 2 = 0 \quad \text{or} \quad x - 1 &= 0 && \text{Zero-factor property (Section 1.4)} \\ x = -2 \quad \text{or} \quad x &= 1 \end{aligned}$$

Substituting -2 for x in equation (2) gives $y = 0$. If $x = 1$, then $y = -3$. The solution set of the given system is $\{(-2, 0), (1, -3)\}$. A graph of the system is shown in **Figure 9.**

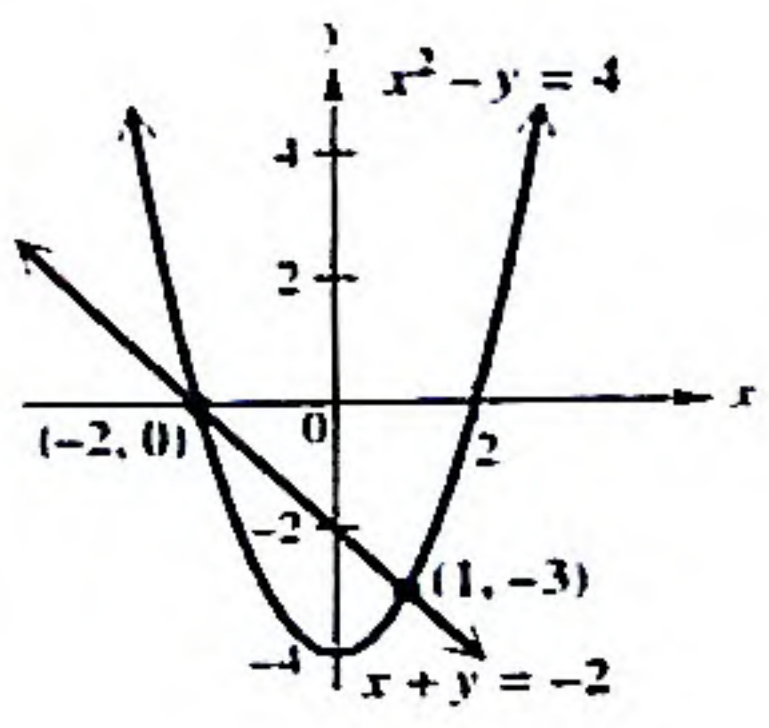


Figure 9

GRAPHING CALCULATOR SOLUTION

Solve each equation for y and graph them in the same viewing window. We obtain

$$Y_1 = X^2 - 4$$

and

$$Y_2 = -X - 2.$$

The screens in **Figure 10,** which indicate that the points of intersection are $(-2, 0)$ and $(1, -3)$, support the solution found algebraically.

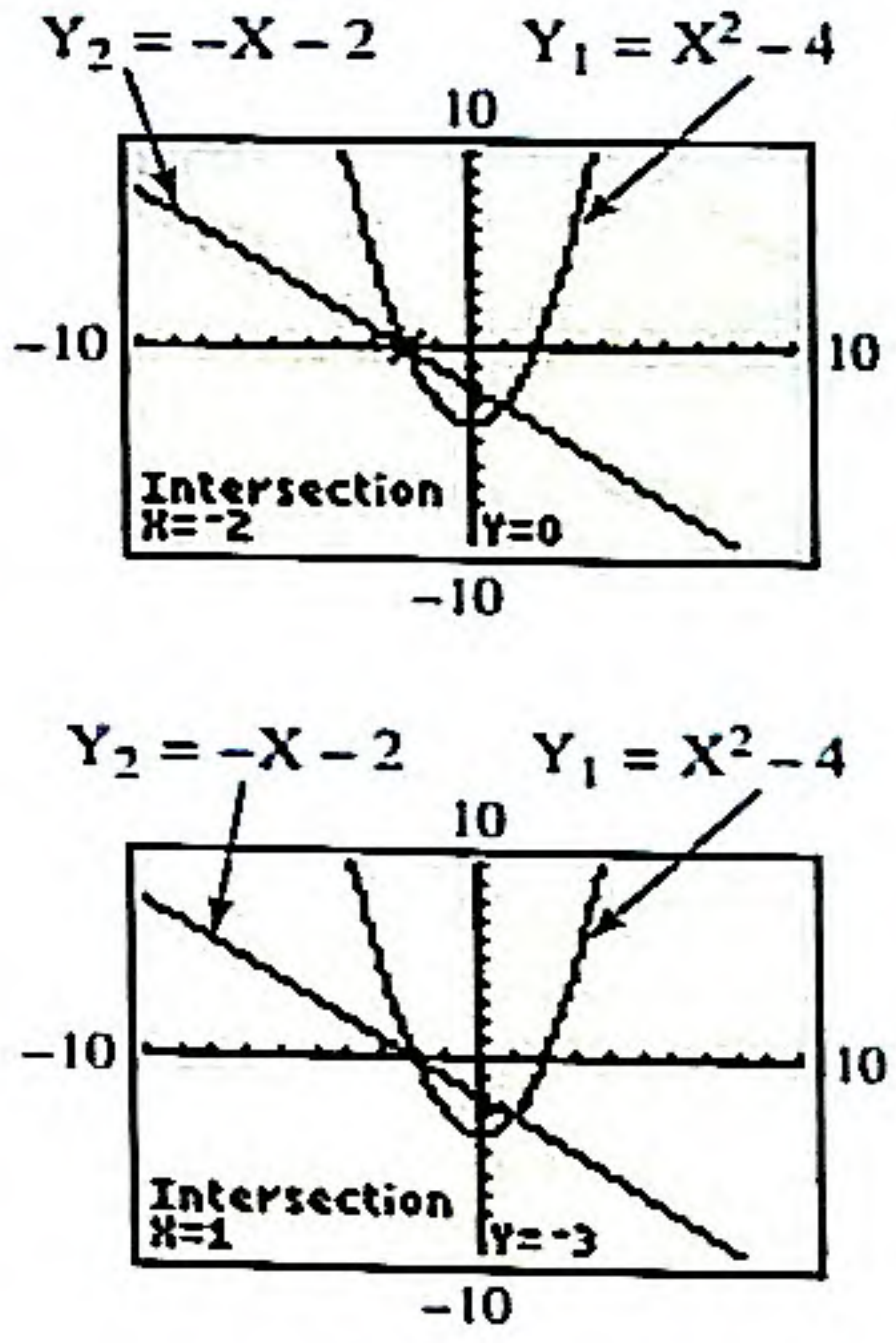


Figure 10

✔ *Now Try Exercise 9.*

The two equations form this system.

$$x^2y = 75 \quad (1)$$

$$x^2 + 4xy = 85 \quad (2)$$

Step 4 **Solve the system.** Solve equation (1) for y to get $y = \frac{75}{x^2}$, and substitute into equation (2).

$$x^2 + 4x\left(\frac{75}{x^2}\right) = 85 \qquad \text{Let } y = \tfrac{75}{x^2} \text{ in (2)}$$

$$x^2 + \frac{300}{x} = 85 \qquad \text{Multiply}$$

$$x^3 + 300 = 85x \qquad \text{Multiply by } x\text{, where } x \neq 0$$

$$x^3 - 85x + 300 = 0 \qquad \text{Subtract } 85x$$

$$\begin{array}{r|rrrr} 5) & 1 & 0 & -85 & 300 \\ & & 5 & 25 & -300 \\ \hline & 1 & 5 & -60 & 0 \end{array}$$

Coefficients of a quadratic polynomial factor
(Section 3.2)

We are restricted to positive values for x, and considering the nature of the problem, any solution should be relatively small. By the rational zeros theorem, factors of 300 are the only possible rational solutions. Using synthetic division, as shown in the margin, we see that 5 is a solution. Therefore, one value of x is 5, and $y = \frac{75}{5^2} = 3$. We must now solve

$$x^2 + 5x - 60 = 0$$

for any other possible positive solutions. Use the quadratic formula to find the positive solution.

$$x = \frac{-5 + \sqrt{5^2 - 4(1)(-60)}}{2(1)} \approx 5.639 \qquad \text{Quadratic formula with } a = 1, b = 5, c = -60 \text{ (Section 1.4)}$$

This value of x leads to $y \approx 2.359$.

Step 5 **State the answer.** There are two possible answers.

First answer: length = width = 5 in.; height = 3 in.

Second answer: length = width ≈ 5.639 in.; height ≈ 2.359 in.

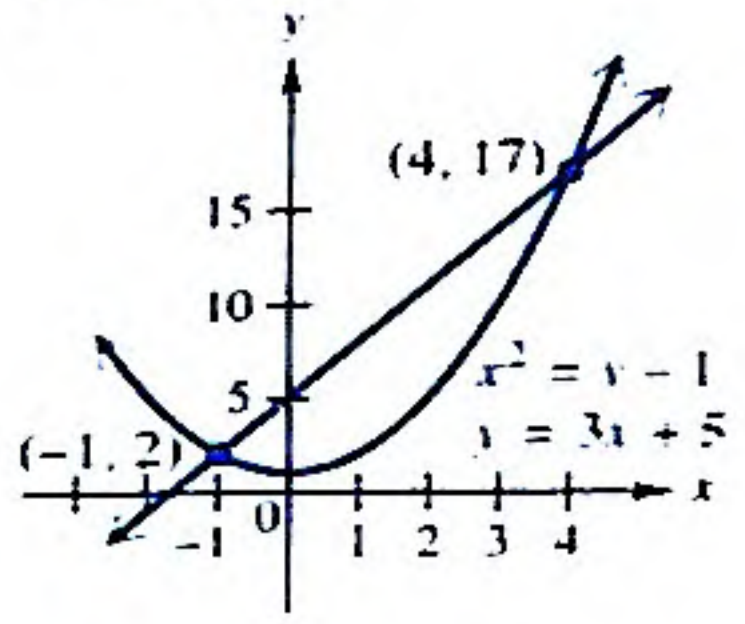
Step 6 **Check.** The check is left for **Exercise 63.**

✔ ***Now Try Exercises 61 and 63.***

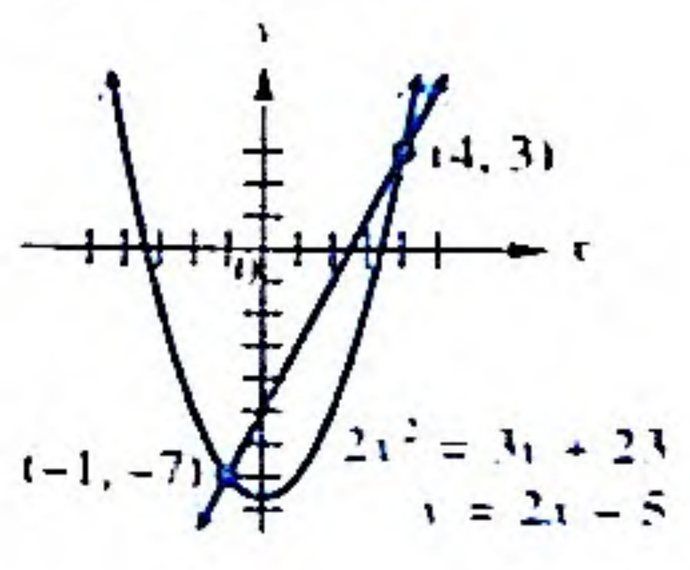
5.5 Exercises

Concept Check In Exercises 1–6 a nonlinear system is given, along with the graphs of both equations in the system. Verify that the points of intersection specified on the graph are solutions of the system by substituting directly into both equations.

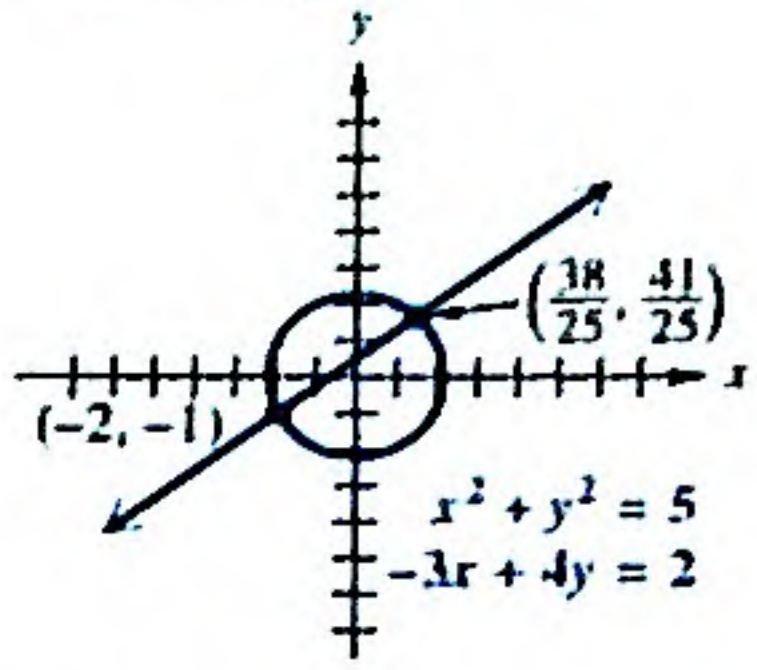
1. $x^2 = y - 1$
$y = 3x + 5$



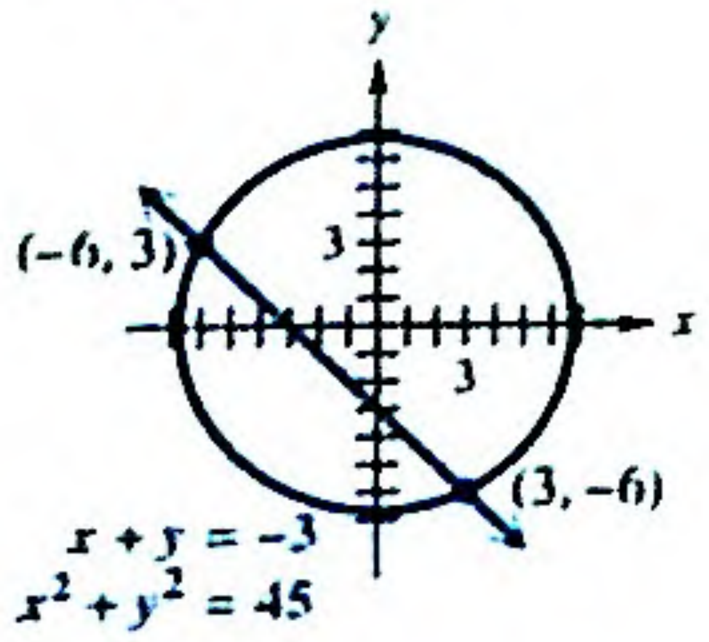
2. $2x^2 = 3y + 23$
$y = 2x - 5$



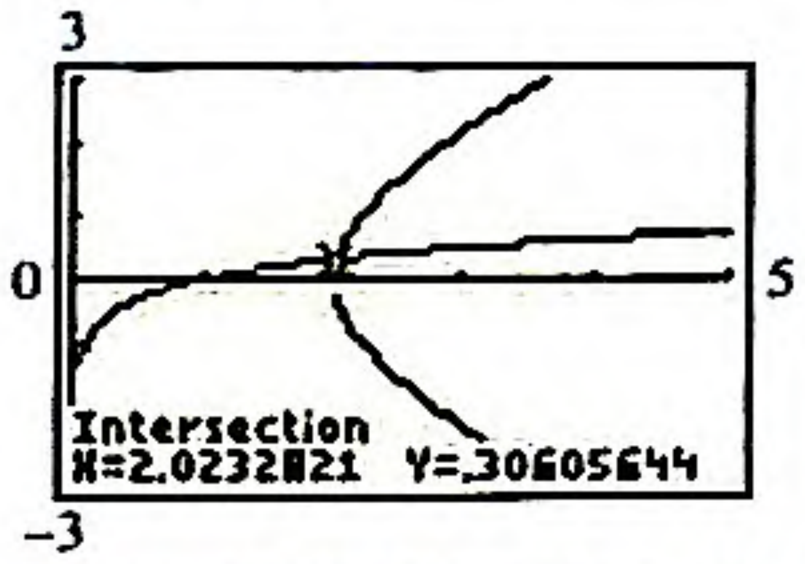
3. $\begin{aligned} x^2 + y^2 &= 5 \\ -3x + 4y &= 2 \end{aligned}$



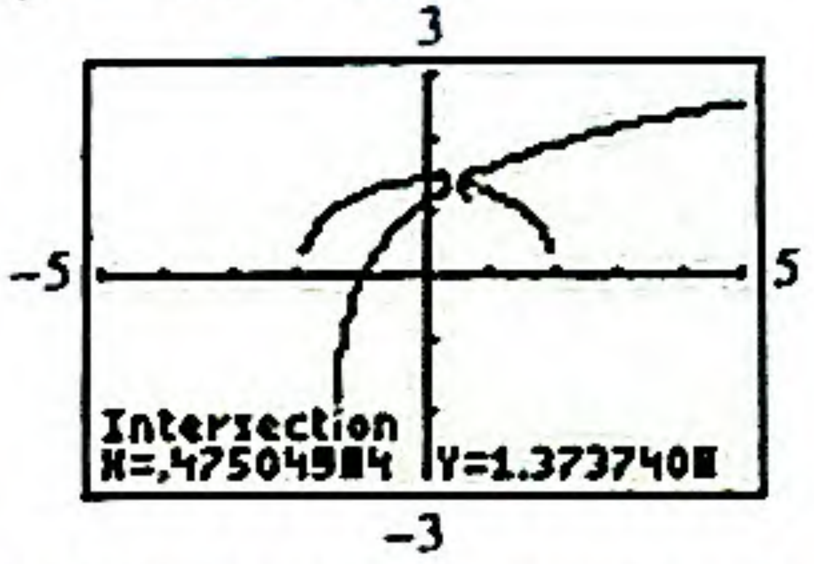
4. $\begin{aligned} x + y &= -3 \\ x^2 + y^2 &= 45 \end{aligned}$



5. $\begin{aligned} y &= \log x \\ x^2 - y^2 &= 4 \end{aligned}$



6. $\begin{aligned} y &= \ln(2x + 3) \\ y &= \sqrt{2 - 0.5x^2} \end{aligned}$



7. *Concept Check* In **Example 1**, we solved the following system. How can we tell, before doing any work, that this system cannot have more than two solutions?

$$\begin{aligned} x^2 - y &= 4 \\ x + y &= -2 \end{aligned}$$

8. *Concept Check* In **Example 5**, there were four solutions to the system, but there were no points of intersection of the graphs. If a nonlinear system has nonreal complex numbers as components of its solutions, will they appear as intersection points of the graphs?

Give all solutions of each nonlinear system of equations, including those with nonreal complex components. See Examples 1–5.

9. $\begin{aligned} x^2 - y &= 0 \\ x + y &= 2 \end{aligned}$

10. $\begin{aligned} x^2 + y &= 2 \\ x - y &= 0 \end{aligned}$

11. $\begin{aligned} y &= x^2 - 2x + 1 \\ x - 3y &= -1 \end{aligned}$

12. $\begin{aligned} y &= x^2 + 6x + 9 \\ x + 2y &= -2 \end{aligned}$

13. $\begin{aligned} y &= x^2 + 4x \\ 2x - y &= -8 \end{aligned}$

14. $\begin{aligned} y &= 6x + x^2 \\ 4x - y &= -3 \end{aligned}$

15. $\begin{aligned} 3x^2 + 2y^2 &= 5 \\ x - y &= -2 \end{aligned}$

16. $\begin{aligned} x^2 + y^2 &= 5 \\ -3x + 4y &= 2 \end{aligned}$

17. $\begin{aligned} x^2 + y^2 &= 8 \\ x^2 - y^2 &= 0 \end{aligned}$

18. $\begin{aligned} x^2 + y^2 &= 10 \\ 2x^2 - y^2 &= 17 \end{aligned}$

19. $\begin{aligned} 5x^2 - y^2 &= 0 \\ 3x^2 + 4y^2 &= 0 \end{aligned}$

20. $\begin{aligned} x^2 + y^2 &= 0 \\ 2x^2 - 3y^2 &= 0 \end{aligned}$

21. $\begin{aligned} 3x^2 + y^2 &= 3 \\ 4x^2 + 5y^2 &= 26 \end{aligned}$

22. $\begin{aligned} x^2 + 2y^2 &= 9 \\ x^2 + y^2 &= 25 \end{aligned}$

23. $\begin{aligned} 2x^2 + 3y^2 &= 5 \\ 3x^2 - 4y^2 &= -1 \end{aligned}$

24. $\begin{aligned} 3x^2 + 5y^2 &= 17 \\ 2x^2 - 3y^2 &= 5 \end{aligned}$

25. $\begin{aligned} 2x^2 + 2y^2 &= 20 \\ 4x^2 + 4y^2 &= 30 \end{aligned}$

26. $\begin{aligned} x^2 + y^2 &= 4 \\ 5x^2 + 5y^2 &= 28 \end{aligned}$

27. $\begin{aligned} 2x^2 - 3y^2 &= 12 \\ 6x^2 + 5y^2 &= 36 \end{aligned}$

28. $\begin{aligned} 5x^2 - 2y^2 &= 25 \\ 10x^2 + y^2 &= 50 \end{aligned}$

29. $\begin{aligned} xy &= -15 \\ 4x + 3y &= 3 \end{aligned}$

30. $\begin{aligned} xy &= 8 \\ 3x + 2y &= -16 \end{aligned}$

31. $\begin{aligned} 2xy + 1 &= 0 \\ x + 16y &= 2 \end{aligned}$

32. $\begin{aligned} -5xy + 2 &= 0 \\ x - 15y &= 5 \end{aligned}$

33. $\begin{aligned} 3x^2 - y^2 &= 11 \\ xy &= 12 \end{aligned}$

34. $\begin{aligned} 5x^2 - 2y^2 &= 6 \\ xy &= 2 \end{aligned}$

35. $\begin{aligned} x^2 - xy + y^2 &= 5 \\ 2x^2 + xy - y^2 &= 10 \end{aligned}$

36. $\begin{aligned} 3x^2 + 2xy - y^2 &= 9 \\ x^2 - xy + y^2 &= 9 \end{aligned}$

37. $\begin{aligned} x^2 + 2xy - y^2 &= 14 \\ x^2 - y^2 &= -16 \end{aligned}$

38. $\begin{aligned} x^2 + 3xy - y^2 &= 12 \\ x^2 - y^2 &= -12 \end{aligned}$

39. $x^2 + y^2 = 25$
$|x| - y = 5$

40. $x^2 + y^2 = 9$
$|x| + y = 3$

41. $x = |y|$
$x^2 + y^2 = 18$

42. $2x + |y| = 4$
$x^2 + y^2 = 5$

43. $2x^2 - y^2 = 4$
$|x| = |y|$

44. $x^2 + y^2 = 9$
$|x| = |y|$

Many nonlinear systems cannot be solved algebraically, so graphical analysis is the only way to determine the solutions of such systems. Use a graphing calculator to solve each nonlinear system. Give x- and y-coordinates to the nearest hundredth.

45. $y = \log(x + 5)$
$y = x^2$

46. $y = 5^x$
$xy = 1$

47. $y = e^{x+1}$
$2x + y = 3$

48. $y = \sqrt[3]{x - 4}$
$x^2 + y^2 = 6$

Solve each problem using a system of equations in two variables. ***See Example 6.***

49. *Unknown Numbers* Find two numbers whose sum is 17 and whose product is 42.

50. *Unknown Numbers* Find two numbers whose sum is 10 and whose squares differ by 20.

51. *Unknown Numbers* Find two numbers whose squares have a sum of 100 and a difference of 28.

52. *Unknown Numbers* Find two numbers whose squares have a sum of 194 and a difference of 144.

53. *Unknown Numbers* Find two numbers whose ratio is 9 to 2 and whose product is 162.

54. *Unknown Numbers* Find two numbers whose ratio is 4 to 3 and are such that the sum of their squares is 100.

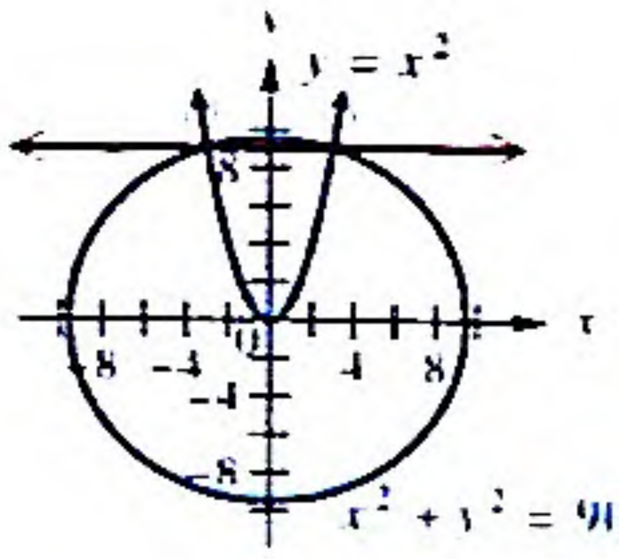
55. *Triangle Dimensions* The longest side of a right triangle is 13 m in length. One of the other sides is 7 m longer than the shortest side. Find the lengths of the two shorter sides of the triangle.

56. *Triangle Dimensions* The longest side of a right triangle is 29 ft in length. One of the other two sides is 1 ft longer than the shortest side. Find the lengths of the two shorter sides of the triangle.

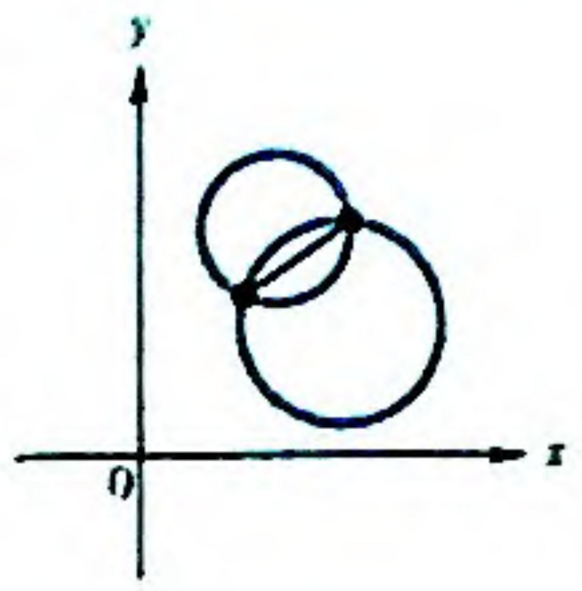
57. Does the straight line $3x - 2y = 9$ intersect the circle $x^2 + y^2 = 25$? (*Hint:* To find out, solve the system formed by these two equations.)

58. For what value(s) of b will the line $x + 2y = b$ touch the circle $x^2 + y^2 = 9$ in only one point?

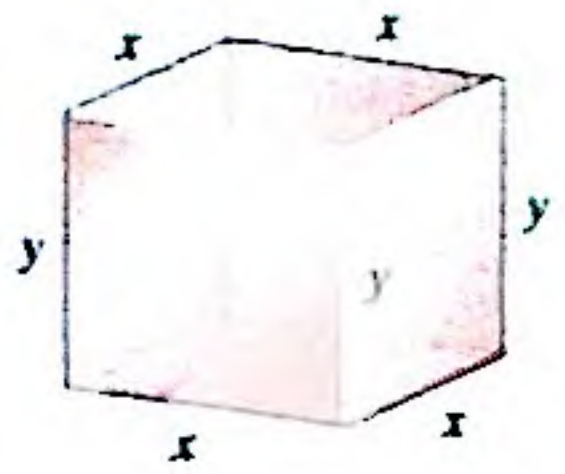
59. Find the equation of the line passing through the points of intersection of the graphs of $y = x^2$ and $x^2 + y^2 = 90$.



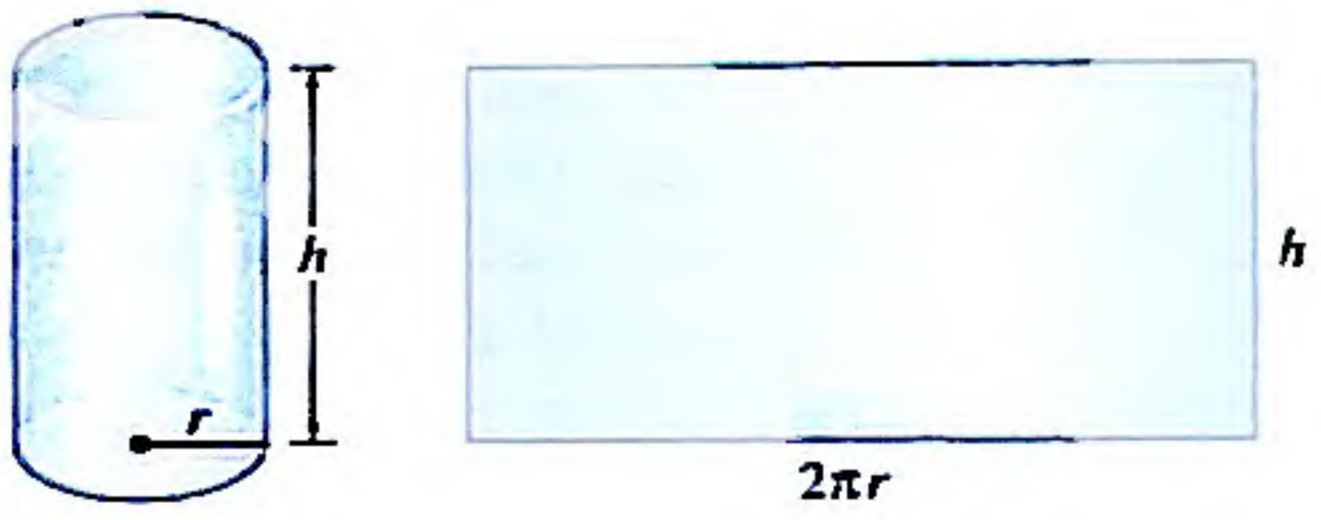
60. Suppose you are given the equations of two circles that are known to intersect in exactly two points. Explain how you would find the equation of the only chord common to these circles.



61. *Dimensions of a Box* A box with an open top has a square base and four sides of equal height. The volume of the box is 360 ft^3. If the surface area is 276 ft^2, find the dimensions of the box. (Round answers to the nearest thousandth, if necessary.)



62. *Dimensions of a Cylinder* Find the radius and height (to the nearest thousandth) of an open-ended cylinder with volume 50 in.^{3} and lateral surface area 65 in.^{2}.



63. *Checking Answers* Check the two answers in **Example 6.**

64. *(Modeling) Equilibrium Demand and Price* The supply and demand equations for a certain commodity are given.

$$\text{supply: } p = \frac{2000}{2000 - q} \quad \text{and} \quad \text{demand: } p = \frac{7000 - 3q}{2q}$$

(a) Find the equilibrium demand.
(b) Find the equilibrium price (in dollars).

65. *(Modeling) Equilibrium Demand and Price* The supply and demand equations for a certain commodity are given.

$$\text{supply: } p = \sqrt{0.1q + 9} - 2 \quad \text{and} \quad \text{demand: } p = \sqrt{25 - 0.1q}$$

(a) Find the equilibrium demand.
(b) Find the equilibrium price (in dollars).

66. *(Modeling) Circuit Gain* In electronics, circuit gain is modeled by

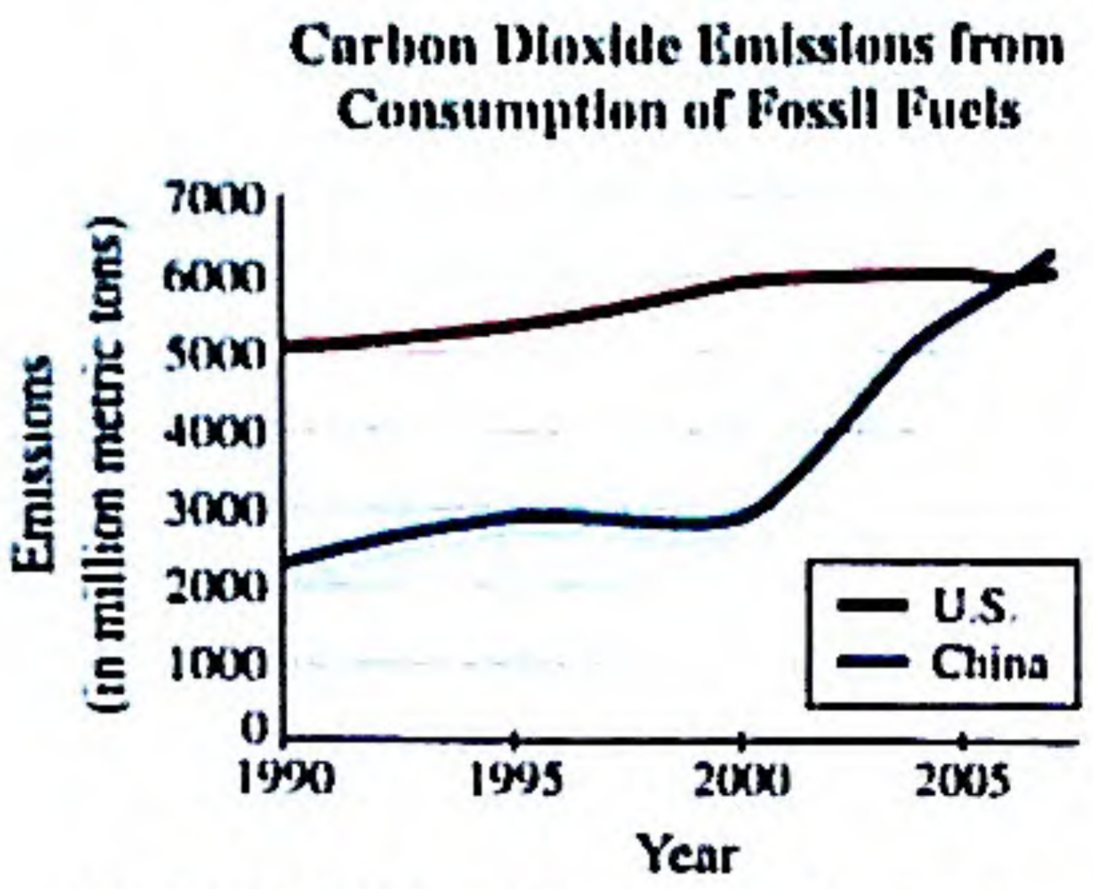
$$G = \frac{Bt}{R + R_t},$$

where R is the value of a resistor, t is temperature, R_t is the value of R at temperature t, and B is a constant. The sensitivity of the circuit to temperature is modeled by

$$S = \frac{BR}{(R + R_t)^2}.$$

If $B = 3.7$ and t is 90 K (kelvins), find the values of R and R_t that will result in the values $G = 0.4$ and $S = 0.001$.

67. ***(Modeling) Carbon Dioxide Emissions*** The emissions of carbon dioxide from consumption of fossil fuels from 1990 through 2007 are modeled in the graph for both the United States and China.



Carbon Dioxide Emissions from Consumption of Fossil Fuels

Source: U.S. Energy Information Administration.

(a) Interpret this graph. How are emissions changing with time?

(b) During what years were the carbon dioxide emissions in China decreasing?

(c) Use the graph to estimate the year and the amount of emissions when the carbon emissions for both countries were equal.

(d) The following equations model carbon dioxide emissions in the United States and in China. Use the equations to determine the year and emissions levels when $U = C$.

$$U = 5092(1.0105)^{(x-1990)} \quad \text{United States}$$

$$C = 2082(1.0638)^{(x-1990)} \quad \text{China}$$

Relating Concepts

For individual or collaborative investigation *(Exercises 68–74)*

Consider the following nonlinear system.

$$y = |x - 1|$$
$$y = x^2 - 4$$

*Work Exercises 68–74 in order, to see how **concepts from previous chapters are related to the graphs and the solutions of this system.***

68. How is the graph of $y = |x - 1|$ obtained by transforming the graph of $y = |x|$?

69. How is the graph of $y = x^2 - 4$ obtained by transforming the graph of $y = x^2$?

70. Use the definition of absolute value to write $y = |x - 1|$ as a piecewise-defined function.

71. Write two quadratic equations that will be used to solve the system. (*Hint:* Set both parts of the piecewise-defined function in Exercise 70 equal to $x^2 - 4$.)

72. Use the quadratic formula to solve both equations from Exercise 71. Pay close attention to the restriction on x.

73. Use the values of x found in Exercise 72 to find the solution set of the system.

74. Check the two answers in Exercise 73.

Summary Exercises on Systems of Equations

This chapter has introduced methods for solving systems of equations, including substitution and elimination, and matrix methods such as the Gauss-Jordan method and Cramer's rule. Use each method at least once when solving the systems below. Include solutions with nonreal complex number components. For systems with infinitely many solutions, write the solution set using an arbitrary variable.

1. $$\begin{aligned} 2x + 5y &= 4 \\ 3x - 2y &= -13 \end{aligned}$$

2. $$\begin{aligned} x - 3y &= 7 \\ -3x + 4y &= -1 \end{aligned}$$

3. $$\begin{aligned} 2x^2 + y^2 &= 5 \\ 3x^2 + 2y^2 &= 10 \end{aligned}$$

4. $$\begin{aligned} 2x - 3y &= -2 \\ x + y &= -16 \\ 3x - 2y + z &= 7 \end{aligned}$$

5. $$\begin{aligned} 6x - y &= 5 \\ xy &= 4 \end{aligned}$$

6. $$\begin{aligned} 4x + 2z &= -12 \\ x + y - 3z &= 13 \\ -3x + y - 2z &= 13 \end{aligned}$$

7. $$\begin{aligned} x + 2y + z &= 5 \\ y + 3z &= 9 \end{aligned}$$

8. $$\begin{aligned} x - 4 &= 3y \\ x^2 + y^2 &= 8 \end{aligned}$$

9. $$\begin{aligned} 3x + 6y - 9z &= 1 \\ 2x + 4y - 6z &= 1 \\ 3x + 4y + 5z &= 0 \end{aligned}$$

10. $$\begin{aligned} x + 2y + z &= 0 \\ x + 2y - z &= 6 \\ 2x - y &= -9 \end{aligned}$$

11. $$\begin{aligned} x^2 + y^2 &= 4 \\ y &= x + 6 \end{aligned}$$

12. $$\begin{aligned} x + 5y &= -23 \\ 4y - 3z &= -29 \\ 2x + 5z &= 19 \end{aligned}$$

13. $$\begin{aligned} y + 1 &= x^2 + 2x \\ y + 2x &= 4 \end{aligned}$$

14. $$\begin{aligned} 3x + 6z &= -3 \\ y - z &= 3 \\ 2x + 4z &= -1 \end{aligned}$$

15. $$\begin{aligned} 2x + 3y + 4z &= 3 \\ -4x + 2y - 6z &= 2 \\ 4x + 3z &= 0 \end{aligned}$$

16. $$\begin{aligned} \frac{3}{x} + \frac{4}{y} &= 4 \\ \frac{1}{x} + \frac{2}{y} &= \frac{2}{3} \end{aligned}$$

17. $$\begin{aligned} -5x + 2y + z &= 5 \\ -3x - 2y - z &= 3 \\ -x + 6y &= 1 \end{aligned}$$

18. $$\begin{aligned} x + 5y + 3z &= 9 \\ 2x + 9y + 7z &= 5 \end{aligned}$$

19. $$\begin{aligned} 2x^2 + y^2 &= 9 \\ 3x - 2y &= -6 \end{aligned}$$

20. $$\begin{aligned} 2x - 4y - 6 &= 0 \\ -x + 2y + 3 &= 0 \end{aligned}$$

21. $$\begin{aligned} x + y - z &= 0 \\ 2y - z &= 1 \\ 2x + 3y - 4z &= -4 \end{aligned}$$

22. $$\begin{aligned} 2y &= 3x - x^2 \\ x + 2y &= 12 \end{aligned}$$

23. $$\begin{aligned} 4x - z &= -6 \\ \frac{3}{5}y + \frac{1}{2}z &= 0 \\ \frac{1}{3}x + \frac{2}{3}z &= -5 \end{aligned}$$

24. $$\begin{aligned} x - y + 3z &= 3 \\ -2x + 3y - 11z &= -4 \\ x - 2y + 8z &= 6 \end{aligned}$$

25. $$\begin{aligned} x^2 + 3y^2 &= 28 \\ y - x &= -2 \end{aligned}$$

26. $$\begin{aligned} 5x - 2z &= 8 \\ 4y + 3z &= -9 \\ \frac{1}{2}x + \frac{2}{3}y &= -1 \end{aligned}$$

27. $$\begin{aligned} 2x^2 + 3y^2 &= 20 \\ x^2 + 4y^2 &= 5 \end{aligned}$$

28. $$\begin{aligned} x + y + z &= -1 \\ 2x + 3y + 2z &= 3 \\ 2x + y + 2z &= -7 \end{aligned}$$

29. $$\begin{aligned} x + 2z &= 9 \\ y + z &= 1 \\ 3x - 2y &= 9 \end{aligned}$$

30. $$\begin{aligned} x^2 - y^2 &= 15 \\ x - 2y &= 2 \end{aligned}$$

31. $$\begin{aligned} -x + y &= -1 \\ x + z &= 4 \\ 6x - 3y + 2z &= 10 \end{aligned}$$

32. $$\begin{aligned} 2x - y - 2z &= -1 \\ -x + 2y + 13z &= 12 \\ 3x + 9z &= 6 \end{aligned}$$

33. $$\begin{aligned} xy &= -3 \\ x + y &= -2 \end{aligned}$$

34. $$\begin{aligned} -3x + 2z &= 1 \\ 4x + y - 2z &= -6 \\ x + y + 4z &= 3 \end{aligned}$$

35. $$\begin{aligned} y &= x^2 + 6x + 9 \\ x + y &= 3 \end{aligned}$$

36. $$\begin{aligned} x^2 + y^2 &= 9 \\ 2x - y &= 3 \end{aligned}$$

37. $$\begin{aligned} 2x - 3y &= -2 \\ x + y - 4z &= -16 \\ 3x - 2y + z &= 7 \end{aligned}$$

38. $$\begin{aligned} 3x - y &= -2 \\ y + 5z &= -4 \\ -2x + 3y - z &= -8 \end{aligned}$$

39. $$\begin{aligned} y &= (x - 1)^2 + 2 \\ y &= 2x - 1 \end{aligned}$$

Robin takes x of the \$0.10 capsules and y of the \$0.20 chewable tablets, and she gets 8 units of vitamin A from each capsule and 2 units of vitamin A from each tablet. Altogether she gets $8x + 2y$ units of A per day. Since she wants at least 16 units,

$$8x + 2y \geq 16.$$

Each capsule and each tablet supplies 1 unit of vitamin B_1. Robin wants at least 5 units per day, so

$$x + y \geq 5.$$

For vitamin C, the inequality is

$$2x + 7y \geq 20.$$

Since Robin cannot take negative numbers of multivitamins,

$$x \geq 0 \quad \text{and} \quad y \geq 0.$$

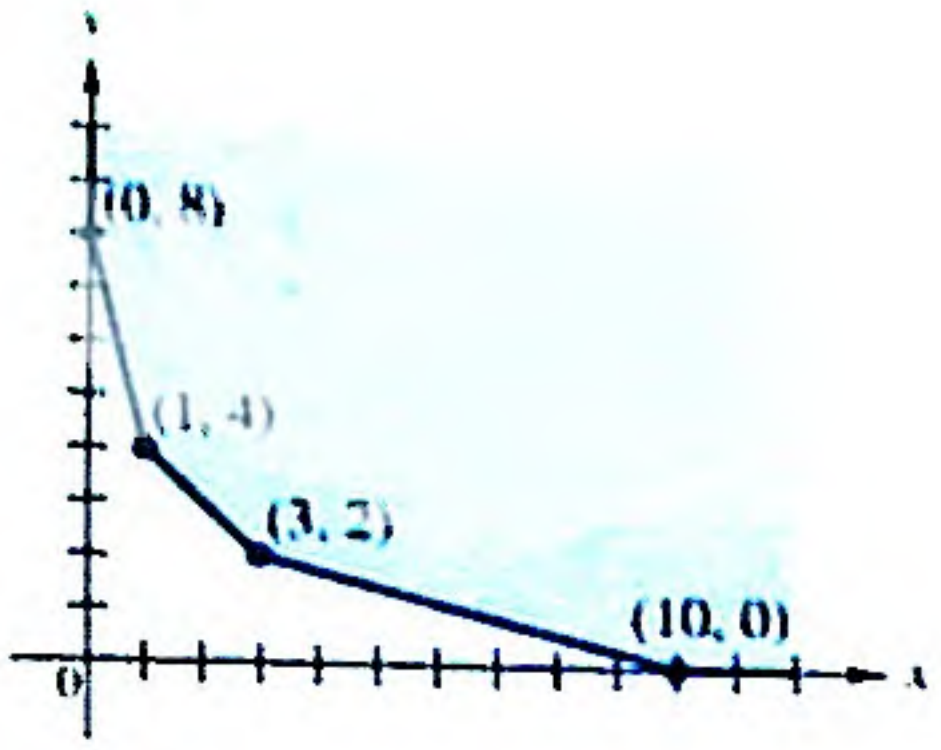


Figure 23

Step 2 **Figure 23** shows the intersection of the graphs of

$$8x + 2y \geq 16, \quad x + y \geq 5, \quad 2x + 7y \geq 20, \quad x \geq 0, \quad \text{and} \quad y \geq 0.$$

Step 3 The vertices are $(0, 8)$, $(1, 4)$, $(3, 2)$, and $(10, 0)$.

Steps 4 and 5 See the table. The minimum cost occurs at $(3, 2)$.

Point	**Cost = $10x + 20y$**	
$(0, 8)$	$10(0) + 20(8) = 160$	
$(1, 4)$	$10(1) + 20(4) = 90$	
$(3, 2)$	$10(3) + 20(2) = 70$	← Minimum
$(10, 0)$	$10(10) + 20(0) = 100$	

Robin's best choice is to take 3 capsules and 2 chewable tablets each day, for a total cost of \$0.70 per day. She receives just the minimum amounts of vitamins B_1 and C, and an excess of vitamin A.

✔ ***Now Try Exercises 71 and 81.***

5.6 Exercises

Graph each inequality. See Examples 1–3.

1. $x + 2y \leq 6$

2. $x - y \geq 2$

3. $2x + 3y \geq 4$

4. $4y - 3x \leq 5$

5. $3x - 5y > 6$

6. $x < 3 + 2y$

7. $5x < 4y - 2$

8. $2x > 3 - 4y$

9. $x \leq 3$

10. $y \leq -2$

11. $y < 3x^2 + 2$

12. $y \leq x^2 - 4$

13. $y > (x - 1)^2 + 2$

14. $y > 2(x + 3)^2 - 1$

15. $x^2 + (y + 3)^2 \leq 16$

16. $(x - 4)^2 + y^2 \geq 9$

17. $y > 2^x + 1$

18. $y \leq \log(x - 1) - 2$

19. Explain how to determine whether the boundary of the graph of an inequality is solid or dashed.

20. When graphing $y \leq 3x - 6$, would you shade above or below the line $y = 3x - 6$? Explain your answer.

Concept Check Work each problem.

21. For $Ax + By \geq C$, if $B > 0$, would you shade above or below the line?

22. For $Ax + By \geq C$, if $B < 0$, would you shade above or below the line?

23. Which one of the following is a description of the graph of the inequality

$$(x - 5)^2 + (y - 2)^2 < 4?$$

A. the region inside a circle with center $(-5, -2)$ and radius 2

B. the region inside a circle with center $(5, 2)$ and radius 2

C. the region inside a circle with center $(-5, -2)$ and radius 4

D. the region outside a circle with center $(5, 2)$ and radius 4

24. Find an example of a linear inequality in two variables whose graph does not intersect the graph of $y \geq 3x + 5$.

Concept Check In Exercises 25–28, match each inequality with the appropriate calculator graph. Do not use your calculator. Instead, use your knowledge of the concepts involved in graphing inequalities.

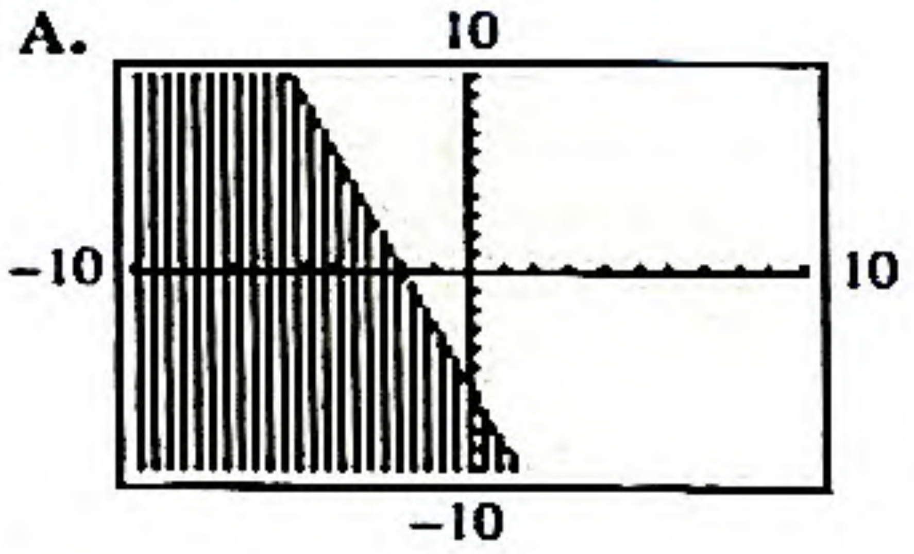
25. $y \leq 3x - 6$

26. $y \geq 3x - 6$

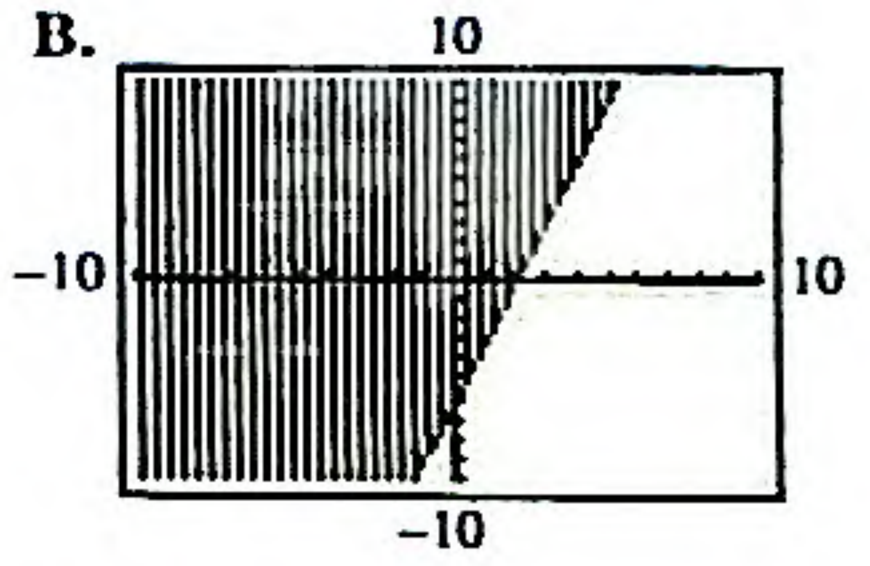
27. $y \leq -3x - 6$

28. $y \geq -3x - 6$

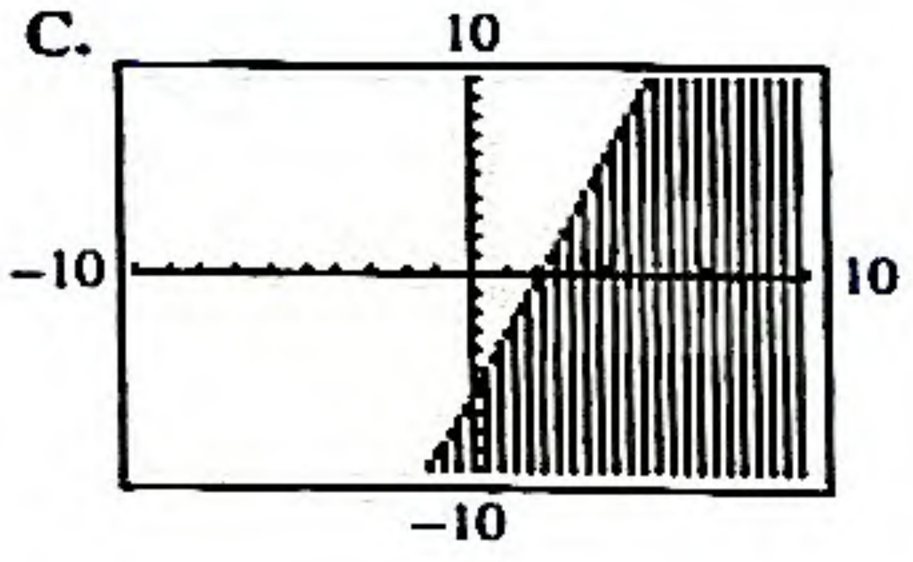
A.



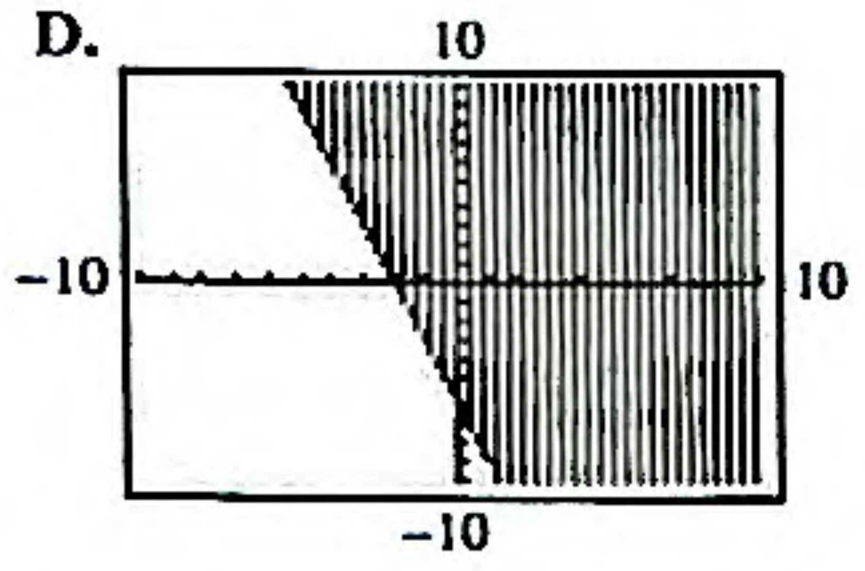
B.



C.



D.



Graph the solution set of each system of inequalities. See Example 3.

29. $x + y \geq 0$
$2x - y \geq 3$

30. $x + y \leq 4$
$x - 2y \geq 6$

31. $2x + y > 2$
$x - 3y < 6$

32. $4x + 3y < 12$
$y + 4x > -4$

33. $3x + 5y \leq 15$
$x - 3y \geq 9$

34. $y \leq x$
$x^2 + y^2 < 1$

35. $4x - 3y \leq 12$
$y \leq x^2$

36. $y \leq -x^2$
$y \geq x^2 - 6$

37. $x + 2y \leq 4$
$y \geq x^2 - 1$

38. $x + y \le 9$
$x \le -y^2$

39. $y \le (x + 2)^2$
$y \ge -2x^2$

40. $x - y < 1$
$-1 < y < 1$

41. $x + y \le 36$
$-4 \le x \le 4$

42. $y \ge (x - 2)^2 + 3$
$y \le -(x - 1)^2 + 6$

43. $y \ge x^2 + 4x + 4$
$y < -x^2$

44. $x \ge 0$
$x + y \le 4$
$2x + y \le 5$

45. $3x - 2y \ge 6$
$x + y \le -5$
$y \le 4$

46. $-2 < x < 3$
$-1 \le y \le 5$
$2x + y < 6$

47. $-2 < x < 2$
$y > 1$
$x - y > 0$

48. $x + y \le 4$
$x - y \le 5$
$4x + y \le -4$

49. $x \le 4$
$x \ge 0$
$y \ge 0$
$x + 2y \ge 2$

50. $2y + x \ge -5$
$y \le 3 + x$
$x \le 0$
$y \le 0$

51. $2x + 3y \le 12$
$2x + 3y > -6$
$3x + y < 4$
$x \ge 0$
$y \ge 0$

52. $y \ge 3^x$
$y \ge 2$

53. $y \le \left(\frac{1}{2}\right)^x$
$y \ge 4$

54. $\ln x - y \ge 1$
$x^2 - 2x - y \le 1$

55. $y \le \log x$
$y \ge |x - 2|$

56. $e^{-x} - y \le 1$
$x - 2y \ge 4$

57. $y > x^3 + 1$
$y \ge -1$

58. $y \le x^3 - x$
$y > -3$

Use the shading capabilities of your graphing calculator to graph each inequality or system of inequalities.

59. $3x + 2y \ge 6$

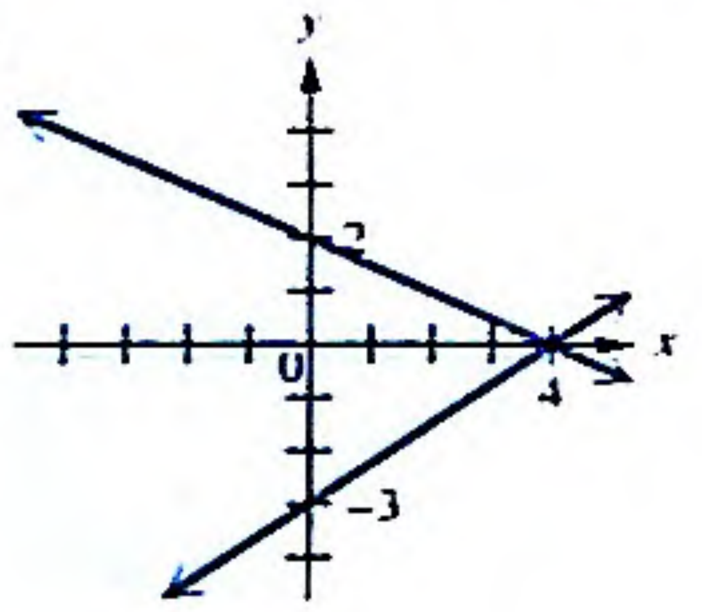
60. $y \le x^2 + 5$

61. $x + y \ge 2$
$x + y \le 6$

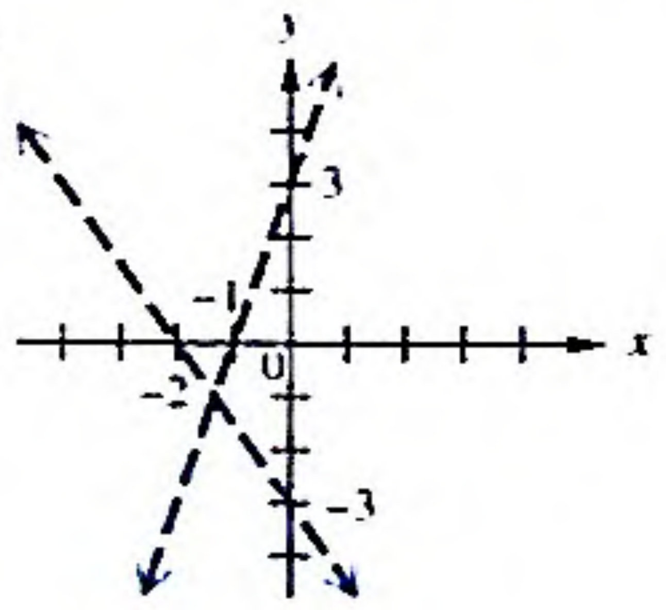
62. $y \ge |x + 2|$
$y \le 6$

Connecting Graphs with Equations *In Exercises 63–66, determine the system of inequalities illustrated in each graph. Write inequalities in standard form.*

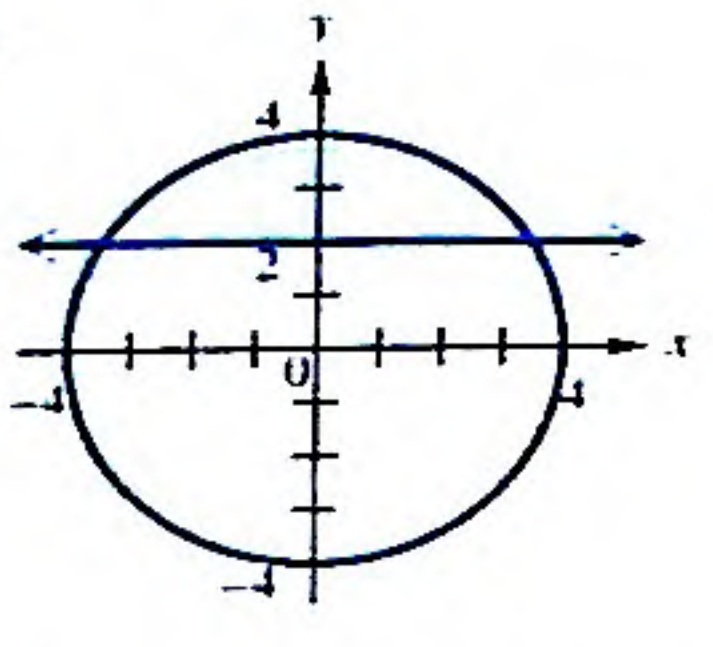
63.



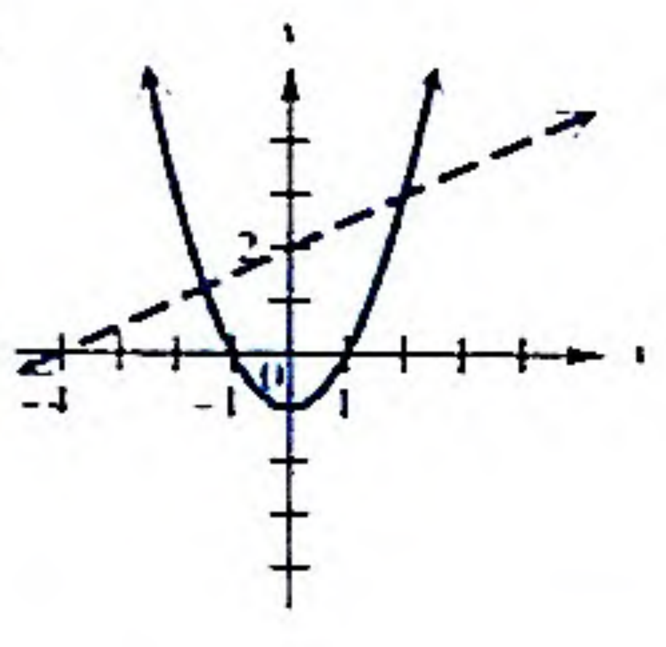
64.



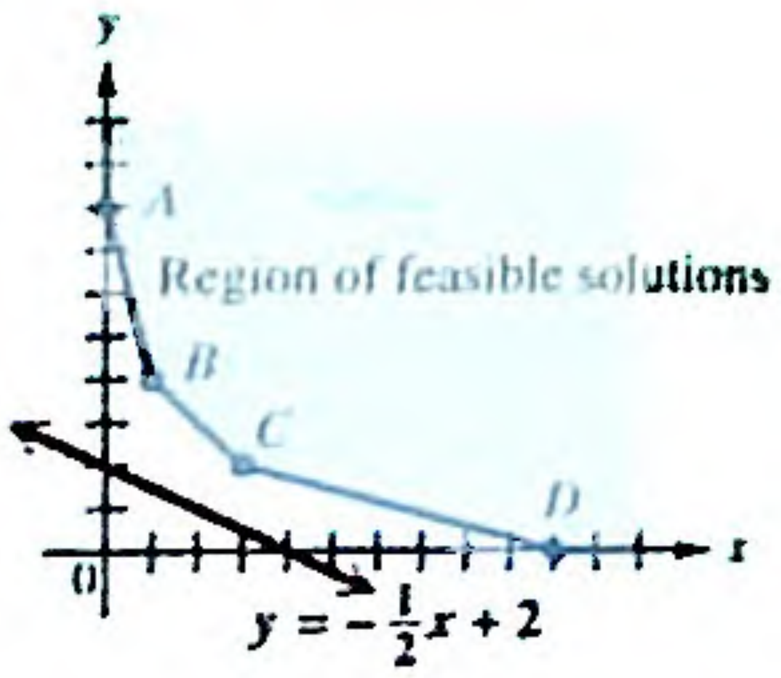
65.



66.



67. *Concept Check* Write a system of inequalities for which the graph is the region in the first quadrant inside and including the circle with radius 2 centered at the origin, and above (not including) the line that passes through the points $(0, -1)$ and $(2, 2)$.



68. ***Cost of Vitamins*** The figure shows the region of feasible solutions for the vitamin problem of Example 5 and the straight-line graph of all combinations of capsules and chewable tablets for which the cost is \$0.40.

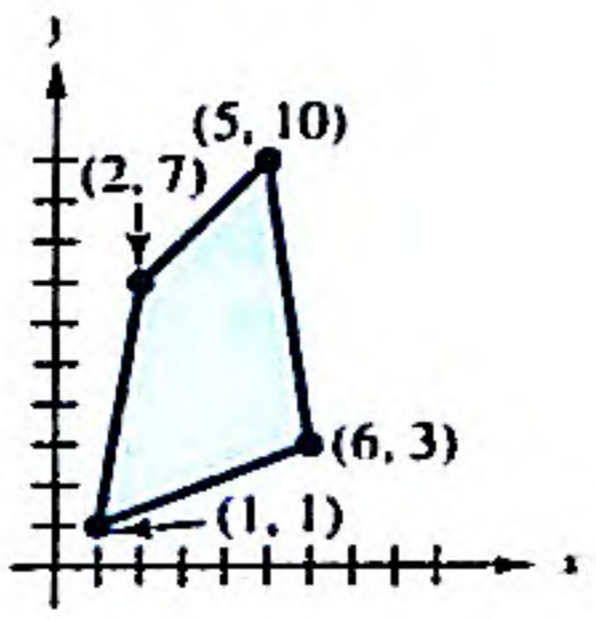
   (a) The cost function is $10x + 20y$. Give the linear equation (in slope-intercept form) of the line of constant cost c.

   (b) As c increases, does the line of constant cost move up or down?

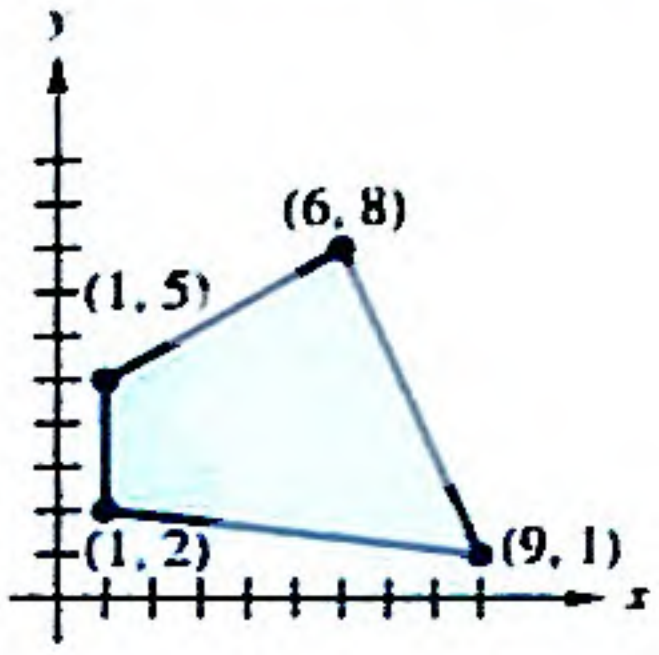
   (c) By inspection, find the vertex of the region of feasible solutions that gives the optimal solution.

The graphs show regions of feasible solutions. Find the maximum and minimum values of each objective function. See Examples 4 and 5.

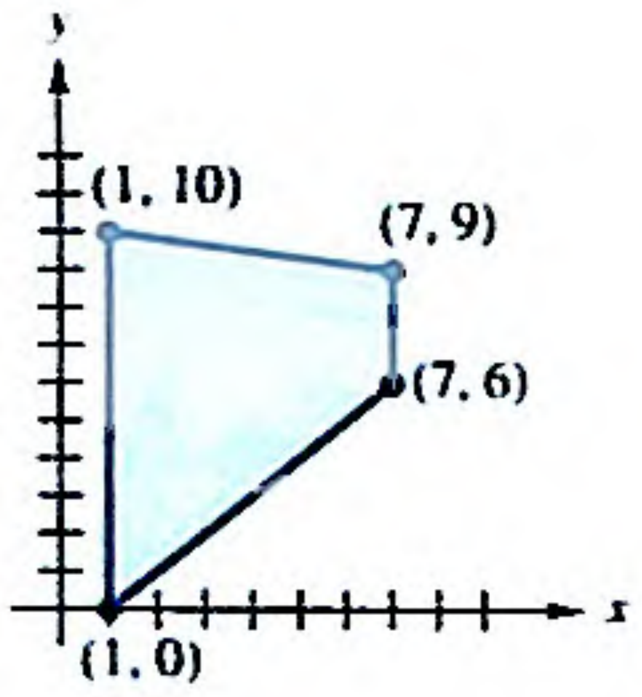
69. objective function $= 3x + 5y$



70. objective function $= 6x + y$



Find the maximum and minimum values of each objective function over the region of feasible solutions shown at the right. See Examples 4 and 5.



71. objective function $= 3x + 5y$

72. objective function $= 5x + 5y$

73. objective function $= 10y$

74. objective function $= 3x - y$

Write a system of inequalities for each problem, and then graph the region of feasible solutions of the system. See Examples 4 and 5.

75. ***Vitamin Requirements*** Jane Lukas was given the following advice. She should supplement her daily diet with at least 6000 USP units of vitamin A, at least 195 mg of vitamin C, and at least 600 USP units of vitamin D. She finds that Mason's Pharmacy carries Brand X and Brand Y vitamins. Each Brand X pill contains 3000 USP units of A, 45 mg of C, and 75 USP units of D, while the Brand Y pills contain 1000 USP units of A, 50 mg of C, and 200 USP units of D.

76. ***Shipping Requirements*** The California Almond Growers have 2400 boxes of almonds to be shipped from their plant in Sacramento to Des Moines and San Antonio. The Des Moines market needs at least 1000 boxes, while the San Antonio market must have at least 800 boxes.

Solve each linear programming problem. See Examples 4 and 5.

77. ***Aid to Disaster Victims*** An agency wants to ship food and clothing to tsunami victims in Japan. Commercial carriers have volunteered to transport the packages, provided they fit in the available cargo space. Each 20-ft^3 box of food weighs 40 lb and each 30-ft^3 box of clothing weighs 10 lb. The total weight cannot exceed 16,000 lb, and the total volume must be at most 18,000 ft^3. Each carton of food will feed 10 people, and each carton of clothing will help 8 people.

(a) How many cartons of food and clothing should be sent to maximize the number of people assisted?

(b) What is the maximum number assisted?

78. ***Aid to Disaster Victims*** Earthquake victims in Haiti need medical supplies and bottled water. Each medical kit measures 1 ft^3 and weighs 10 lb. Each container of water is also 1 ft^3 but weighs 20 lb. The plane can carry only 80,000 lb with a total volume of 6000 ft^3. Each medical kit will aid 4 people, and each container of water will serve 10 people.

(a) How many of each should be sent to maximize the number of people helped?

(b) If each medical kit could aid 6 people instead of 4, how would the results from part (a) change?

79. ***Storage Capacity*** An office manager wants to buy some filing cabinets. He knows that cabinet A costs $10 each, requires 6 ft^2 of floor space, and holds 8 ft^3 of files. Cabinet B costs $20 each, requires 8 ft^2 of floor space, and holds 12 ft^3 of files. He can spend no more than $140 due to budget limitations, and his office has room for no more than 72 ft^2 of cabinets. He wants to maximize storage capacity within the limits imposed by funds and space. How many of each type of cabinet should he buy?

80. ***Gasoline Revenues*** The manufacturing process requires that oil refineries manufacture at least 2 gal of gasoline for each gallon of fuel oil. To meet the winter demand for fuel oil, at least 3 million gal per day must be produced. The demand for gasoline is no more than 6.4 million gal per day. If the price of gasoline is $2.90 per gal and the price of fuel oil is $2.50 per gal, how much of each should be produced to maximize revenue?

81. ***Diet Requirements*** Theo, who is dieting, requires two food supplements, I and II. He can get these supplements from two different products, *A* and *B*, as shown in the table. Theo's physician has recommended that he include at least 15 g of each supplement in his daily diet. If product *A* costs $0.25 per serving and product *B* costs $0.40 per serving, how can he satisfy his requirements most economically?

Supplement (g/serving)	I	II
Product *A*	3	2
Product *B*	2	4

82. ***Profit from Televisions*** Seall Manufacturing Company makes television monitors. It produces a bargain monitor that sells for $100 profit and a deluxe monitor that sells for $150 profit. On the assembly line the bargain monitor requires 3 hr, and the deluxe monitor takes 5 hr. The cabinet shop spends 1 hr on the cabinet for the bargain monitor and 3 hr on the cabinet for the deluxe monitor. Both models require 2 hr of time for testing and packing. On a particular production run, the Seall Company has available 3900 work hours on the assembly line, 2100 work hours in the cabinet shop, and 2200 work hours in the testing and packing department. How many of each model should it produce to make the maximum profit? What is the maximum profit?

5.7 Properties of Matrices

- Basic Definitions
- Adding Matrices
- Special Matrices
- Subtracting Matrices
- Multiplying Matrices
- Applying Matrix Algebra

We used matrix notation to solve a system of linear equations earlier in this chapter. In this section and the next, we discuss algebraic properties of matrices.

Basic Definitions It is customary to use capital letters to name matrices and to use subscript notation to name elements of a matrix, as in the following matrix A.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

The first row, first column element is a_{11} (read "a-sub-one-one"); the second row, third column element is a_{23}; and in general, the ith row, jth column element is a_{ij}.

Certain matrices have special names. An $n \times n$ matrix is a **square matrix** because the number of rows is equal to the number of columns. A matrix with just one row is a **row matrix**, and a matrix with just one column is a **column matrix**.

Two matrices are equal if they have the same dimension and if corresponding elements, position by position, are equal. Using this definition, the matrices

$$\begin{bmatrix} 2 & 1 \\ 3 & -5 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 \\ -5 & 3 \end{bmatrix} \text{ are } \textit{not} \text{ equal}$$

(even though they contain the same elements and have the same dimension), since the corresponding elements differ.

EXAMPLE 1 Finding Values to Make Two Matrices Equal

Find the values of the variables for which each statement is true, if possible.

(a) $\begin{bmatrix} 2 & 1 \\ p & q \end{bmatrix} = \begin{bmatrix} x & y \\ -1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$

SOLUTION

(a) From the definition of equality given above, the only way that the statement can be true is if $2 = x$, $1 = y$, $p = -1$, and $q = 0$.

(b) This statement can never be true since the two matrices have different dimensions. (One is 2×1 and the other is 3×1.)

✔ ***Now Try Exercises 1 and 7.***

Adding Matrices Addition of matrices is defined as follows.

Addition of Matrices

To add two matrices of the same dimension, add corresponding elements. *Only matrices of the same dimension can be added.*

It can be shown that matrix addition satisfies the commutative, associative, closure, identity, and inverse properties. (See Exercises 87 and 88.)

Applying Matrix Algebra

EXAMPLE 8 Using Matrix Multiplication to Model Plans for a Subdivision

A contractor builds three kinds of houses, models A, B, and C, with a choice of two styles, colonial or ranch. Matrix P below shows the number of each kind of house the contractor is planning to build for a new 100-home subdivision. The amounts for each of the main materials used depend on the style of the house. These amounts are shown in matrix Q, while matrix R gives the cost in dollars for each kind of material. Concrete is measured here in cubic yards, lumber in 1000 board feet, brick in 1000s, and shingles in 100 square feet.

$$\begin{array}{l} \\ \text{Model A} \\ \text{Model B} \\ \text{Model C} \end{array} \begin{array}{c} \begin{array}{cc} \text{Colonial} & \text{Ranch} \end{array} \\ \begin{bmatrix} 0 & 30 \\ 10 & 20 \\ 20 & 20 \end{bmatrix} \end{array} = P$$

$$\begin{array}{l} \\ \text{Colonial} \\ \text{Ranch} \end{array} \begin{array}{c} \begin{array}{cccc} \text{Concrete} & \text{Lumber} & \text{Brick} & \text{Shingles} \end{array} \\ \begin{bmatrix} 10 & 2 & 0 & 2 \\ 50 & 1 & 20 & 2 \end{bmatrix} \end{array} = Q \qquad \begin{array}{l} \\ \text{Concrete} \\ \text{Lumber} \\ \text{Brick} \\ \text{Shingles} \end{array} \begin{array}{c} \text{Cost per Unit} \\ \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix} \end{array} = R$$

(a) What is the total cost of materials for all houses of each model?

(b) How much of each of the four kinds of material must be ordered?

(c) What is the total cost of the materials?

SOLUTION

(a) To find the materials cost for each model, first find matrix PQ, which will show the total amount of each material needed for all houses of each model.

$$PQ = \begin{bmatrix} 0 & 30 \\ 10 & 20 \\ 20 & 20 \end{bmatrix} \begin{bmatrix} 10 & 2 & 0 & 2 \\ 50 & 1 & 20 & 2 \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} \text{Concrete} & \text{Lumber} & \text{Brick} & \text{Shingles} \end{array} \\ \begin{bmatrix} 1500 & 30 & 600 & 60 \\ 1100 & 40 & 400 & 60 \\ 1200 & 60 & 400 & 80 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Model A} \\ \text{Model B} \\ \text{Model C} \end{array}$$

Multiplying PQ and the cost matrix R gives the total cost of materials for each model.

$$(PQ)R = \begin{bmatrix} 1500 & 30 & 600 & 60 \\ 1100 & 40 & 400 & 60 \\ 1200 & 60 & 400 & 80 \end{bmatrix} \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix} = \begin{array}{c} \text{Cost} \\ \begin{bmatrix} 72{,}900 \\ 54{,}700 \\ 60{,}800 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Model A} \\ \text{Model B} \\ \text{Model C} \end{array}$$

(b) To find how much of each kind of material to order, refer to the columns of matrix PQ. The sums of the elements of the columns will give a matrix whose elements represent the total amounts of all materials needed for the subdivision. Call this matrix T, and write it as a row matrix.

$$T = \begin{bmatrix} 3800 & 130 & 1400 & 200 \end{bmatrix}$$

(c) The total cost of all the materials is given by the product of matrix T, the total amounts matrix, and matrix R, the cost matrix. To multiply these matrices and get a 1×1 matrix, representing the total cost, requires multiplying a 1×4 matrix and a 4×1 matrix. This is why in part (b) a row matrix was written rather than a column matrix.

The total materials cost is given by TR, so

$$TR = \begin{bmatrix} 3800 & 130 & 1400 & 200 \end{bmatrix} \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix} = \begin{bmatrix} 188{,}400 \end{bmatrix}.$$

The total cost of materials is $188,400. This total may also be found by summing the elements of the column matrix $(PQ)R$.

✓ *Now Try Exercise 81.*

5.7 Exercises

Find the values of the variables for which each statement is true, if possible. See Examples 1 and 2.

1. $\begin{bmatrix} -3 & a \\ b & 5 \end{bmatrix} = \begin{bmatrix} c & 0 \\ 4 & d \end{bmatrix}$

2. $\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$

3. $\begin{bmatrix} x+2 & y-6 \\ z-3 & w+5 \end{bmatrix} = \begin{bmatrix} -2 & 8 \\ 0 & 3 \end{bmatrix}$

4. $\begin{bmatrix} 6 & a+3 \\ b+2 & 9 \end{bmatrix} = \begin{bmatrix} c-3 & 4 \\ -2 & d-4 \end{bmatrix}$

5. $\begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} 21 & 6 \end{bmatrix}$

6. $\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \end{bmatrix}$

7. $\begin{bmatrix} 0 & 5 & x \\ -1 & 3 & y+2 \\ 4 & 1 & z \end{bmatrix} = \begin{bmatrix} 0 & w & 6 \\ -1 & 3 & 0 \\ 4 & 1 & 8 \end{bmatrix}$

8. $\begin{bmatrix} 5 & x-4 & 9 \\ 2 & -3 & 8 \\ 6 & 0 & 5 \end{bmatrix} = \begin{bmatrix} y+3 & 2 & 9 \\ z+4 & -3 & 8 \\ 6 & 0 & w \end{bmatrix}$

9. $\begin{bmatrix} -7+z & 4r & 8s \\ 6p & 2 & 5 \end{bmatrix} + \begin{bmatrix} -9 & 8r & 3 \\ 2 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 36 & 27 \\ 20 & 7 & 12a \end{bmatrix}$

10. $\begin{bmatrix} a+2 & 3z+1 & 5m \\ 8k & 0 & 3 \end{bmatrix} + \begin{bmatrix} 3a & 2z & 5m \\ 2k & 5 & 6 \end{bmatrix} = \begin{bmatrix} 10 & -14 & 80 \\ 10 & 5 & 9 \end{bmatrix}$

11. ***Concept Check*** Two matrices are equal if they have the same ______ and if corresponding elements are ______.

12. ***Concept Check*** In order to add two matrices, they must have the same ______.

Concept Check *Find the dimension of each matrix. Identify any square, column, or row matrices.*

13. $\begin{bmatrix} -4 & 8 \\ 2 & 3 \end{bmatrix}$

14. $\begin{bmatrix} -9 & 6 & 2 \\ 4 & 1 & 8 \end{bmatrix}$

15. $\begin{bmatrix} -6 & 8 & 0 & 0 \\ 4 & 1 & 9 & 2 \\ 3 & -5 & 7 & 1 \end{bmatrix}$

16. $\begin{bmatrix} 8 & -2 & 4 & 6 & 3 \end{bmatrix}$

17. $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

18. $\begin{bmatrix} -9 \end{bmatrix}$

19. Your friend missed the lecture on adding matrices. Explain to him how to add two matrices.

20. Explain how to subtract two matrices.

Perform each operation when possible. ***See Examples 2 and 3.***

21. $\begin{bmatrix} -4 & 3 \\ 12 & -6 \end{bmatrix} + \begin{bmatrix} 2 & -8 \\ 5 & 10 \end{bmatrix}$

22. $\begin{bmatrix} 9 & 4 \\ -8 & 2 \end{bmatrix} + \begin{bmatrix} -3 & 2 \\ -4 & 7 \end{bmatrix}$

23. $\begin{bmatrix} 6 & -9 & 2 \\ 4 & 1 & 3 \end{bmatrix} + \begin{bmatrix} -8 & 2 & 5 \\ 6 & -3 & 4 \end{bmatrix}$

24. $\begin{bmatrix} 4 & -3 \\ 7 & 2 \\ -6 & 8 \end{bmatrix} + \begin{bmatrix} 9 & -10 \\ 0 & 5 \\ -1 & 6 \end{bmatrix}$

25. $\begin{bmatrix} 2 & 4 & 6 \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}$

26. $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -6 \end{bmatrix}$

27. $\begin{bmatrix} -6 & 8 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -4 & -2 \end{bmatrix}$

28. $\begin{bmatrix} 11 & 0 \\ -4 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 12 \\ 0 & -14 \end{bmatrix}$

29. $\begin{bmatrix} 12 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \\ -1 \end{bmatrix}$

30. $\begin{bmatrix} 10 & -4 & 6 \end{bmatrix} - \begin{bmatrix} -2 & 5 & 3 \end{bmatrix}$

31. $\begin{bmatrix} -4 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 8 & 2 \end{bmatrix}$

32. $\begin{bmatrix} 4 & 6 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

33. $\begin{bmatrix} \sqrt{3} & -4 \\ 2 & -\sqrt{5} \\ -8 & \sqrt{8} \end{bmatrix} - \begin{bmatrix} 2\sqrt{3} & 9 \\ -2 & \sqrt{5} \\ -7 & 3\sqrt{2} \end{bmatrix}$

34. $\begin{bmatrix} 2 & \sqrt{7} \\ 3\sqrt{28} & -6 \end{bmatrix} - \begin{bmatrix} -1 & 5\sqrt{7} \\ 2\sqrt{7} & 2 \end{bmatrix}$

35. $\begin{bmatrix} 3x + y & -2y \\ x + 2y & 3y \end{bmatrix} + \begin{bmatrix} 2x & 3y \\ 5x & x \end{bmatrix}$

36. $\begin{bmatrix} 4k - 8y \\ 6z - 3x \\ 2k + 5a \\ -4m + 2n \end{bmatrix} - \begin{bmatrix} 5k + 6y \\ 2z + 5x \\ 4k + 6a \\ 4m - 2n \end{bmatrix}$

Let $A = \begin{bmatrix} -2 & 4 \\ 0 & 3 \end{bmatrix}$ *and* $B = \begin{bmatrix} -6 & 2 \\ 4 & 0 \end{bmatrix}$. *Find each of the following.* ***See Example 4.***

37. $2A$

38. $-3B$

39. $\frac{3}{2}B$

40. $-\frac{3}{2}A$

41. $2A - 3B$

42. $-2A + 4B$

43. $-A + \frac{1}{2}B$

44. $\frac{3}{4}A - B$

Suppose that matrix A has dimension 2 × 3, B has dimension 3 × 5, and C has dimension 5 × 2. Decide whether the given product can be calculated. If it can, determine its dimension. ***See Example 5.***

45. AB

46. CA

47. BA

48. AC

49. BC

50. CB

Find each matrix product when possible. ***See Examples 5–7.***

51. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 7 \end{bmatrix}$

52. $\begin{bmatrix} -1 & 5 \\ 7 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

53. $\begin{bmatrix} 3 & -4 & 1 \\ 5 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$

54. $\begin{bmatrix} -6 & 3 & 5 \\ 2 & 9 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$

55. $\begin{bmatrix} \sqrt{2} & \sqrt{2} & -\sqrt{18} \\ \sqrt{3} & \sqrt{27} & 0 \end{bmatrix} \begin{bmatrix} 8 & -10 \\ 9 & 12 \\ 0 & 2 \end{bmatrix}$

56. $\begin{bmatrix} -9 & 2 & 1 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{5} \\ \sqrt{20} \\ -2\sqrt{5} \end{bmatrix}$

57. $\begin{bmatrix} \sqrt{3} & 1 \\ 2\sqrt{5} & 3\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & -\sqrt{6} \\ 4\sqrt{3} & 0 \end{bmatrix}$

58. $\begin{bmatrix} \sqrt{7} & 0 \\ 2 & \sqrt{28} \end{bmatrix} \begin{bmatrix} 2\sqrt{3} & -\sqrt{7} \\ 0 & -6 \end{bmatrix}$

59. $\begin{bmatrix} -3 & 0 & 2 & 1 \\ 4 & 0 & 2 & 6 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 0 & 1 \end{bmatrix}$

60. $\begin{bmatrix} -1 & 2 & 4 & 1 \\ 0 & 2 & -3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ -2 & 5 & 1 \end{bmatrix}$

61. $\begin{bmatrix} -2 & 4 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 & 4 \\ 2 & 1 & 0 \\ 0 & -1 & 4 \end{bmatrix}$

62. $\begin{bmatrix} 0 & 3 & -4 \end{bmatrix}\begin{bmatrix} -2 & 6 & 3 \\ 0 & 4 & 2 \\ -1 & 1 & 4 \end{bmatrix}$

63. $\begin{bmatrix} -2 & -3 & -4 \\ 2 & -1 & 0 \\ 4 & -2 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix}$

64. $\begin{bmatrix} -1 & 2 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ 0 & 2 & 1 \\ 3 & 0 & -1 \end{bmatrix}$

Given $A = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 1 \\ 0 & -2 \\ 3 & 7 \end{bmatrix}$, *and* $C = \begin{bmatrix} -5 & 4 & 1 \\ 0 & 3 & 6 \end{bmatrix}$, *find each product when possible. See Examples 5–7.*

65. BA

66. AC

67. BC

68. CB

69. AB

70. CA

71. A^2

72. A^3 (*Hint:* $A^3 = A^2 \cdot A$)

73. *Concept Check* Compare the answers to Exercises 65 and 69, 67 and 68, and 66 and 70. How do they show that matrix multiplication is not commutative?

74. Your friend missed the lecture on multiplying matrices. Explain to her the process of matrix multiplication.

For each pair of matrices A and B, find (a) AB and (b) BA. See Example 7.

75. $A = \begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}, B = \begin{bmatrix} 6 & 0 \\ 5 & -2 \end{bmatrix}$

76. $A = \begin{bmatrix} 0 & -5 \\ -4 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 \\ -5 & 4 \end{bmatrix}$

77. $A = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

78. $A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

79. *Concept Check* In Exercise 77, $AB = A$ and $BA = A$. For this pair of matrices, B acts the same way for matrix multiplication as the number ________ acts for multiplication of real numbers.

80. *Concept Check* Find AB and BA for

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

What do you notice? Matrix B acts as the multiplicative ________ element for 2×2 square matrices.

Solve each problem. See Example 8.

81. *Income from Yogurt* Yagel's Yogurt sells three types of yogurt: nonfat, regular, and super creamy, at three locations. Location I sells 50 gal of nonfat, 100 gal of regular, and 30 gal of super creamy each day. Location II sells 10 gal of nonfat, and Location III sells 60 gal of nonfat each day. Daily sales of regular yogurt are 90 gal at Location II and 120 gal at Location III. At Location II, 50 gal of super creamy are sold each day, and 40 gal of super creamy are sold each day at Location III.

(a) Write a 3×3 matrix that shows the sales figures for the three locations, with the rows representing the three locations.

(b) The incomes per gallon for nonfat, regular, and super creamy are \$12, \$10, and \$15, respectively. Write a 1×3 or 3×1 matrix displaying the incomes.

(c) Find a matrix product that gives the daily income at each of the three locations.

(d) What is Yagel's Yogurt's total daily income from the three locations?

82. *Purchasing Costs* The Bread Box, a small neighborhood bakery, sells four main items: sweet rolls, bread, cakes, and pies. The amount of each ingredient (in cups, except for eggs) required for these items is given by matrix A.

$$\begin{array}{l} \\ \text{Rolls (doz)} \\ \text{Bread (loaf)} \\ \text{Cake} \\ \text{Pie (crust)} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Eggs} & \text{Flour} & \text{Sugar} & \text{Shortening} & \text{Milk} \end{array} \\ \begin{bmatrix} 1 & 4 & \frac{1}{4} & \frac{1}{4} & 1 \\ 0 & 3 & 0 & \frac{1}{4} & 0 \\ 4 & 3 & 2 & 1 & 1 \\ 0 & 1 & 0 & \frac{1}{3} & 0 \end{bmatrix} \end{array} = A$$

The cost (in cents) for each ingredient when purchased in large lots or small lots is given by matrix B.

$$\begin{array}{l} \\ \\ \text{Eggs} \\ \text{Flour} \\ \text{Sugar} \\ \text{Shortening} \\ \text{Milk} \end{array} \begin{array}{c} \text{Cost} \\ \begin{array}{cc} \text{Large Lot} & \text{Small Lot} \end{array} \\ \begin{bmatrix} 5 & 5 \\ 8 & 10 \\ 10 & 12 \\ 12 & 15 \\ 5 & 6 \end{bmatrix} \end{array} = B$$

(a) Use matrix multiplication to find a matrix giving the comparative cost per bakery item for the two purchase options.

(b) Suppose a day's orders consist of 20 dozen sweet rolls, 200 loaves of bread, 50 cakes, and 60 pies. Write the orders as a 1×4 matrix, and, using matrix multiplication, write as a matrix the amount of each ingredient needed to fill the day's orders.

(c) Use matrix multiplication to find a matrix giving the costs under the two purchase options to fill the day's orders.

83. *(Modeling) Northern Spotted Owl Population* Several years ago, mathematical ecologists created a model to analyze population dynamics of the endangered northern spotted owl in the Pacific Northwest. The ecologists divided the female owl population into three categories: juvenile (up to 1 yr old), subadult (1 to 2 yr old), and adult (over 2 yr old). They concluded that the change in the makeup of the northern spotted owl population in successive years could be described by the following matrix equation.

$$\begin{bmatrix} j_{n+1} \\ s_{n+1} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.33 \\ 0.18 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix} \begin{bmatrix} j_n \\ s_n \\ a_n \end{bmatrix}$$

The numbers in the column matrices give the numbers of females in the three age groups after n years and $n + 1$ years. Multiplying the matrices yields the following.

$j_{n+1} = 0.33a_n$	Each year 33 juvenile females are born for each 100 adult females.
$s_{n+1} = 0.18j_n$	Each year 18% of the juvenile females survive to become subadults.
$a_{n+1} = 0.71s_n + 0.94a_n$	Each year 71% of the subadults survive to become adults, and 94% of the adults survive.

(*Source:* Lamberson, R. H., R. McKelvey, B. R. Noon, and C. Voss, "A Dynamic Analysis of Northern Spotted Owl Viability in a Fragmented Forest Landscape," *Conservation Biology*, Vol. 6, No. 4.)

(a) Suppose there are currently 3000 female northern spotted owls made up of 690 juveniles, 210 subadults, and 2100 adults. Use the matrix equation on the preceding page to determine the total number of female owls for each of the next 5 yr.

(b) Using advanced techniques from linear algebra, we can show that in the long run,

$$\begin{bmatrix} j_{n+1} \\ s_{n+1} \\ a_{n+1} \end{bmatrix} \approx 0.98359 \begin{bmatrix} j_n \\ s_n \\ a_n \end{bmatrix}.$$

What can we conclude about the long-term fate of the northern spotted owl?

(c) In the model, the main impediment to the survival of the northern spotted owl is the number 0.18 in the second row of the 3×3 matrix. This number is low for two reasons.

- The first year of life is precarious for most animals living in the wild.
- In addition, juvenile owls must eventually leave the nest and establish their own territory. If much of the forest near their original home has been cleared, then they are vulnerable to predators while searching for a new home.

Suppose that, thanks to better forest management, the number 0.18 can be increased to 0.3. Rework part (a) under this new assumption.

84. ***(Modeling) Predator-Prey Relationship*** In certain parts of the Rocky Mountains, deer provide the main food source for mountain lions. When the deer population is large, the mountain lions thrive. However, a large mountain lion population reduces the size of the deer population. Suppose the fluctuations of the two populations from year to year can be modeled with the matrix equation

$$\begin{bmatrix} m_{n+1} \\ d_{n+1} \end{bmatrix} = \begin{bmatrix} 0.51 & 0.4 \\ -0.05 & 1.05 \end{bmatrix} \begin{bmatrix} m_n \\ d_n \end{bmatrix}.$$

The numbers in the column matrices give the numbers of animals in the two populations after n years and $n + 1$ years, where the number of deer is measured in hundreds.

(a) Give the equation for d_{n+1} obtained from the second row of the square matrix. Use this equation to determine the rate at which the deer population will grow from year to year if there are no mountain lions.

(b) Suppose we start with a mountain lion population of 2000 and a deer population of 500,000 (that is, 5000 hundred deer). How large would each population be after 1 yr? 2 yr?

(c) Consider part (b) but change the initial mountain lion population to 4000. Show that the populations would both grow at a steady annual rate of 1.01.

85. ***Northern Spotted Owl Population*** Refer to Exercise 83(b). Show that the number 0.98359 is an approximate zero of the polynomial represented by

$$\begin{vmatrix} -x & 0 & 0.33 \\ 0.18 & -x & 0 \\ 0 & 0.71 & 0.94 - x \end{vmatrix}.$$

86. ***Predator-Prey Relationship*** Refer to Exercise 84(c). Show that the number 1.01 is a zero of the polynomial represented by

$$\begin{vmatrix} 0.51 - x & 0.4 \\ -0.05 & 1.05 - x \end{vmatrix}.$$

For Exercises 87–94, let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix},$$

where all the elements are real numbers. Use these matrices to show that each statement is true for 2 × 2 matrices.

87. $A + B = B + A$
(commutative property)

88. $A + (B + C) = (A + B) + C$
(associative property)

89. $(AB)C = A(BC)$
(associative property)

90. $A(B + C) = AB + AC$
(distributive property)

91. $c(A + B) = cA + cB$,
for any real number c.

92. $(c + d)A = cA + dA$,
for any real numbers c and d.

93. $(cA)d = (cd)A$, for any real numbers c and d.

94. $(cd)A = c(dA)$, for any real numbers c and d.

5.8 Matrix Inverses

- Identity Matrices
- Multiplicative Inverses
- Solving Systems Using Inverse Matrices

In the previous section, we saw several parallels between the set of real numbers and the set of matrices. Another similarity is that both sets have identity and inverse elements for multiplication.

Identity Matrices

By the identity property for real numbers,

$$a \cdot 1 = a \quad \text{and} \quad 1 \cdot a = a \quad \text{(Section R.2)}$$

for any real number a. If there is to be a multiplicative identity matrix I, such that

$$AI = A \quad \text{and} \quad IA = A,$$

for any matrix A, then A and I must be square matrices of the same dimension.

2 × 2 Identity Matrix

If I_2 represents the 2 × 2 identity matrix, then

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

To verify that I_2 is the 2 × 2 identity matrix, we must show that $AI = A$ and $IA = A$ for any 2 × 2 matrix A. Let

$$A = \begin{bmatrix} x & y \\ z & w \end{bmatrix}.$$

Then

$$AI = \begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x \cdot 1 + y \cdot 0 & x \cdot 0 + y \cdot 1 \\ z \cdot 1 + w \cdot 0 & z \cdot 0 + w \cdot 1 \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix} = A,$$

and

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 \cdot x + 0 \cdot z & 1 \cdot y + 0 \cdot w \\ 0 \cdot x + 1 \cdot z & 0 \cdot y + 1 \cdot w \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix} = A.$$

Generalizing, there is an $n \times n$ identity matrix for every $n \times n$ square matrix. The $n \times n$ identity matrix has 1s on the main diagonal and 0s elsewhere.

$n \times n$ Identity Matrix

The $n \times n$ identity matrix is I_n, where

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & a_{ij} & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

The element $a_{ij} = 1$ when $i = j$ (the diagonal elements), and $a_{ij} = 0$ otherwise.

EXAMPLE 1 Verifying the Identity Property of I_3

Let $A = \begin{bmatrix} -2 & 4 & 0 \\ 3 & 5 & 9 \\ 0 & 8 & -6 \end{bmatrix}$. Give the 3×3 identity matrix I_3 and show that $AI_3 = A$.

ALGEBRAIC SOLUTION

The 3×3 identity matrix is

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

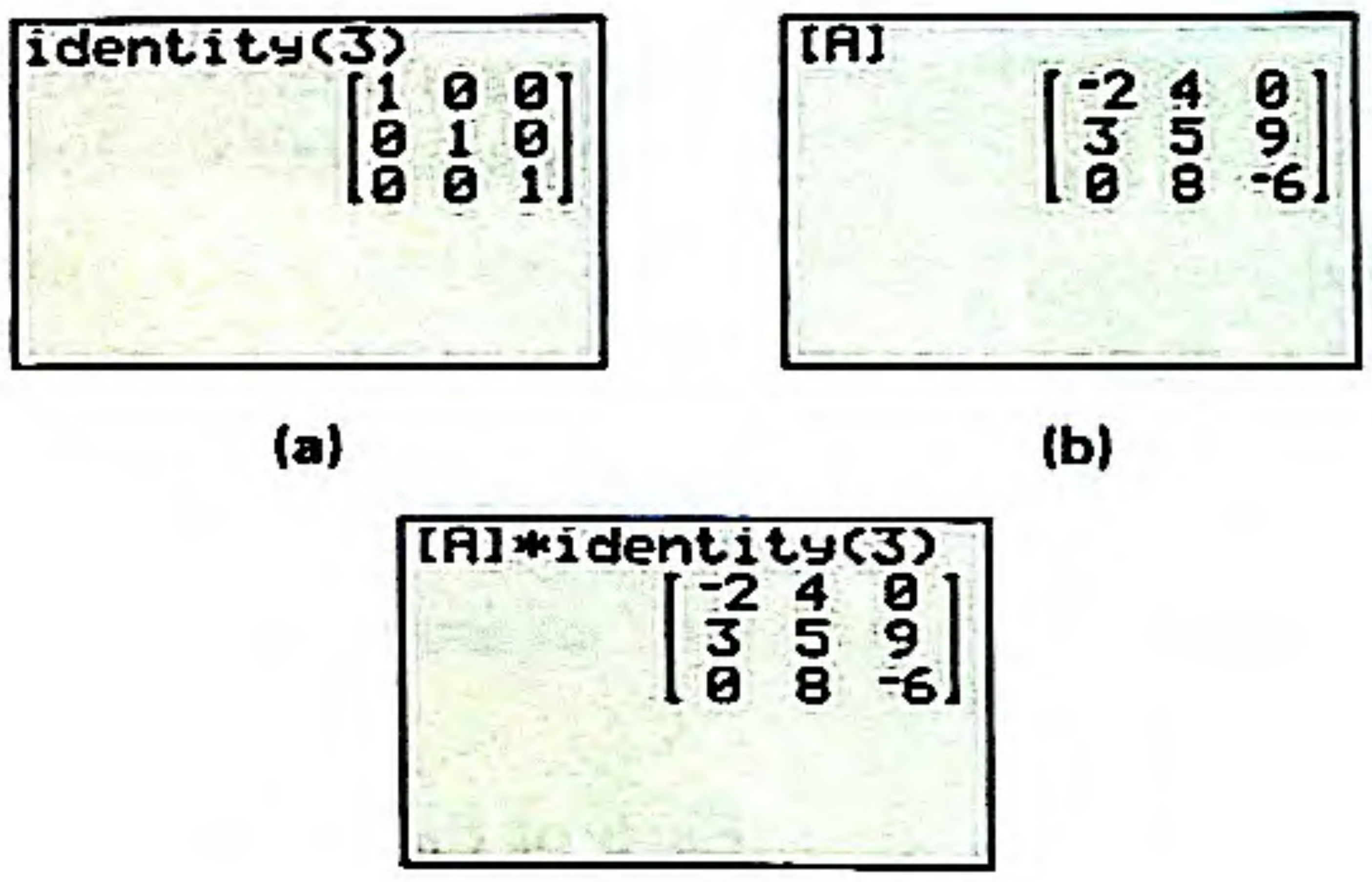
By the definition of matrix multiplication,

$$AI_3 = \begin{bmatrix} -2 & 4 & 0 \\ 3 & 5 & 9 \\ 0 & 8 & -6 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 4 & 0 \\ 3 & 5 & 9 \\ 0 & 8 & -6 \end{bmatrix} = A.$$

GRAPHING CALCULATOR SOLUTION

The calculator screen in **Figure 27(a)** shows the identity matrix for $n = 3$. The screens in **Figures 27(b) and (c)** support the algebraic result.



```
identity(3)
              [[1 0 0]
               [0 1 0]
               [0 0 1]]
```

(a)

```
[A]
           [[-2 4 0 ]
            [3  5 9 ]
            [0  8 -6]]
```

(b)

```
[A]*identity(3)
           [[-2 4 0 ]
            [3  5 9 ]
            [0  8 -6]]
```

(c)

Figure 27

✓ *Now Try Exercise 1.*

Multiplicative Inverses For every *nonzero* real number a, there is a multiplicative inverse $\frac{1}{a}$ that satisfies both of the following.

$$a \cdot \frac{1}{a} = 1 \quad \text{and} \quad \frac{1}{a} \cdot a = 1 \qquad \text{(Section R.2)}$$

(Recall: $\frac{1}{a}$ is also written a^{-1}.) In a similar way, if A is an $n \times n$ matrix, then its **multiplicative inverse**, written A^{-1}, must satisfy both of the following.

$$AA^{-1} = I_n \quad \text{and} \quad A^{-1}A = I_n$$

This means that only a square matrix can have a multiplicative inverse.

CAUTION Although $a^{-1} = \frac{1}{a}$ for any nonzero real number a, if A is a matrix, then $A^{-1} \neq \frac{1}{A}$. *We do not use the symbol $\frac{1}{A}$, since* 1 *is a number and* A *is a matrix.*

$$\begin{aligned} A^{-1}(AX) &= A^{-1}B \\ (A^{-1}A)X &= A^{-1}B \quad \text{Associative property (Section 5.7)} \\ I_3X &= A^{-1}B \quad \text{Inverse property} \\ X &= A^{-1}B \quad \text{Identity property} \end{aligned}$$

Matrix $A^{-1}B$ gives the solution of the system.

Solution of the Matrix Equation $AX = B$

If A is an $n \times n$ matrix with inverse A^{-1}, X is an $n \times 1$ matrix of variables, and B is an $n \times 1$ matrix, then the matrix equation

$$AX = B$$

has the solution $$X = A^{-1}B.$$

This method of using matrix inverses to solve systems of equations is useful when the inverse is already known or when many systems of the form $AX = B$ must be solved and only B changes.

EXAMPLE 4 Solving Systems of Equations Using Matrix Inverses

Use the inverse of the coefficient matrix to solve each system.

(a) $$\begin{aligned} 2x - 3y &= 4 \\ x + 5y &= 2 \end{aligned}$$

(b) $$\begin{aligned} x + z &= -1 \\ 2x - 2y - z &= 5 \\ 3x &= 6 \end{aligned}$$

ALGEBRAIC SOLUTION

(a) The system can be written in matrix form as

$$\begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix},$$

where

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

An equivalent matrix equation is $AX = B$ with solution $X = A^{-1}B$. Use the methods described in this section to determine that

$$A^{-1} = \begin{bmatrix} \frac{5}{13} & \frac{3}{13} \\ -\frac{1}{13} & \frac{2}{13} \end{bmatrix}.$$

Now, find $A^{-1}B$.

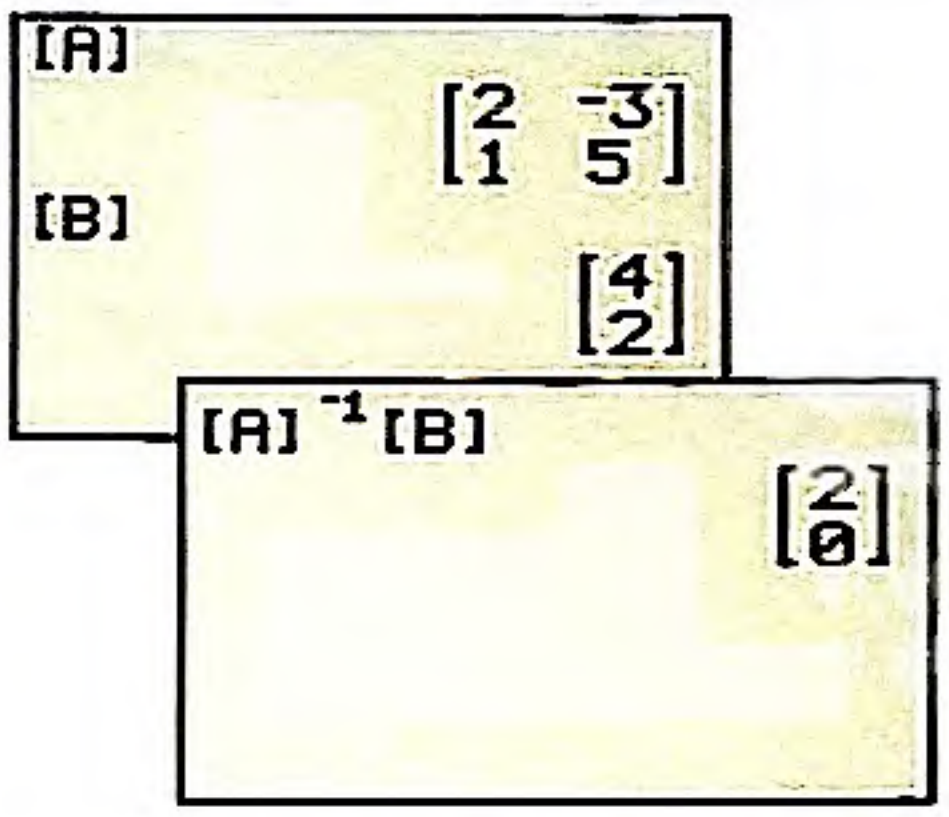
$$A^{-1}B = \begin{bmatrix} \frac{5}{13} & \frac{3}{13} \\ -\frac{1}{13} & \frac{2}{13} \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Since $X = A^{-1}B$, $$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

The final matrix shows that the solution set of the system is $\{(2, 0)\}$.

GRAPHING CALCULATOR SOLUTION

(a) Enter [A] and [B], and then find the product $[A]^{-1}[B]$ as shown in **Figure 29**.



```
[A]
                [[2 -3]
                 [1 5 ]]
[B]
                  [[4]
                   [2]]
[A]⁻¹[B]
                  [[2]
                   [0]]
```

Figure 29

The column matrix indicates the solution set, $\{(2, 0)\}$.

(b) The coefficient matrix A for this system is

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix},$$

and its inverse A^{-1} was found in Example 2. Let

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 \\ 5 \\ 6 \end{bmatrix}.$$

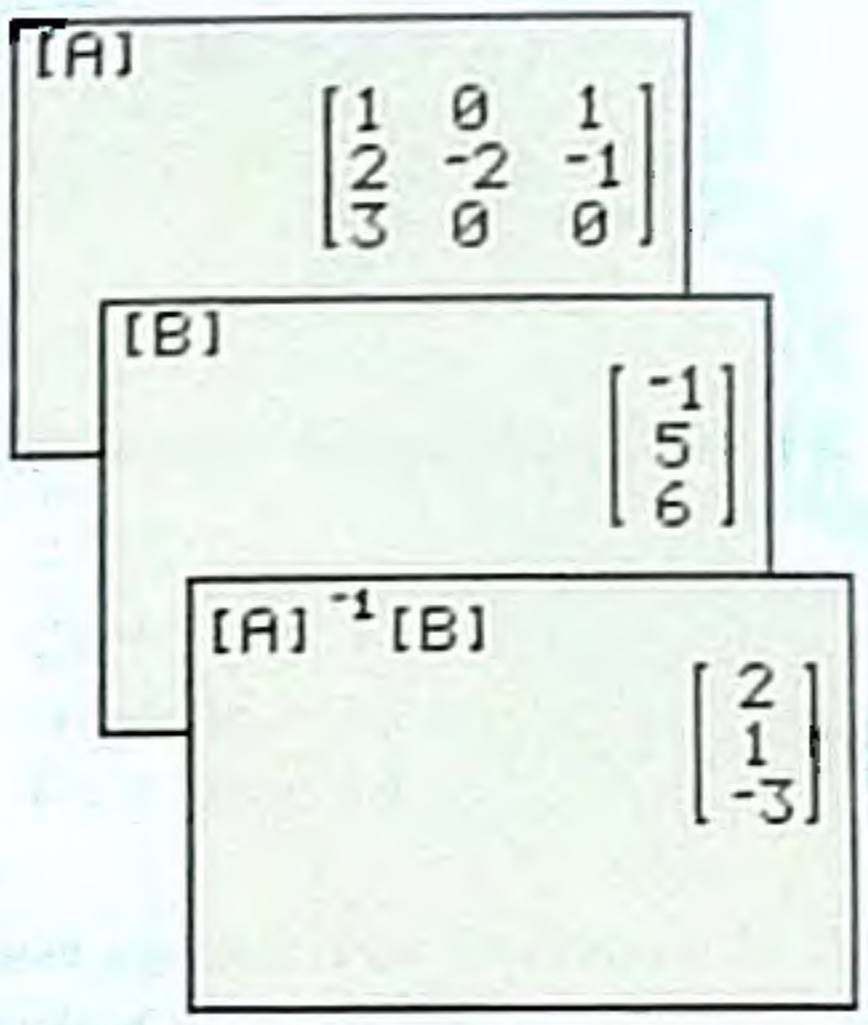
Since $X = A^{-1}B$, we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}}_{A^{-1} \text{ from Example 2}} \begin{bmatrix} -1 \\ 5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}.$$

The solution set is $\{(2, 1, -3)\}$.

(b) Figure 30 shows the coefficient matrix [A] and the column matrix of constants [B]. Note that when $[\text{A}]^{-1}[\text{B}]$ is determined, it is not necessary to display $[\text{A}]^{-1}$. As long as the inverse exists, the product will be computed.



```
[A]
              [1  0  1 ]
              [2 -2 -1 ]
              [3  0  0 ]
[B]
                    [-1]
                    [ 5]
                    [ 6]
[A]⁻¹[B]
                    [ 2]
                    [ 1]
                    [-3]
```

Figure 30

✓ ***Now Try Exercises 37 and 51.***

5.8 Exercises

1. Show that for

$$A = \begin{bmatrix} -2 & 4 & 0 \\ 3 & 5 & 9 \\ 0 & 8 & -6 \end{bmatrix} \quad \text{and} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$I_3A = A$. (This result, along with that of Example 1, illustrates that the commutative property holds when one of the matrices is an identity matrix.)

2. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Show that $AI_2 = I_2A = A$, thus proving that I_2 is the identity element for matrix multiplication for 2 × 2 square matrices.

Decide whether or not the given matrices are inverses of each other. (Hint: Check to see whether their products are the identity matrix I_n.)

3. $\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$ and $\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$

4. $\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$

5. $\begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}$ and $\begin{bmatrix} -5 & -2 \\ -3 & -1 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$

7. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -2 \\ 1 & -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Find the inverse, if it exists, for each matrix. See Examples 2 and 3.

11. $\begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$ **12.** $\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$ **13.** $\begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix}$

14. $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$ **15.** $\begin{bmatrix} 5 & 10 \\ -3 & -6 \end{bmatrix}$ **16.** $\begin{bmatrix} -6 & 4 \\ -3 & 2 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ **18.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ **19.** $\begin{bmatrix} 2 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

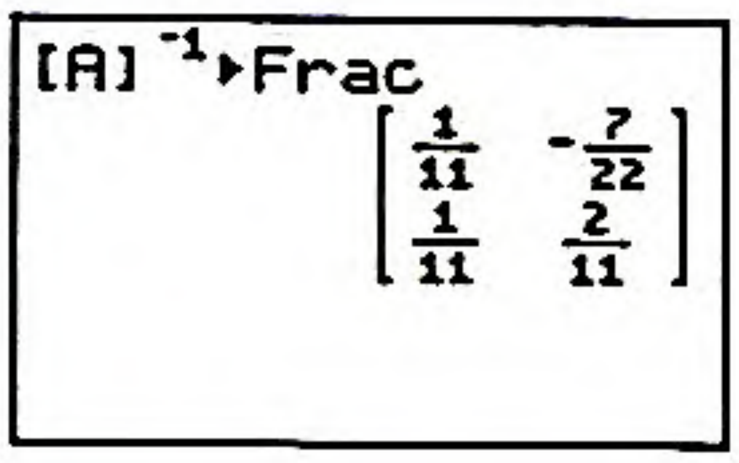
20. $\begin{bmatrix} -2 & 2 & 4 \\ -3 & 4 & 5 \\ 1 & 0 & 2 \end{bmatrix}$ **21.** $\begin{bmatrix} 2 & 2 & -4 \\ 2 & 6 & 0 \\ -3 & -3 & 5 \end{bmatrix}$ **22.** $\begin{bmatrix} 2 & 4 & 6 \\ -1 & -4 & -3 \\ 0 & 1 & -1 \end{bmatrix}$

23. $\begin{bmatrix} 1 & 1 & 0 & 2 \\ 2 & -1 & 1 & -1 \\ 3 & 3 & 2 & -2 \\ 1 & 2 & 1 & 0 \end{bmatrix}$ **24.** $\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & -1 & 1 \\ -2 & 2 & -2 & 4 \\ 0 & 2 & -3 & 1 \end{bmatrix}$ **25.** $\begin{bmatrix} 3 & 2 & 0 & -1 \\ 2 & 0 & 1 & 2 \\ 1 & 2 & -1 & 0 \\ 2 & -1 & 1 & 1 \end{bmatrix}$

26. Explain the process for finding the inverse of a matrix.

Each graphing calculator screen shows A^{-1} for some matrix A. Find each matrix A. (Hint: $(A^{-1})^{-1} = A$.)

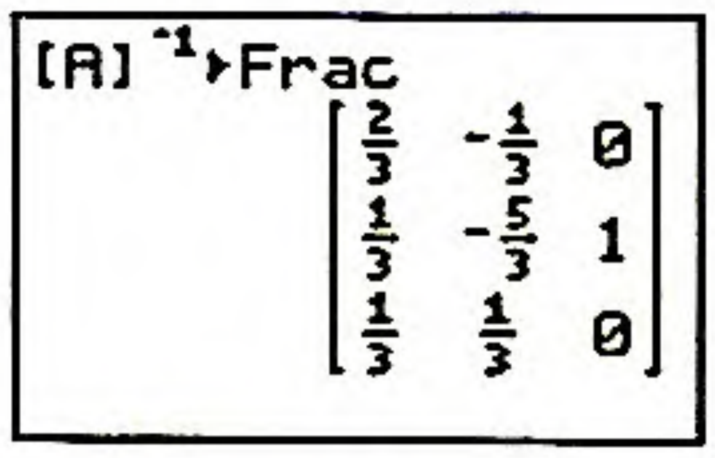
27.



[A]$^{-1}$▸Frac

$\begin{bmatrix} \frac{1}{11} & -\frac{7}{22} \\ \frac{1}{11} & \frac{2}{11} \end{bmatrix}$

28.



[A]$^{-1}$▸Frac

$\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{5}{3} & 1 \\ \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$

Relating Concepts

For individual or collaborative investigation *(Exercises 29–34)*

It can be shown that the inverse of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is*

$$A^{-1} = \begin{bmatrix} \dfrac{d}{ad - bc} & \dfrac{-b}{ad - bc} \\ \dfrac{-c}{ad - bc} & \dfrac{a}{ad - bc} \end{bmatrix}.$$

***Work Exercises 29–34 in order,** to discover a connection between the inverse and the determinant of a 2×2 matrix.*

29. With respect to the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, what do we call $ad - bc$?

30. Refer to A^{-1} as given above, and write it using determinant notation.

31. Write A^{-1} using scalar multiplication, where the scalar is $\frac{1}{|A|}$.

32. Explain how the inverse of matrix A can be found using a determinant.

33. Use the method described here to find the inverse of $A = \begin{bmatrix} 4 & 2 \\ 7 & 3 \end{bmatrix}$.

34. Complete the following statement: The inverse of a 2×2 matrix A does not exist if the determinant of A has value ______. (*Hint:* Look at the denominators in A^{-1} as given earlier.)

Solve each system by using the inverse of the coefficient matrix. See Example 4.

35. $-x + y = 1$
$2x - y = 1$

36. $x + y = 5$
$x - y = -1$

37. $2x - y = -8$
$3x + y = -2$

38. $x + 3y = -12$
$2x - y = 11$

39. $3x + 4y = -3$
$-5x + 8y = 16$

40. $2x - 3y = 10$
$2x + 2y = 5$

41. $6x + 9y = 3$
$-8x + 3y = 6$

42. $5x - 3y = 0$
$10x + 6y = -4$

43. $0.2x + 0.3y = -1.9$
$0.7x - 0.2y = 4.6$

44. $0.5x + 0.2y = 0.8$
$0.3x - 0.1y = 0.7$

45. $\frac{1}{2}x + \frac{1}{3}y = \frac{49}{18}$
$\frac{1}{2}x + 2y = \frac{4}{3}$

46. $\frac{1}{5}x + \frac{1}{7}y = \frac{12}{5}$
$\frac{1}{10}x + \frac{1}{3}y = \frac{5}{6}$

Concept Check *Show that the matrix inverse method cannot be used to solve each system.*

47. $7x - 2y = 3$
$14x - 4y = 1$

48. $x - 2y + 3z = 4$
$2x - 4y + 6z = 8$
$3x - 6y + 9z = 14$

Solve each system by using the inverse of the coefficient matrix. For Exercises 51–56, the inverses were found in Exercises 19–24. See Example 4.

49. $x + y + z = 6$
$2x + 3y - z = 7$
$3x - y - z = 6$

50. $2x + 5y + 2z = 9$
$4x - 7y - 3z = 7$
$3x - 8y - 2z = 9$

51. $2x + 3y + 3z = 1$
$x + 4y + 3z = 0$
$x + 3y + 4z = -1$

52. $-2x + 2y + 4z = 3$
$-3x + 4y + 5z = 1$
$x + 2z = 2$

53. $2x + 2y - 4z = 12$
$2x + 6y = 16$
$-3x - 3y + 5z = -20$

54. $2x + 4y + 6z = 4$
$-x - 4y - 3z = 8$
$y - z = -4$

55. $x + y + 2w = 3$
$2x - y + z - w = 3$
$3x + 3y + 2z - 2w = 5$
$x + 2y + z = 3$

56. $x - 2y + 3z = 1$
$y - z + w = -1$
$-2x + 2y - 2z + 4w = 2$
$2y - 3z + w = -3$

(Modeling) *Solve each problem.*

57. *Plate-Glass Sales* The amount of plate-glass sales S (in millions of dollars) can be affected by the number of new building contracts B issued (in millions) and automobiles A produced (in millions). A plate-glass company in California wants to forecast future sales by using the past three years of sales. The totals for the three years are given in the table. To describe the relationship among these variables, we can use the equation

S	A	B
602.7	5.543	37.14
656.7	6.933	41.30
778.5	7.638	45.62

$$S = a + bA + cB,$$

where the coefficients a, b, and c are constants that must be determined before the equation can be used. (*Source:* Makridakis, S., and S. Wheelwright, *Forecasting Methods for Management,* John Wiley and Sons.)

(a) Substitute the values for S, A, and B for each year from the table into the equation $S = a + bA + cB$, and obtain three linear equations involving a, b, and c.

(b) Use a graphing calculator to solve this linear system for a, b, and c. Use matrix inverse methods.

(c) Write the equation for S using these values for the coefficients.

(d) For the next year it is estimated that $A = 7.752$ and $B = 47.38$. Predict S. (The actual value for S was \$77.6.)

(e) It is predicted that in 6 yr, $A = 8.9$ and $B = 66.25$. Find the value of S in this situation and discuss its validity.

58. ***Tire Sales*** The number of automobile tire sales is dependent on several variables. In one study the relationship among annual tire sales S (in thousands of dollars), automobile registrations R (in millions), and personal disposable income I (in millions of dollars) was investigated. The results for three years are given in the table. To describe the relationship among these variables, we can use the equation

S	R	I
10,170	112.9	307.5
15,305	132.9	621.63
21,289	155.2	1937.13

$$S = a + bR + cI,$$

where the coefficients a, b, and c are constants that must be determined before the equation can be used. (*Source:* Jarrett, J., *Business Forecasting Methods*, Basil Blackwell, Ltd.)

(a) Substitute the values for S, R, and I for each year from the table into the equation $S = a + bR + cI$, and obtain three linear equations involving a, b, and c.

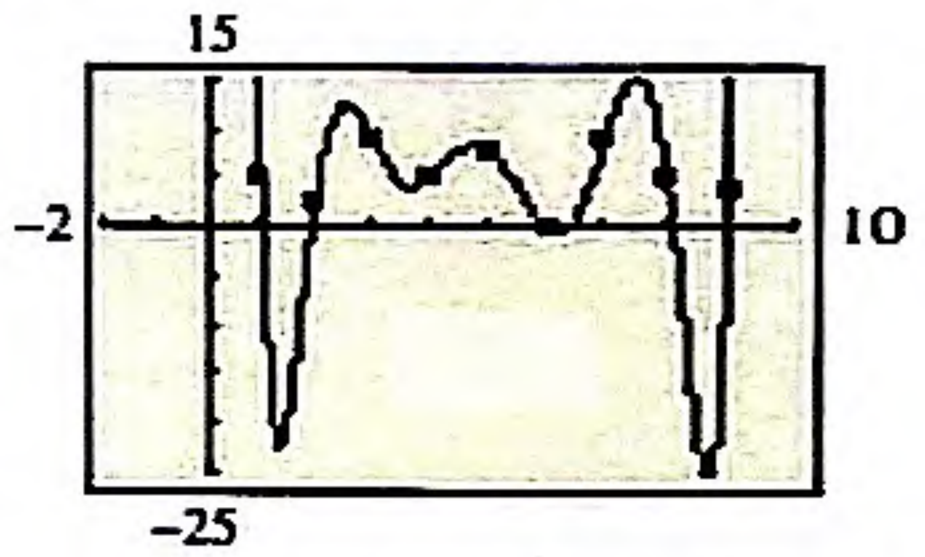
(b) Use a graphing calculator to solve this linear system for a, b, and c. Use matrix inverse methods.

(c) Write the equation for S using these values for the coefficients.

(d) If $R = 117.6$ and $I = 310.73$, predict S. (The actual value for S was 11,314.)

(e) If $R = 143.8$ and $I = 829.06$, predict S. (The actual value for S was 18,481.)

59. ***Social Security Numbers*** In Exercise 89 of Section 3.4, construction of your own personal Social Security polynomial was discussed. It is also possible to find a polynomial that goes through a given set of points in the plane by using a process called **polynomial interpolation.**



Recall that three points define a second-degree polynomial, four points define a third-degree polynomial, and so on. The only restriction on the points, since polynomials define functions, is that no two distinct points can have the same x-coordinate. Using the same SSN (539-58-0954) as in that exercise, we can find an eighth-degree polynomial that lies on the nine points with x-coordinates 1 through 9 and y-coordinates that are digits of the SSN: $(1, 5)$, $(2, 3)$, $(3, 9)$, ..., $(9, 4)$. This is done by writing a system of nine equations with nine variables, which is then solved by the inverse matrix method. The graph of this polynomial is shown. Find such a polynomial using your own SSN.

Use a graphing calculator to find the inverse of each matrix. Give as many decimal places as the calculator shows. See Example 2.

60. $\begin{bmatrix} \sqrt{2} & 0.5 \\ -17 & \frac{1}{2} \end{bmatrix}$

61. $\begin{bmatrix} \frac{2}{3} & 0.7 \\ 22 & \sqrt{3} \end{bmatrix}$

62. $\begin{bmatrix} 1.4 & 0.5 & 0.59 \\ 0.84 & 1.36 & 0.62 \\ 0.56 & 0.47 & 1.3 \end{bmatrix}$

63. $\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \\ 0 & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}$

Use a graphing calculator and the method of matrix inverses to solve each system. Give as many decimal places as the calculator shows. See Example 4.

64.
$$\begin{aligned} x - \sqrt{2}y &= 2.6 \\ 0.75x + y &= -7 \end{aligned}$$

65.
$$\begin{aligned} 2.1x + y &= \sqrt{5} \\ \sqrt{2}x - 2y &= 5 \end{aligned}$$

66.
$$\begin{aligned} \pi x + ey + \sqrt{2}z &= 1 \\ ex + \pi y + \sqrt{2}z &= 2 \\ \sqrt{2}x + ey + \pi z &= 3 \end{aligned}$$

67.
$$\begin{aligned} (\log 2)x + (\ln 3)y + (\ln 4)z &= 1 \\ (\ln 3)x + (\log 2)y + (\ln 8)z &= 5 \\ (\log 12)x + (\ln 4)y + (\ln 8)z &= 9 \end{aligned}$$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, *and let O be the* 2×2 *zero matrix. Show that the statements in Exercises 68 and 69 are true.*

68. $A \cdot O = O \cdot A = O$

69. For square matrices A and B of the same dimension, if $AB = O$ and if A^{-1} exists, then $B = O$.

70. Prove that any square matrix has no more than one inverse.

71. Give an example of two matrices A and B, where $(AB)^{-1} \neq A^{-1}B^{-1}$.

72. Suppose A and B are matrices, where A^{-1}, B^{-1}, and AB all exist. Show that $(AB)^{-1} = B^{-1}A^{-1}$.

73. Let $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, where a, b, and c are nonzero real numbers. Find A^{-1}.

74. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$. Show that $A^3 = I_3$, and use this result to find the inverse of A.

75. What are the inverses of I_n, $-A$ (in terms of A), and kA (k a scalar)?

76. Discuss the similarities and differences between solving the linear equation $ax = b$ and solving the matrix equation $AX = B$.

Chapter 5 Test Prep

Key Terms

5.1
- linear equation (first-degree equation) in *n* unknowns
- system of equations
- solutions of a system of equations
- system of linear equations (linear system)
- consistent system
- independent equations
- inconsistent system
- dependent equations
- equivalent systems
- ordered triple

5.2
- matrix (matrices)
- element (of a matrix)
- augmented matrix
- dimension (of a matrix)

5.3
- determinant
- minor
- cofactor
- expansion by a row or column
- Cramer's rule

5.4
- partial fraction decomposition
- partial fraction

5.5
- nonlinear system

5.6
- half-plane
- boundary
- linear inequality in two variables
- system of inequalities
- linear programming
- constraints
- objective function
- region of feasible solutions
- vertex (corner point)

5.7
- square matrix
- row matrix
- column matrix
- zero matrix
- additive inverse (negative) of a matrix
- scalar

5.8
- identity matrix
- multiplicative inverse (of a matrix)

New Symbols

(a, b, c)	ordered triple	D, D_x, D_y	determinants used in Cramer's rule
$[A]$	matrix A (graphing calculator symbolism)	I_2, I_3	identity matrices
$\lvert A \rvert$	determinant of matrix A	A^{-1}	multiplicative inverse of matrix A
a_{ij}	the element in row i, column j, of a matrix		

(continued)

Concepts	Examples

Fundamental Theorem of Linear Programming
The optimal value for a linear programming problem occurs at a vertex of the region of feasible solutions.

Solving a Linear Programming Problem

Step 1 Write the objective function and all necessary constraints.

Step 2 Graph the region of feasible solutions.

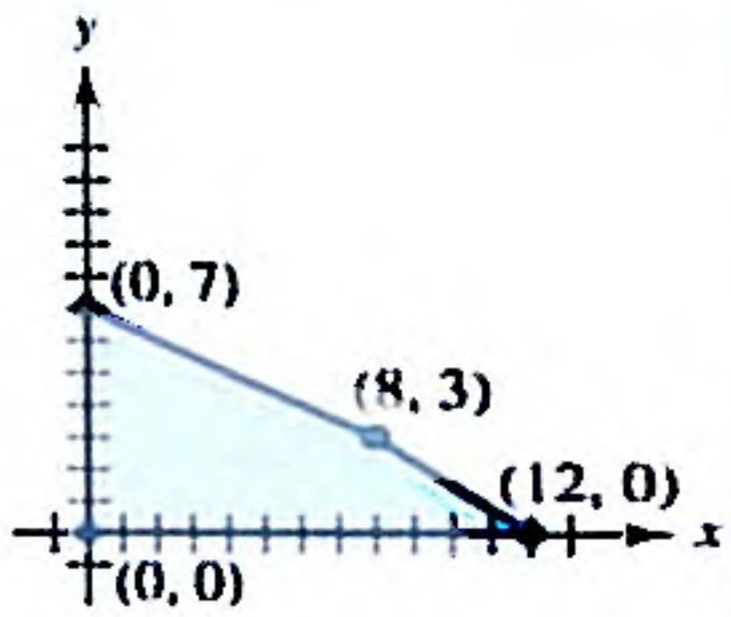
Step 3 Identify all vertices (corner points).

Step 4 Find the value of the objective function at each vertex.

Step 5 The solution is given by the vertex producing the optimal value of the objective function.

The region of feasible solutions for

$$\begin{aligned} x + 2y &\le 14 \\ 3x + 4y &\le 36 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$



is given in the figure. Maximize the objective function $8x + 12y$.

Vertex Point	Value of $8x + 12y$
(0, 0)	0
(0, 7)	84
(8, 3)	100 ← Maximum
(12, 0)	96

The objective function is maximized for $x = 8$ and $y = 3$.

5.7 Properties of Matrices

Addition and Subtraction of Matrices
To add (subtract) matrices of the same dimension, add (subtract) the corresponding elements.

Find the sum or difference.

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & 4 & 9 \end{bmatrix} + \begin{bmatrix} -8 & 12 & 1 \\ 5 & 3 & -3 \end{bmatrix} = \begin{bmatrix} -6 & 15 & 0 \\ 5 & 7 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -1 \\ -8 & 8 \end{bmatrix} - \begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ -11 & 14 \end{bmatrix}$$

Scalar Multiplication
To multiply a matrix by a scalar, multiply each element of the matrix by the scalar.

Find the scalar product.

$$3\begin{bmatrix} 6 & 2 \\ 1 & -2 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 18 & 6 \\ 3 & -6 \\ 0 & 24 \end{bmatrix}$$

Multiplication of Matrices
The product AB of an $m \times n$ matrix A and an $n \times p$ matrix B is found as follows. To get the ith row, jth column element of the $m \times p$ matrix AB, multiply each element in the ith row of A by the corresponding element in the jth column of B. The sum of these products will give the element of row i, column j of AB.

Find the matrix product.

$$\underset{3 \times 3}{\begin{bmatrix} 1 & -2 & 3 \\ 5 & 0 & 4 \\ -8 & 7 & -7 \end{bmatrix}} \underset{3 \times 1}{\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}} = \begin{bmatrix} 1(1) + (-2)(-2) + 3(3) \\ 5(1) + 0(-2) + 4(3) \\ -8(1) + 7(-2) + (-7)(3) \end{bmatrix}$$

$$= \underset{3 \times 1}{\begin{bmatrix} 14 \\ 17 \\ -43 \end{bmatrix}}$$

Concepts | **Examples**

5.8 Matrix Inverses

Finding an Inverse Matrix

To obtain A^{-1} for any $n \times n$ matrix A for which A^{-1} exists, follow these steps.

Step 1 Form the augmented matrix $[A \,|\, I_n]$, where I_n is the $n \times n$ identity matrix.

Step 2 Perform row transformations on $[A \,|\, I_n]$ to obtain a matrix of the form $[I_n \,|\, B]$.

Step 3 Matrix B is A^{-1}.

Find A^{-1} if $A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 5 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & -2 \\ 2 & 1 & 0 & 1 \end{array}\right] \quad -2\text{R}2 + \text{R}1$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 5 \end{array}\right] \quad -2\text{R}1 + \text{R}2$$

$$\underbrace{}_{I_2} \qquad \underbrace{}_{A^{-1}}$$

Therefore, $A^{-1} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$.

Chapter 5 Review Exercises

Use the substitution or elimination method to solve each linear system. Identify any inconsistent systems or systems with infinitely many solutions. If the system has infinitely many solutions, write the solution set with y arbitrary.

1. $\begin{aligned} 2x + 6y &= 6 \\ 5x + 9y &= 9 \end{aligned}$

2. $\begin{aligned} 3x - 5y &= 7 \\ 2x + 3y &= 30 \end{aligned}$

3. $\begin{aligned} x + 5y &= 9 \\ 2x + 10y &= 18 \end{aligned}$

4. $\begin{aligned} \frac{1}{6}x + \frac{1}{3}y &= 8 \\ \frac{1}{4}x + \frac{1}{2}y &= 12 \end{aligned}$

5. $\begin{aligned} y &= -x + 3 \\ 2x + 2y &= 1 \end{aligned}$

6. $\begin{aligned} 0.2x + 0.5y &= 6 \\ 0.4x + y &= 9 \end{aligned}$

7. $\begin{aligned} 3x - 2y &= 0 \\ 9x + 8y &= 7 \end{aligned}$

8. $\begin{aligned} 6x + 10y &= -11 \\ 9x + 6y &= -3 \end{aligned}$

9. $\begin{aligned} 2x - 5y + 3z &= -1 \\ x + 4y - 2z &= 9 \\ -x + 2y + 4z &= 5 \end{aligned}$

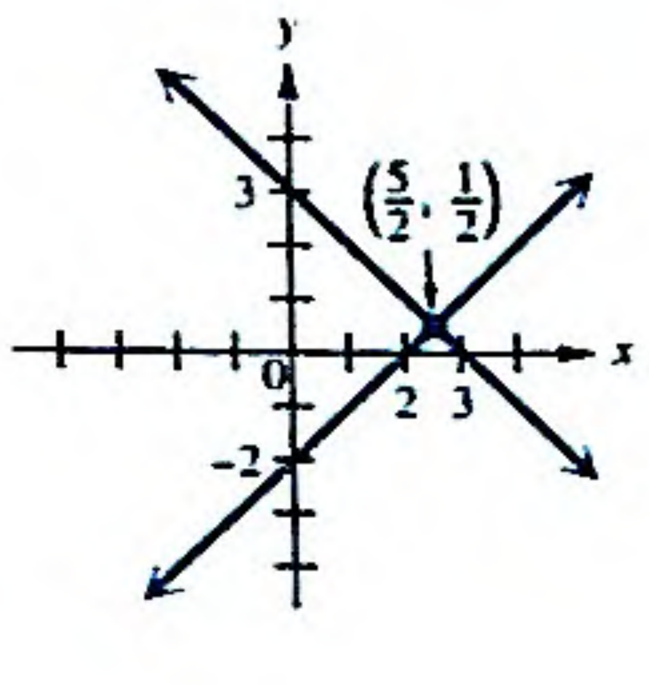
10. $\begin{aligned} 4x + 3y + z &= -8 \\ 3x + y - z &= -6 \\ x + y + 2z &= -2 \end{aligned}$

11. $\begin{aligned} 5x - y &= 26 \\ 4y + 3z &= -4 \\ 3x + 3z &= 15 \end{aligned}$

12. $\begin{aligned} x + z &= 2 \\ 2y - z &= 2 \\ -x + 2y &= -4 \end{aligned}$

13. ***Concept Check*** Create your own inconsistent system of two equations.

14. ***Connecting Graphs with Equations*** Determine the system of equations illustrated in the graph. Write equations in standard form.



Write a system of linear equations, and then use the system to solve the problem.

15. *Meal Planning* A cup of uncooked rice contains 15 g of protein and 810 calories. A cup of uncooked soybeans contains 22.5 g of protein and 270 calories. How many cups of each should be used for a meal containing 9.5 g of protein and 324 calories?

16. *Order Quantities* A company sells recordable CDs for \$0.80 each and play-only CDs for \$0.60 each. The company receives \$76.00 for an order of 100 CDs. However, the customer neglected to specify how many of each type to send. Determine the number of each type of CD that should be sent.

17. *Indian Weavers* The Waputi Indians make woven blankets, rugs, and skirts. Each blanket requires 24 hr for spinning the yarn, 4 hr for dyeing the yarn, and 15 hr for weaving. Rugs require 30, 5, and 18 hr and skirts 12, 3, and 9 hr, respectively. If there are 306, 59, and 201 hr available for spinning, dyeing, and weaving, respectively, how many of each item can be made? (*Hint:* Simplify the equations you write, if possible, before solving the system.)

18. *(Modeling) Populations of Minorities in the United States* The current and estimated resident populations (in percent) of blacks and Hispanics in the United States for the years 1990–2050 are modeled by the following linear functions.

$$y = 0.0515x + 12.3 \quad \text{Blacks}$$
$$y = 0.255x + 9.01 \quad \text{Hispanics}$$

In each case, x represents the number of years since 1990. (*Source:* U.S. Census Bureau.)

(a) Solve the system to find the year when these population percents will be equal.
(b) What percent of the U.S. resident population will be black or Hispanic in the year found in part (a)?
(c) Use a calculator graph of the system to support your algebraic solution.
(d) Which population is increasing more rapidly? (*Hint:* Consider the slopes of the lines.)

19. *(Modeling) Heart Rate* In a study, a group of athletes was exercised to exhaustion. Let x represent an athlete's heart rate 5 sec after stopping exercise and y this rate 10 sec after stopping. It was found that the maximum heart rate H for these athletes satisfied the two equations

$$H = 0.491x + 0.468y + 11.2$$
$$H = -0.981x + 1.872y + 26.4.$$

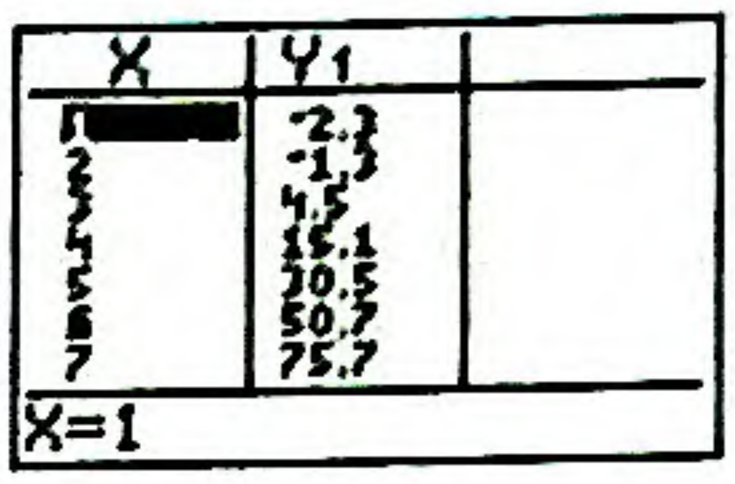
If an athlete had maximum heart rate $H = 180$, determine x and y graphically. Interpret your answer. (*Source:* Thomas, V., *Science and Sport*, Faber and Faber.)

20. *(Modeling) Equilibrium Supply and Demand* Let the supply and demand equations for units of backpacks be

$$\text{supply} \quad p = \frac{3}{2}q \quad \text{and} \quad \text{demand} \quad p = 81 - \frac{3}{4}q$$

(a) Graph these equations on the same axes.
(b) Find the equilibrium demand.
(c) Find the equilibrium price.

21. *(Modeling)* Find the equation of the vertical parabola that passes through the points shown in the table.



X	Y1	
1	-2.3	
2	-1.3	
3	4.5	
4	15.1	
5	30.5	
6	50.7	
7	75.7	

X=1

22. *(Modeling)* Knowing that the equation of a circle may be written in the form

$$x^2 + y^2 + ax + by + c = 0,$$

find the equation of the circle passing through the points $(-3, -7)$, $(4, -8)$, and $(1, 1)$.

Find solutions for each system with the specified arbitrary variable.

23. $\begin{aligned} 3x - 4y + z &= 2 \\ 2x + y &= 1 \end{aligned}$; x

24. $\begin{aligned} 2x - 6y + 4z &= 5 \\ 5x + y - 3z &= 1 \end{aligned}$; z

Use the Gauss-Jordan method to solve each system.

25. $\begin{aligned} 5x + 2y &= -10 \\ 3x - 5y &= -6 \end{aligned}$

26. $\begin{aligned} 2x + 3y &= 10 \\ -3x + y &= 18 \end{aligned}$

27. $\begin{aligned} 3x + y &= -7 \\ x - y &= -5 \end{aligned}$

28. $\begin{aligned} 2x - y + 4z &= -1 \\ -3x + 5y - z &= 5 \\ 2x + 3y + 2z &= 3 \end{aligned}$

29. $\begin{aligned} x - z &= -3 \\ y + z &= 6 \\ 2x - 3z &= -9 \end{aligned}$

30. $\begin{aligned} 2x - y + z &= 4 \\ x + 2y - z &= 0 \\ 3x + y - 2z &= 1 \end{aligned}$

Solve each problem by using the Gauss-Jordan method to solve a system of equations.

31. ***Mixing Teas*** Three kinds of tea worth \$4.60, \$5.75, and \$6.50 per lb are to be mixed to get 20 lb of tea worth \$5.25 per lb. The amount of \$4.60 tea used is to be equal to the total amount of the other two kinds together. How many pounds of each tea should be used?

32. ***Mixing Solutions*** A 5% solution of a drug is to be mixed with some 15% solution and some 10% solution to get 20 ml of 8% solution. The amount of 5% solution used must be 2 ml more than the sum of the other two solutions. How many milliliters of each solution should be used?

33. *(Modeling)* ***Master's Degrees*** During the period 1970–2008, the numbers of master's degrees awarded to both males and females grew, but degrees earned by females grew at a greater rate. If $x = 0$ represents 1970 and $x = 38$ represents 2008, the number of master's degrees earned (in thousands) are closely modeled by the following system.

$$\begin{aligned} y &= 2.81x + 121 \quad \text{Males} \\ y &= 7.25x + 65.5 \quad \text{Females} \end{aligned}$$

Solve the system to find the year in which males and females earned the same number of master's degrees. What was the total number of master's degrees earned in that year? *(Source:* U.S. Census Bureau.)

34. *(Modeling) Comparing Prices* One refrigerator sells for \$700 and uses \$85 worth of electricity per year. A second refrigerator is \$100 more expensive but costs only \$25 per year to run. Assuming that there are no repair costs, the costs to run the refrigerators over a 10-yr period are given by the following system of equations. Here, y represents the total cost in dollars, and x is time in years.

$$\begin{aligned} y &= 700 + 85x \\ y &= 800 + 25x \end{aligned}$$

In how many years will the costs for the two refrigerators be equal? What are the equivalent costs at that time?

Evaluate each determinant.

35. $\begin{vmatrix} -1 & 8 \\ 2 & 9 \end{vmatrix}$

36. $\begin{vmatrix} -2 & 4 \\ 0 & 3 \end{vmatrix}$

37. $\begin{vmatrix} x & 4x \\ 2x & 8x \end{vmatrix}$

38. $\begin{vmatrix} -2 & 4 & 1 \\ 3 & 0 & 2 \\ -1 & 0 & 3 \end{vmatrix}$

39. $\begin{vmatrix} -1 & 2 & 3 \\ 4 & 0 & 3 \\ 5 & -1 & 2 \end{vmatrix}$

40. $\begin{vmatrix} -3 & 2 & 7 \\ 6 & -4 & -14 \\ 7 & 1 & 4 \end{vmatrix}$

Solve each determinant equation.

41. $\begin{vmatrix} 3x & 7 \\ -x & 4 \end{vmatrix} = 8$

42. $\begin{vmatrix} 6x & 2 & 0 \\ 1 & 5 & 3 \\ x & 2 & -1 \end{vmatrix} = 2x$

Solve each system by Cramer's rule if possible. Identify any inconsistent systems or systems with infinitely many solutions. (Use another method if Cramer's rule cannot be used.)

43. $\begin{aligned} 3x + 7y &= 2 \\ 5x - y &= -22 \end{aligned}$

44. $\begin{aligned} 3x + y &= -1 \\ 5x + 4y &= 10 \end{aligned}$

45. $\begin{aligned} 6x + y &= -3 \\ 12x + 2y &= 1 \end{aligned}$

46. $\begin{aligned} 3x + 2y + z &= 2 \\ 4x - y + 3z &= -16 \\ x + 3y - z &= 12 \end{aligned}$

47. $\begin{aligned} x + y &= -1 \\ 2y + z &= 5 \\ 3x - 2z &= -28 \end{aligned}$

48. $\begin{aligned} 5x - 2y - z &= 8 \\ -5x + 2y + z &= -8 \\ x - 4y - 2z &= 0 \end{aligned}$

Find the partial fraction decomposition for each rational expression.

49. $\dfrac{2}{3x^2 - 5x + 2}$

50. $\dfrac{11 - 2x}{x^2 - 8x + 16}$

51. $\dfrac{5 - 2x}{(x^2 + 2)(x - 1)}$

52. $\dfrac{x^3 + 2x^2 - 3}{x^4 - 4x^2 + 4}$

Solve each nonlinear system.

53. $\begin{aligned} y &= 2x + 10 \\ x^2 + y &= 13 \end{aligned}$

54. $\begin{aligned} x^2 &= 2y - 3 \\ x + y &= 3 \end{aligned}$

55. $\begin{aligned} x^2 + y^2 &= 17 \\ 2x^2 - y^2 &= 31 \end{aligned}$

56. $\begin{aligned} 2x^2 + 3y^2 &= 30 \\ x^2 + y^2 &= 13 \end{aligned}$

57. $\begin{aligned} xy &= -10 \\ x + 2y &= 1 \end{aligned}$

58. $\begin{aligned} xy + 2 &= 0 \\ y - x &= 3 \end{aligned}$

59. $\begin{aligned} x^2 + 2xy + y^2 &= 4 \\ x - 3y &= -2 \end{aligned}$

60. $\begin{aligned} x^2 + 2xy &= 15 + 2x \\ xy - 3x + 3 &= 0 \end{aligned}$

61. $\begin{aligned} 2x^2 - 3y^2 &= 18 \\ 2x^2 - 2y^2 &= 14 \end{aligned}$

62. Find all values of b such that the straight line $3x - y = b$ touches the circle $x^2 + y^2 = 25$ at only one point.

63. Do the circle $x^2 + y^2 = 144$ and the line $x + 2y = 8$ have any points in common? If so, what are they?

64. Find the equation of the line passing through the points of intersection of the graphs of $x^2 + y^2 = 20$ and $x^2 - y = 0$.

Graph the solution set of each system of inequalities.

65. $x + y \le 6$
$2x - y \ge 3$

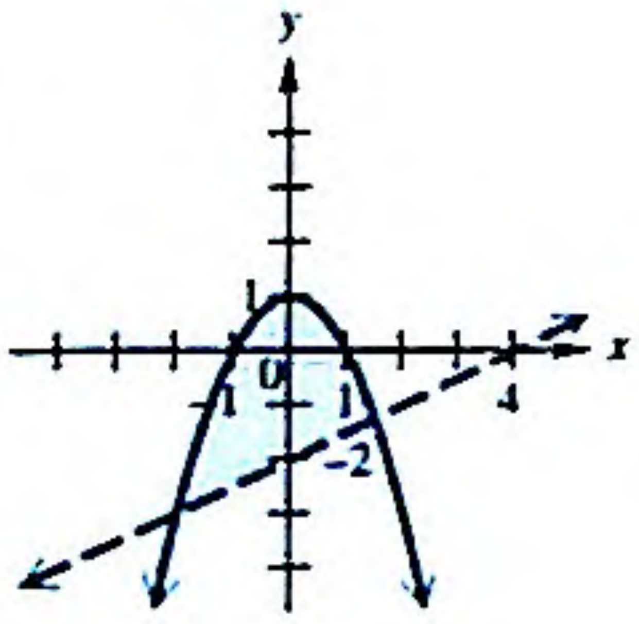
66. $y \le \frac{1}{3}x - 2$
$y^2 \le 16 - x^2$

67. Find $x \ge 0$ and $y \ge 0$ such that

$$3x + 2y \le 12$$
$$5x + y \ge 5$$

and $2x + 4y$ is maximized.

68. ***Connecting Graphs with Equations*** Determine the system of inequalities illustrated in the graph. Write them in standard form.



Solve each linear programming problem.

69. ***Cost of Nutrients*** Certain laboratory animals must have at least 30 g of protein and at least 20 g of fat per feeding period. These nutrients come from food A, which costs \$0.18 per unit and supplies 2 g of protein and 4 g of fat; and from food B, which costs \$0.12 per unit and has 6 g of protein and 2 g of fat. Food B is purchased under a long-term contract requiring that at least 2 units of B be used per serving. How much of each food must be purchased to produce the minimum cost per serving? What is the minimum cost?

70. ***Profit from Farm Animals*** A 4-H member raises only geese and pigs. She wants to raise no more than 16 animals, including no more than 10 geese. She spends \$5 to raise a goose and \$15 to raise a pig, and she has \$180 available for this project. Each goose produces \$6 in profit, and each pig produces \$20 in profit. How many of each animal should she raise to maximize her profit? What is her maximum profit?

Find the value of each variable.

71. $\begin{bmatrix} 5 & x+2 \\ -6y & z \end{bmatrix} = \begin{bmatrix} a & 3x-1 \\ 5y & 9 \end{bmatrix}$

72. $\begin{bmatrix} -6+k & 2 & a+3 \\ -2+m & 3p & 2r \end{bmatrix} + \begin{bmatrix} 3-2k & 5 & 7 \\ 5 & 8p & 5r \end{bmatrix} = \begin{bmatrix} 5 & y & 6a \\ 2m & 11 & -35 \end{bmatrix}$

Perform each operation when possible.

73. $\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} - \begin{bmatrix} 8 \\ -4 \\ 6 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

74. $4\begin{bmatrix} 3 & -4 & 2 \\ 5 & -1 & 6 \end{bmatrix} + \begin{bmatrix} -3 & 2 & 5 \\ 1 & 0 & 4 \end{bmatrix}$

75. $\begin{bmatrix} 2 & 5 & 8 \\ 1 & 9 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 4 \\ 7 & 1 \end{bmatrix}$

76. $\begin{bmatrix} -3 & 4 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 2 & 5 \end{bmatrix}$

77. $\begin{bmatrix} -1 & 0 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} -3 & 4 \\ 2 & 8 \end{bmatrix}$

78. $\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -6 & 5 \end{bmatrix}\begin{bmatrix} 4 & 8 \\ -1 & 2 \end{bmatrix}$

79. $\begin{bmatrix} 3 & 2 & -1 \\ 4 & 0 & 6 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 2 \\ 3 & 1 \end{bmatrix}$

80. $\begin{bmatrix} 1 & -2 & 4 & 2 \\ 0 & 1 & -1 & 8 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

81. $\begin{bmatrix} -2 & 5 & 5 \\ 0 & 1 & 4 \\ 3 & -4 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix}$

82. $\begin{bmatrix} 0 & 1 & 4 \\ 7 & -2 & 9 \\ 10 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 \\ -4 & 7 & 6 \end{bmatrix}$

Find the inverse of each matrix that has an inverse.

83. $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$

84. $\begin{bmatrix} -4 & 2 \\ 0 & 3 \end{bmatrix}$

85. $\begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & -2 & 0 \end{bmatrix}$

86. $\begin{bmatrix} 2 & 3 & 5 \\ -2 & -3 & -5 \\ 1 & 4 & 2 \end{bmatrix}$

Use the method of matrix inverses to solve each system.

87. $\begin{aligned} 5x - 4y &= 1 \\ x + 4y &= 3 \end{aligned}$

88. $\begin{aligned} 2x + y &= 5 \\ 3x - 2y &= 4 \end{aligned}$

89. $\begin{aligned} 3x + 2y + z &= -5 \\ x - y + 3z &= -5 \\ 2x + 3y + z &= 0 \end{aligned}$

90. $\begin{aligned} x + y + z &= 1 \\ 2x - y &= -2 \\ 3y + z &= 2 \end{aligned}$

Chapter 5 Test

Use substitution or elimination to solve each system. Identify any system that is inconsistent or has infinitely many solutions. If a system has infinitely many solutions, express the solution set with y arbitrary.

1. $\begin{aligned} 3x - y &= 9 \\ x + 2y &= 10 \end{aligned}$

2. $\begin{aligned} 6x + 9y &= -21 \\ 4x + 6y &= -14 \end{aligned}$

3. $\begin{aligned} \frac{1}{4}x - \frac{1}{3}y &= -\frac{5}{12} \\ \frac{1}{10}x + \frac{1}{5}y &= \frac{1}{2} \end{aligned}$

4. $\begin{aligned} x - 2y &= 4 \\ -2x + 4y &= 6 \end{aligned}$

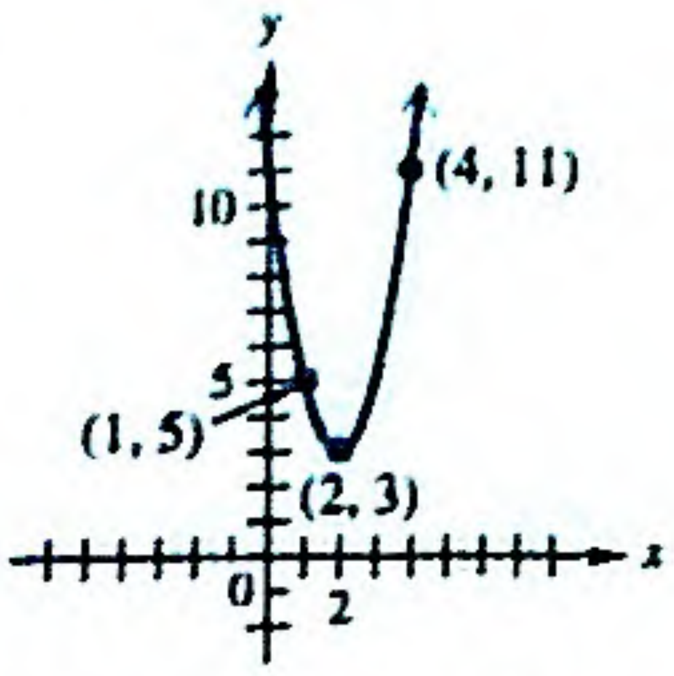
5. $\begin{aligned} 2x + y + z &= 3 \\ x + 2y - z &= 3 \\ 3x - y + z &= 5 \end{aligned}$

Use the Gauss-Jordan method to solve each system.

6. $\begin{aligned} 3x - 2y &= 13 \\ 4x - y &= 19 \end{aligned}$

7. $\begin{aligned} 3x - 4y + 2z &= 15 \\ 2x - y + z &= 13 \\ x + 2y - z &= 5 \end{aligned}$

8. *Connecting Graphs with Equations* Find the equation that defines the parabola shown.



9. *Ordering Supplies* A knitting shop orders yarn from three suppliers in Toronto, Montreal, and Ottawa. One month the shop ordered a total of 100 units of yarn from these suppliers. The delivery costs were $80, $50, and $65 per unit for the orders from Toronto, Montreal, and Ottawa, respectively, with total delivery costs of $5990. The shop ordered the same amount from Toronto and Ottawa. How many units were ordered from each supplier?

Evaluate each determinant.

10. $\begin{vmatrix} 6 & 8 \\ 2 & -7 \end{vmatrix}$

11. $\begin{vmatrix} 2 & 0 & 8 \\ -1 & 7 & 9 \\ 12 & 5 & -3 \end{vmatrix}$

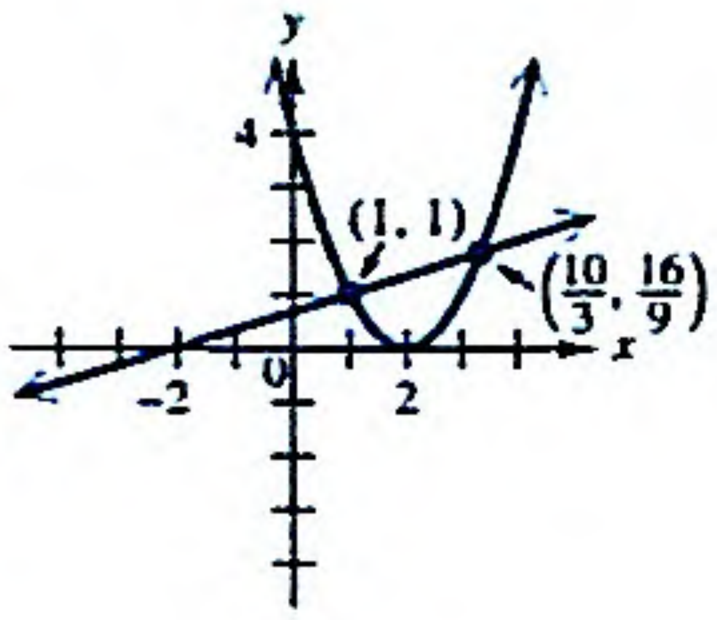
Solve each system by Cramer's rule.

12. $2x - 3y = -33$
$4x + 5y = 11$

13. $x + y - z = -4$
$2x - 3y - z = 5$
$x + 2y + 2z = 3$

14. Find the partial fraction decomposition of $\dfrac{x+2}{x^3 + 2x^2 + x}$.

15. *Connecting Graphs with Equations* Determine the system of equations illustrated in the graph. Write equations in standard form.



Solve each nonlinear system of equations.

16. $2x^2 + y^2 = 6$
$x^2 - 4y^2 = -15$

17. $x^2 + y^2 = 25$
$x + y = 7$

18. *Unknown Numbers* Find two numbers such that their sum is -1 and the sum of their squares is 61.

19. Graph the solution set.

$$x - 3y \geq 6$$
$$y^2 \leq 16 - x^2$$

20. Find $x \geq 0$ and $y \geq 0$ such that

$$x + 2y \leq 24$$
$$3x + 4y \leq 60$$

and $2x + 3y$ is maximized.

21. *Jewelry Profits* The J. J. Gravois Company designs and sells two types of rings: the VIP and the SST. The company can produce up to 24 rings each day using up to 60 total hours of labor. It takes 3 hr to make one VIP ring and 2 hr to make one SST ring. How many of each type of ring should be made daily in order to maximize the company's profit, if the profit on one VIP ring is $30 and the profit on one SST ring is $40? What is the maximum profit?

22. Find the value of each variable in the equation $\begin{bmatrix} 5 & x+6 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} y-2 & 4-x \\ 0 & w+7 \end{bmatrix}$.

Perform each operation when possible.

23. $3\begin{bmatrix} 2 & 3 \\ 1 & -4 \\ 5 & 9 \end{bmatrix} - \begin{bmatrix} -2 & 6 \\ 3 & -1 \\ 0 & 8 \end{bmatrix}$

24. $\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ -6 \end{bmatrix} + \begin{bmatrix} 2 & 8 \\ -7 & 5 \end{bmatrix}$

25. $\begin{bmatrix} 2 & 1 & -3 \\ 4 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 3 & -2 \end{bmatrix}$

26. $\begin{bmatrix} 2 & -4 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix}$

27. *Concept Check* Which of the following properties does not apply to multiplication of matrices?

A. commutative **B.** associative **C.** distributive **D.** identity

Find the inverse, if it exists, of each matrix.

28. $\begin{bmatrix} -8 & 5 \\ 3 & -2 \end{bmatrix}$

29. $\begin{bmatrix} 4 & 12 \\ 2 & 6 \end{bmatrix}$

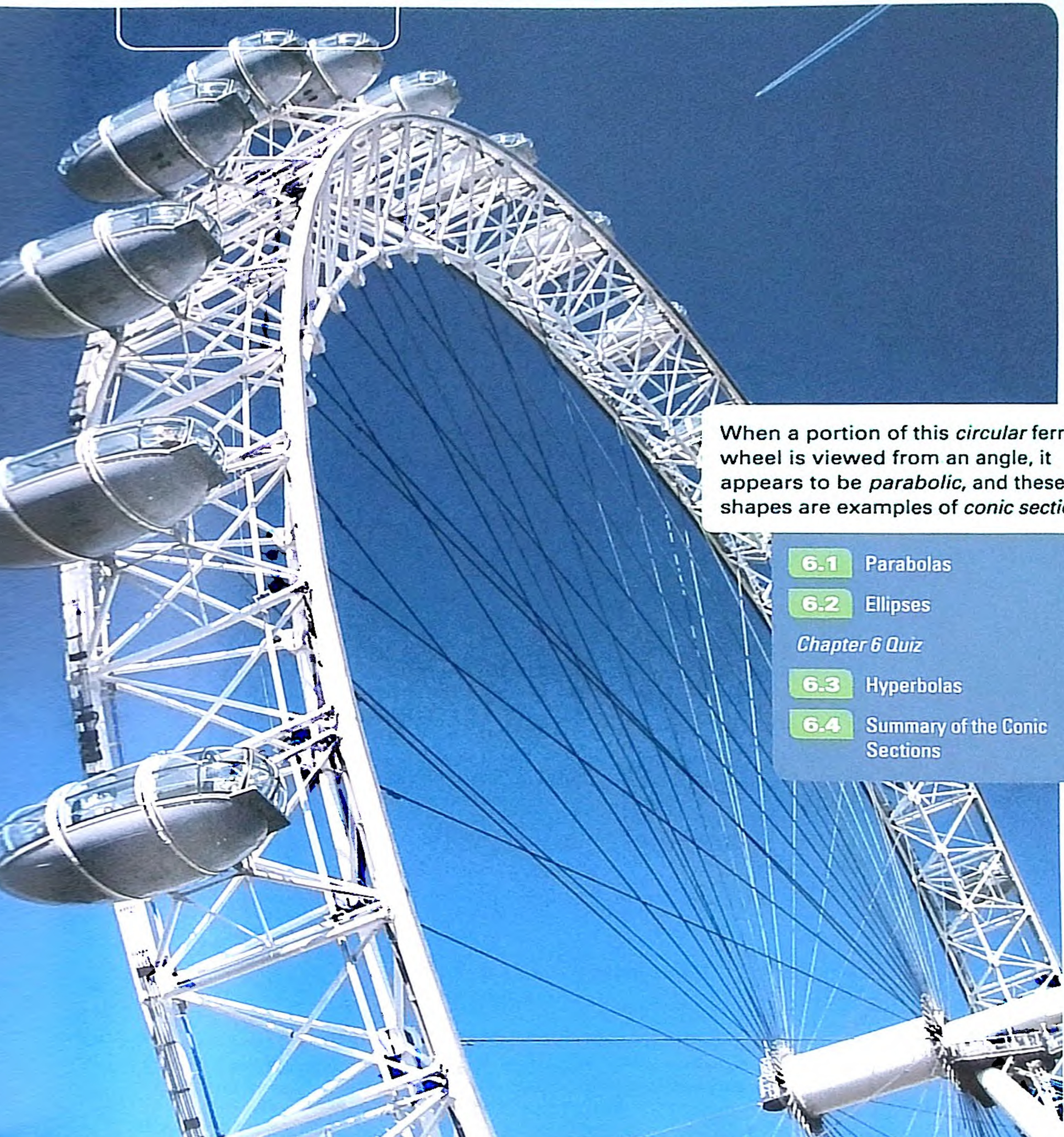
30. $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 7 & 8 \\ -2 & -5 & -7 \end{bmatrix}$

Use matrix inverses to solve each system.

31. $2x + y = -6$
$3x - y = -29$

32. $x + y = 5$
$y - 2z = 23$
$x + 3z = -27$

6 Analytic Geometry



When a portion of this *circular* ferris wheel is viewed from an angle, it appears to be *parabolic*, and these shapes are examples of *conic sections.*

6.1 Parabolas

6.2 Ellipses

Chapter 6 Quiz

6.3 Hyperbolas

6.4 Summary of the Conic Sections

6.1 Exercises

1. ***Concept Check*** Match each equation of a parabola in Column I with its description in Column II.

I	II
(a) $y - 2 = (x + 4)^2$	A. vertex $(-2, 4)$; opens down
(b) $y - 4 = (x + 2)^2$	B. vertex $(-2, 4)$; opens up
(c) $y - 2 = -(x + 4)^2$	C. vertex $(-4, 2)$; opens down
(d) $y - 4 = -(x + 2)^2$	D. vertex $(-4, 2)$; opens up
(e) $x - 2 = (y + 4)^2$	E. vertex $(2, -4)$; opens left
(f) $x - 4 = (y + 2)^2$	F. vertex $(2, -4)$; opens right
(g) $x - 2 = -(y + 4)^2$	G. vertex $(4, -2)$; opens left
(h) $x - 4 = -(y + 2)^2$	H. vertex $(4, -2)$; opens right

2. ***Concept Check*** Match each equation of a parabola in Column I with the appropriate description in Column II.

I	II
(a) $y = 2x^2 + 3x + 9$	A. opens right
(b) $y = -3x^2 + 4x - 2$	B. opens up
(c) $x = 2y^2 - 3y + 9$	C. opens left
(d) $x = -3y^2 - 4y + 2$	D. opens down

Graph each horizontal parabola, and give the domain and range. See Examples 1 and 2.

3. $x + 4 = y^2$
4. $x - 2 = y^2$
5. $x = (y - 3)^2$
6. $x = (y + 1)^2$
7. $x - 2 = (y - 4)^2$
8. $x + 1 = (y + 2)^2$
9. $x - 2 = -3(y - 1)^2$
10. $x - 4 = \frac{1}{2}(y - 1)^2$
11. $-\frac{1}{2}x = (y + 3)^2$
12. $-\frac{1}{3}x = (y - 2)^2$
13. $x = y^2 + 4y + 2$
14. $x = 2y^2 - 4y + 6$
15. $x = -4y^2 - 4y + 3$
16. $x = -2y^2 + 2y - 3$
17. $2x - y^2 + 4y - 6 = 0$
18. $x + 3y^2 + 18y + 22 = 0$
19. $-x = 3y^2 + 6y + 2$
20. $-x = 2y^2 + 4y - 1$

Give the focus, directrix, and axis of symmetry for each parabola. See Example 3.

21. $x^2 = 24y$
22. $x^2 = \frac{1}{8}y$
23. $y = -4x^2$
24. $y = -\frac{1}{9}x^2$
25. $y^2 = -4x$
26. $y^2 = 16x$
27. $x = -32y^2$
28. $x = -16y^2$
29. $(y - 3)^2 = 12(x - 1)$
30. $(y - 2)^2 = 24(x - 3)$
31. $(x - 7)^2 = 16(y + 5)$
32. $(x + 2)^2 = 20(y - 2)$

Write an equation for each parabola with vertex at the origin. See Example 4.

33. focus $(5, 0)$
34. focus $\left(-\frac{1}{2}, 0\right)$
35. directrix $y = -\frac{1}{4}$
36. directrix $x = \frac{1}{3}$
37. through $\left(\sqrt{3}, 3\right)$, opening up
38. through $\left(-2, -2\sqrt{2}\right)$, opening left

39. through $(3, 2)$, symmetric with respect to the x-axis

40. through $(2, -4)$, symmetric with respect to the y-axis

Write an equation for each parabola. See Example 5.

41. vertex $(4, 3)$, focus $(4, 5)$

42. vertex $(-2, 1)$, focus $(-2, -3)$

43. vertex $(-5, 6)$, directrix $x = -12$

44. vertex $(1, 2)$, directrix $x = 4$

Determine the two equations necessary to graph each horizontal parabola using a graphing calculator, and graph it in the viewing window specified. See Example 2.

45. $x = 3y^2 + 6y - 4$; $[-10, 2]$ by $[-4, 4]$

46. $x = -2y^2 + 4y + 3$; $[-10, 6]$ by $[-4, 4]$

47. $x + 2 = -(y + 1)^2$; $[-10, 2]$ by $[-4, 4]$

48. $x - 5 = 2(y - 2)^2$; $[-2, 12]$ by $[-2, 6]$

Relating Concepts

For individual or collaborative investigation *(Exercises 49–52)*

(Modeling) Given three noncollinear points, we can find an equation of the form $x = ay^2 + by + c$ of the horizontal parabola joining them by solving a system of equations. **Work Exercises 49–52 in order,** *to find the equation of the horizontal parabola containing the points $(-5, 1)$, $(-14, -2)$, and $(-10, 2)$.*

49. Write three equations in a, b, and c, by substituting the given values of x and y into the equation $x = ay^2 + by + c$.

50. Solve the system of three equations determined in **Exercise 49.**

51. Does the horizontal parabola open to the left or to the right? Why?

52. Write the equation of the horizontal parabola.

Solve each problem. See Example 6.

53. ***(Modeling) Radio Telescope Design*** The U.S. Naval Research Laboratory designed a giant radio telescope that had diameter 300 ft and maximum depth 44 ft. (*Source:* Mar, J., and H. Liebowitz, *Structure Technology for Large Radio and Radar Telescope Systems,* The MIT Press.)

(a) Find the equation of a parabola that models the cross section of the dish if the vertex is placed at the origin and the parabola opens up.

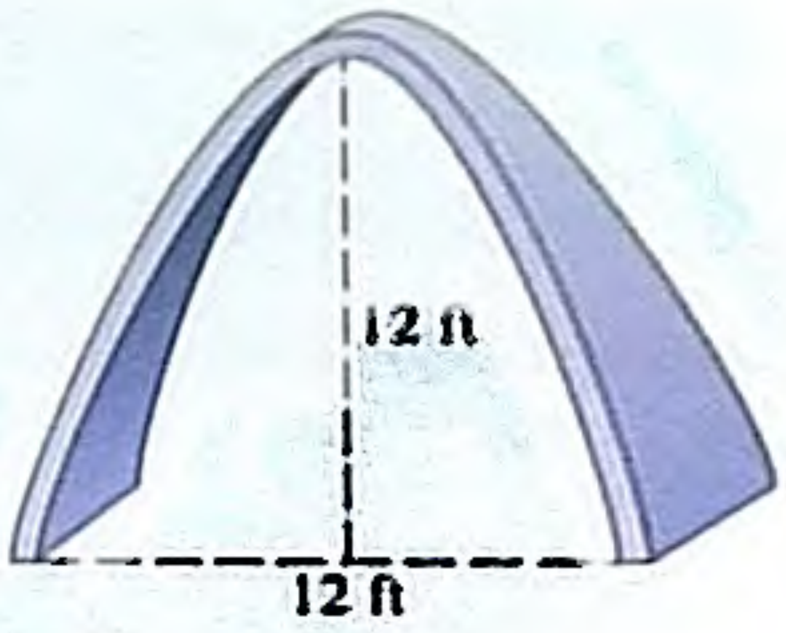
(b) The receiver must be placed at the focus of the parabola. How far from the vertex should the receiver be located?

54. ***(Modeling) Radio Telescope Design*** Suppose the telescope in Exercise 53 had diameter 400 ft and maximum depth 50 ft.

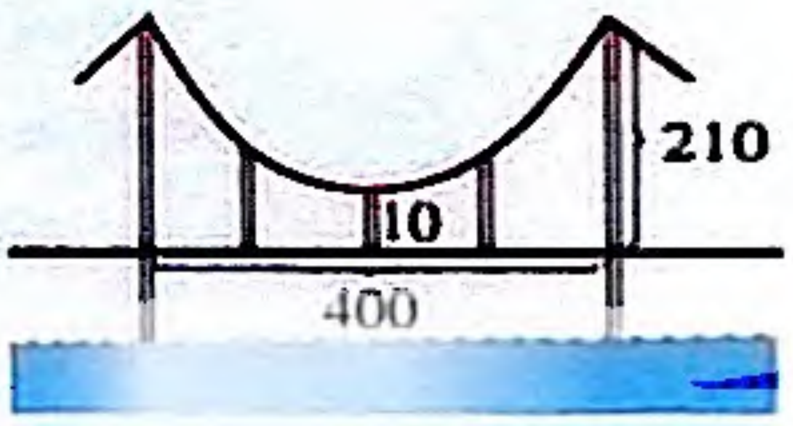
(a) Find the equation of this parabola

(b) The receiver must be placed at the focus of the parabola. How far from the vertex should the receiver be located?

55. ***Parabolic Arch*** An arch in the shape of a parabola has the dimensions shown in the figure. How wide is the arch 9 ft up?

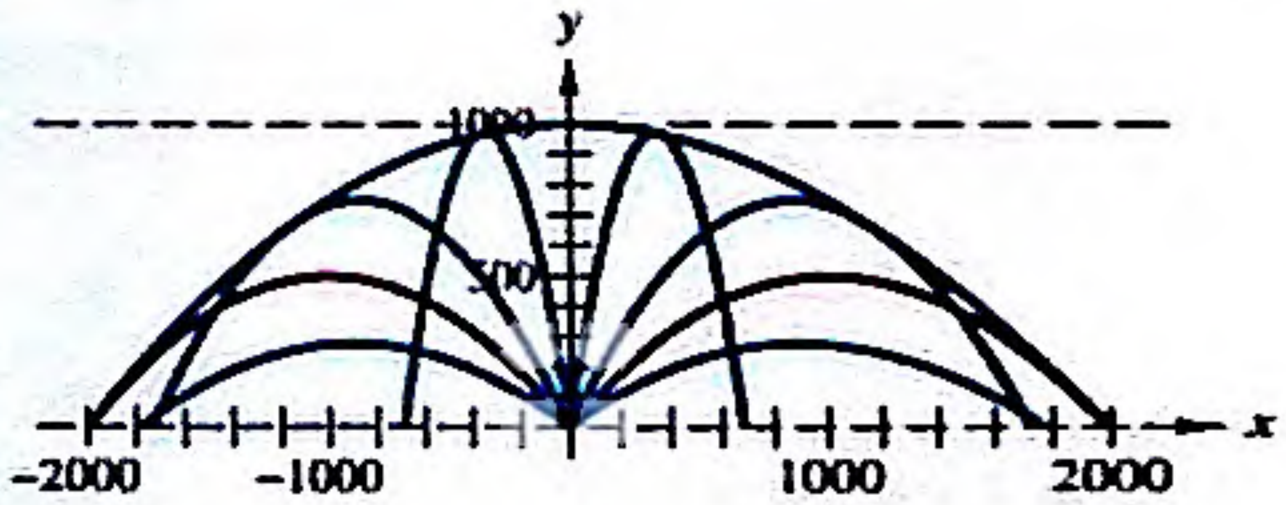


56. ***Height of Bridge Cable Supports*** The cable in the center portion of a bridge is supported as shown in the figure to form a parabola. The center vertical cable is 10 ft high, the supports are 210 ft high, and the distance between the two supports is 400 ft. Find the height of the remaining vertical cables, if the vertical cables are evenly spaced. (Ignore the width of the supports and cables.)



57. ***(Modeling) Path of a Cannon Shell*** The physicist Galileo observed that certain projectiles follow a parabolic path. For instance, if a cannon fires a shell at a 45° angle with a speed of v feet per second, then the path of the shell (see the top figure) is modeled by the following equation.

$$y = x - \frac{32}{v^2}x^2$$



The bottom figure shows the paths of shells all fired at the same speed but at different angles. The greatest distance is achieved with a 45° angle. The outline, or envelope, of this family of curves is another parabola with the cannon as focus. The horizontal line through the vertex of the envelope parabola is a directrix for all the other parabolas. Suppose all the shells are fired at a speed of 252.982 ft per sec.

(a) What is the greatest distance that a shell can be fired?
(b) What is the equation of the envelope parabola?
(c) Can a shell reach a helicopter 1500 ft due east of the cannon flying at a height of 450 ft?

58. ***(Modeling) Path of an Object*** When an object moves under the influence of a constant force (without air resistance), its path is parabolic. This would occur if a ball were thrown near the surface of a planet or other celestial object. Suppose two balls are simultaneously thrown upward at a 45° angle on two different planets. If their initial velocities are both 30 mph, then their xy-coordinates in feet at time x in seconds can be modeled by the following equation.

$$y = x - \frac{g}{1922}x^2$$

Here g is the acceleration due to gravity. The value of g will vary depending on the mass and size of the planet. (*Source*: Zeilik, M., and S. Gregory, *Introductory Astronomy and Astrophysics*, Fourth Edition, Brooks/Cole.)

(a) For Earth $g = 32.2$, while for Mars $g = 12.6$. Find the two equations, and graph on the same screen of a graphing calculator the paths of the two balls thrown on Earth and Mars. Use the window $[0, 180]$ by $[0, 120]$. (*Hint*: If possible, set the mode on your graphing calculator to simultaneous.)
(b) Determine the difference in the horizontal distances traveled by the two balls.

59. *(Modeling) Path of an Object* **(Refer to Exercise 58.)** Suppose the two balls are now thrown upward at a 60° angle on Mars and the moon. If their initial velocity is 60 mph, then their *xy*-coordinates in feet at time x in seconds can be modeled by the following equation.

$$y = \frac{19}{11}x - \frac{g}{3872}x^2$$

(*Source:* Zeilik, M., and S. Gregory, *Introductory Astronomy and Astrophysics*, Fourth Edition, Brooks/Cole.)

(a) Graph on the same coordinate axes the paths of the balls if $g = 5.2$ for the moon. Use the window $[0, 1500]$ by $[0, 1000]$.

(b) Determine the maximum height of each ball to the nearest foot.

60. Explain how you can tell, just by looking at the equation of a parabola, whether it has a horizontal or a vertical axis of symmetry.

61. Prove that the parabola with focus $(p, 0)$ and directrix $x = -p$ has the equation $y^2 = 4px$.

62. Write a short paragraph describing the appearances of parabolic shapes in your everyday surroundings.

6.2 Ellipses

- Equations and Graphs of Ellipses
- Translated Ellipses
- Eccentricity
- Applications of Ellipses

Equations and Graphs of Ellipses Like the circle and the parabola, the ellipse is defined as a set of points in a plane.

Ellipse

An **ellipse** is the set of all points in a plane the sum of whose distances from two fixed points is constant. Each fixed point is a **focus** (plural, **foci**) of the ellipse.

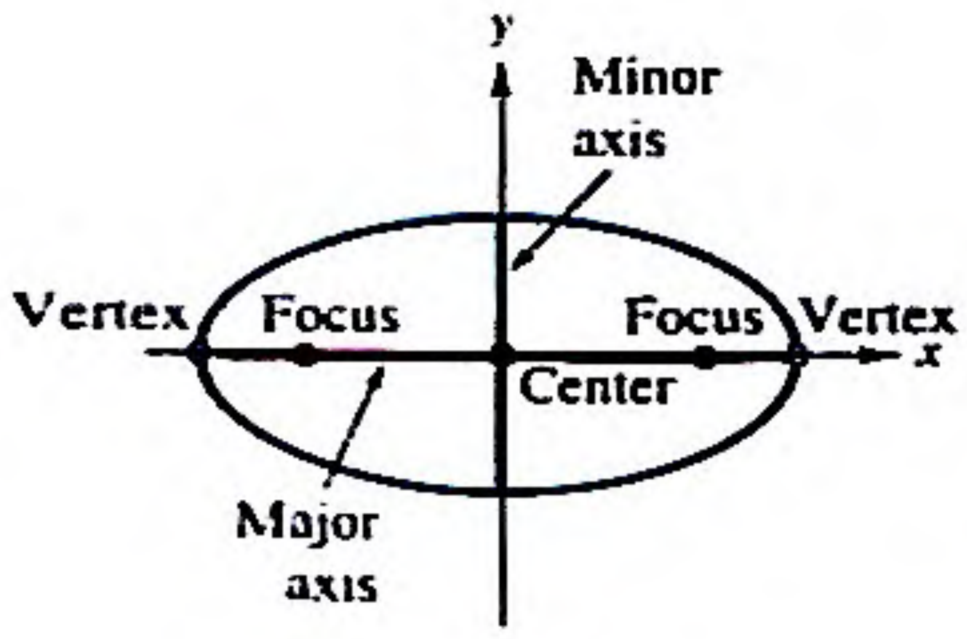


Figure 14

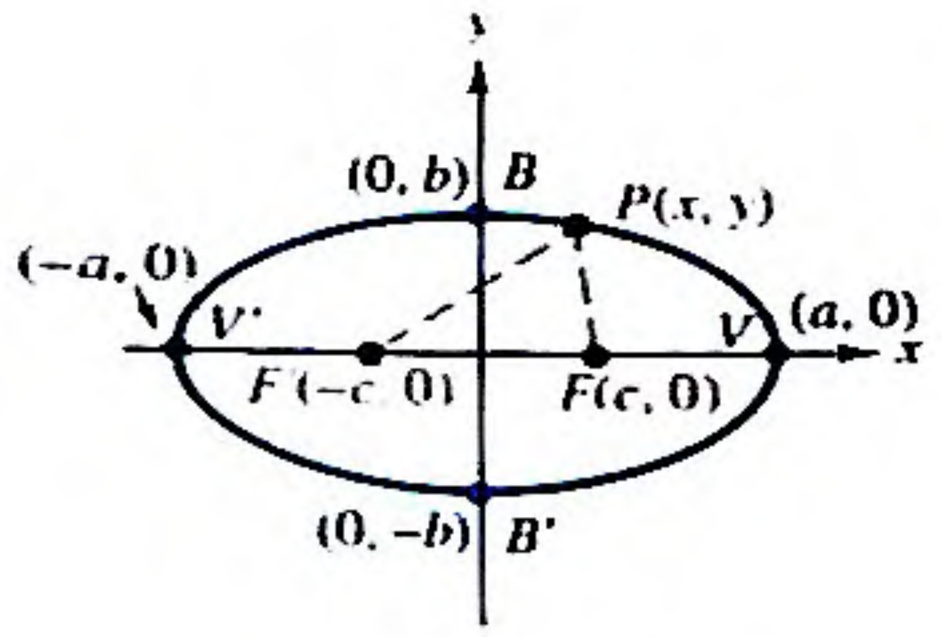


Figure 15

As shown in **Figure 14**, an ellipse has two axes of symmetry, the **major axis** (the longer one) and the **minor axis** (the shorter one). The foci are always located on the major axis. The midpoint of the major axis is the **center** of the ellipse, and the endpoints of the major axis are the **vertices** of the ellipse. *The graph of an ellipse is not the graph of a function.* It fails the vertical line test.

The ellipse in **Figure 15** has its center at the origin, foci $F(c, 0)$ and $F'(-c, 0)$, and vertices $V(a, 0)$ and $V'(-a, 0)$. From **Figure 15**, the distance from V to F is $a - c$ and the distance from V to F' is $a + c$. The sum of these distances is $2a$. Since V is on the ellipse, this sum is the constant referred to in the definition of an ellipse. Thus, for any point $P(x, y)$ on the ellipse,

$$d(P, F) + d(P, F') = 2a.$$

By the distance formula,

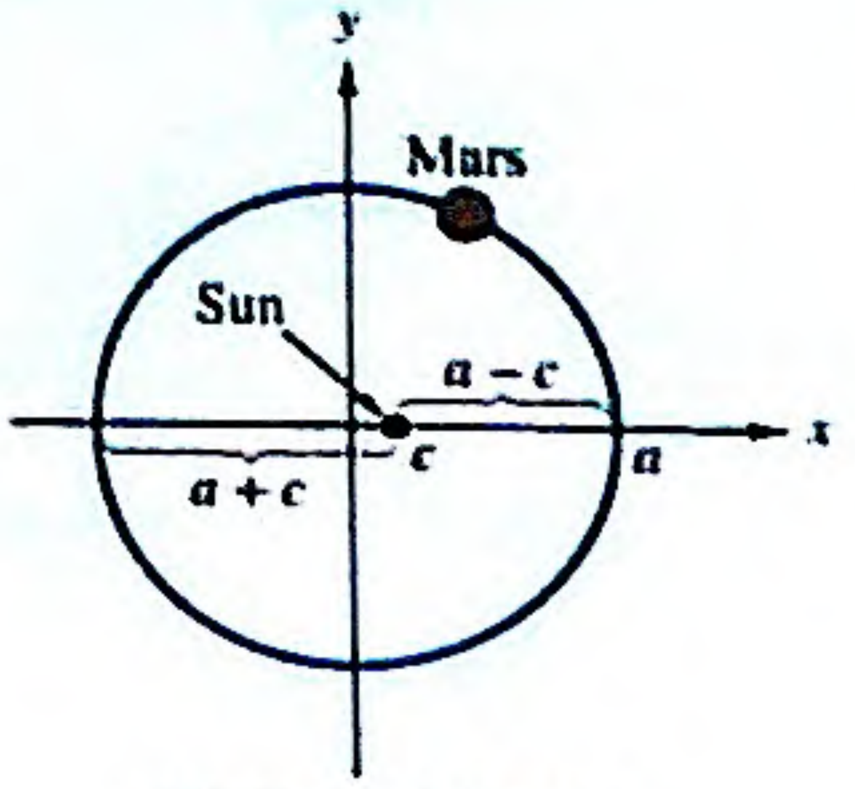
$$d(P, F) = \sqrt{(x - c)^2 + y^2}$$

and $$d(P, F') = \sqrt{[x - (-c)]^2 + y^2} = \sqrt{(x + c)^2 + y^2}.$$

Applications of Ellipses

EXAMPLE 6 Applying the Equation of an Ellipse to the Orbit of a Planet

The orbit of the planet Mars is an ellipse with the sun at one focus. The eccentricity of the ellipse is 0.0935, and the closest distance that Mars comes to the sun is 128.5 million mi. (*Source: World Almanac and Book of Facts.*) Find the maximum distance of Mars from the sun.



NOT TO SCALE

Figure 24

SOLUTION **Figure 24** shows the orbit of Mars with the origin at the center of the ellipse and the sun at one focus. Mars is closest to the sun when Mars is at the right endpoint of the major axis and farthest from the sun when Mars is at the left endpoint. Therefore, the least distance is $a - c$, and the greatest distance is $a + c$.

$$\frac{a - 128.5}{a} = 0.0935 \qquad \text{Since } a - c = 128.5\text{, it follows that } c = a - 128.5\text{. Use } e = \frac{c}{a}\text{.}$$

$$a - 128.5 = 0.0935a \qquad \text{Multiply by } a\text{.}$$

$$a \approx 141.8 \qquad \text{Solve for } a$$

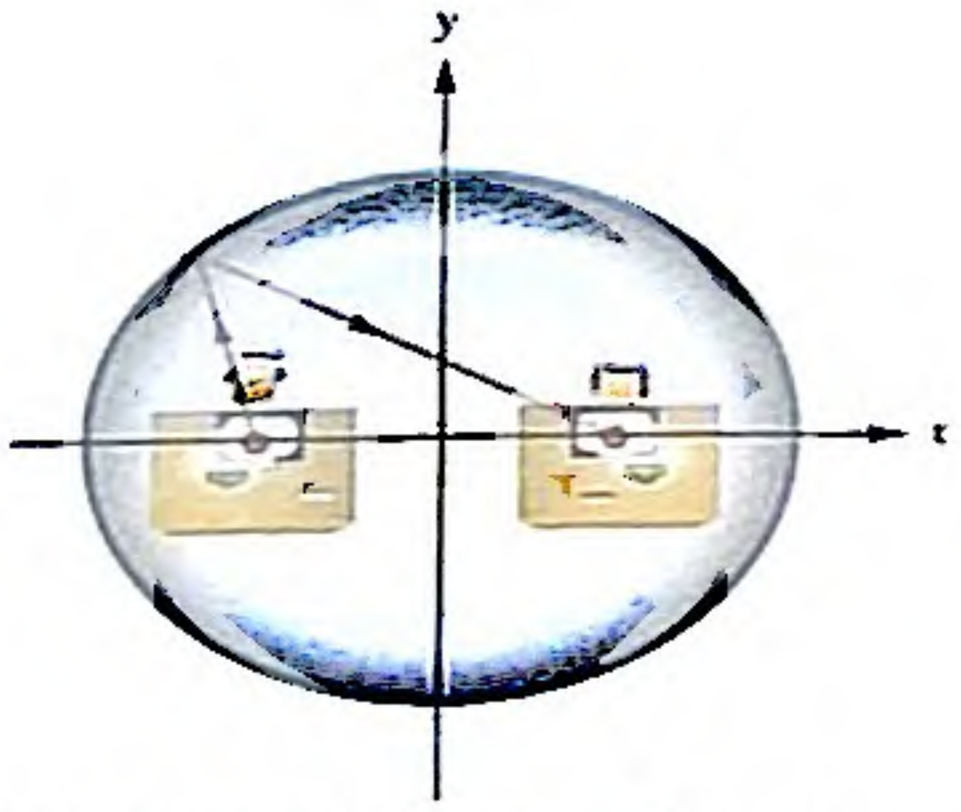
Then $\quad c = 141.8 - 128.5 = 13.3 \quad$ and $\quad a + c = 141.8 + 13.3 = 155.1.$

The maximum distance of Mars from the sun is about 155.1 million mi.

✔ ***Now Try Exercise 45.***

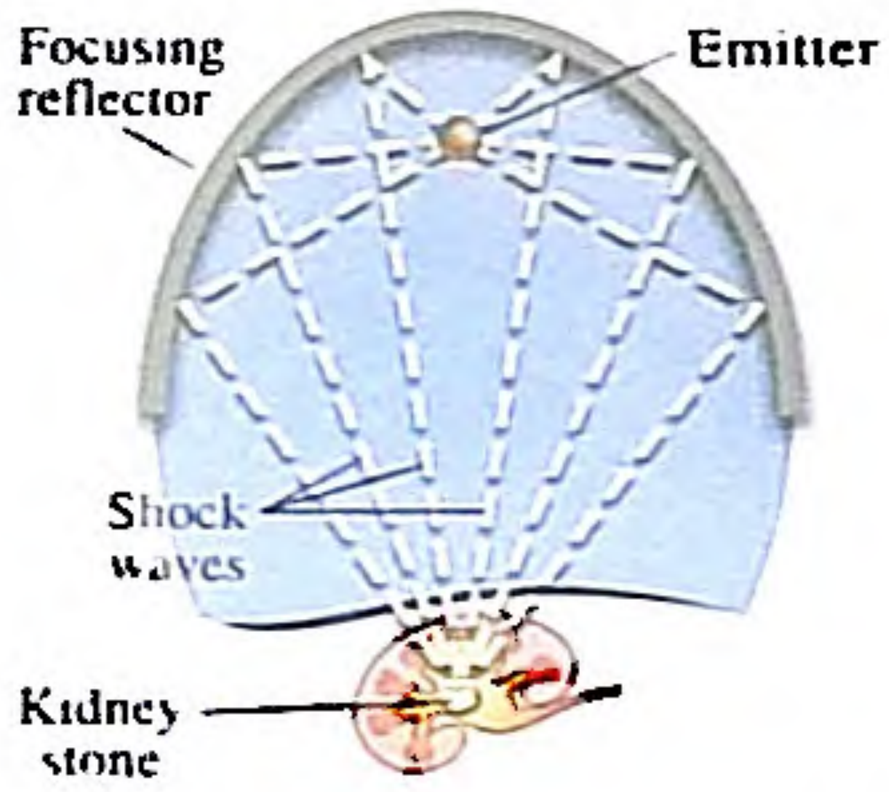
When a ray of light or sound emanating from one focus of an ellipse bounces off the ellipse, it passes through the other focus. See **Figure 25.** This reflecting property is responsible for "whispering galleries." In a whispering gallery, a person whispering at a certain point in the room, though barely audible close by, can be heard clearly at another point across the room. The U.S. statesman John Quincy Adams was able to listen in on his opponents' conversations in the old House Chamber (Statuary Hall) because his desk was positioned at one of the foci beneath the ellipsoidal ceiling and his opponents were located across the room at the other focus.

A lithotripter is a machine used to crush kidney stones using shock waves. The patient is placed in an elliptical tub with the kidney stone at one focus of the ellipse. A beam is projected from the other focus to the tub so that it reflects to hit the kidney stone. See **Figure 26** below and **Figure 27** on the next page.



Aerial view of Old House Chamber

Figure 25



The top of an ellipse is illustrated in this depiction of how a lithotripter crushes a kidney stone

Figure 26

EXAMPLE 7 **Modeling the Reflective Property of Ellipses**

If a lithotripter is based on the ellipse

$$\frac{x^2}{36}+\frac{y^2}{27}=1,$$

determine how many units both the kidney stone and the source of the beam must be placed from the center of the ellipse.

SOLUTION The kidney stone and the source of the beam must be placed at the foci, $(c, 0)$ and $(-c, 0)$. Here $a^2 = 36$ and $b^2 = 27$.

$$c=\sqrt{a^2-b^2} \qquad \text{Equation for } c$$

$$c=\sqrt{36-27} \qquad \text{Substitute.}$$

$$c=\sqrt{9}, \text{ or } 3 \qquad \text{Subtract and simplify.}$$

Thus, the foci are $(3, 0)$ and $(-3, 0)$. The kidney stone and the source both must be placed on a line 3 units from the center. See **Figure 27.**

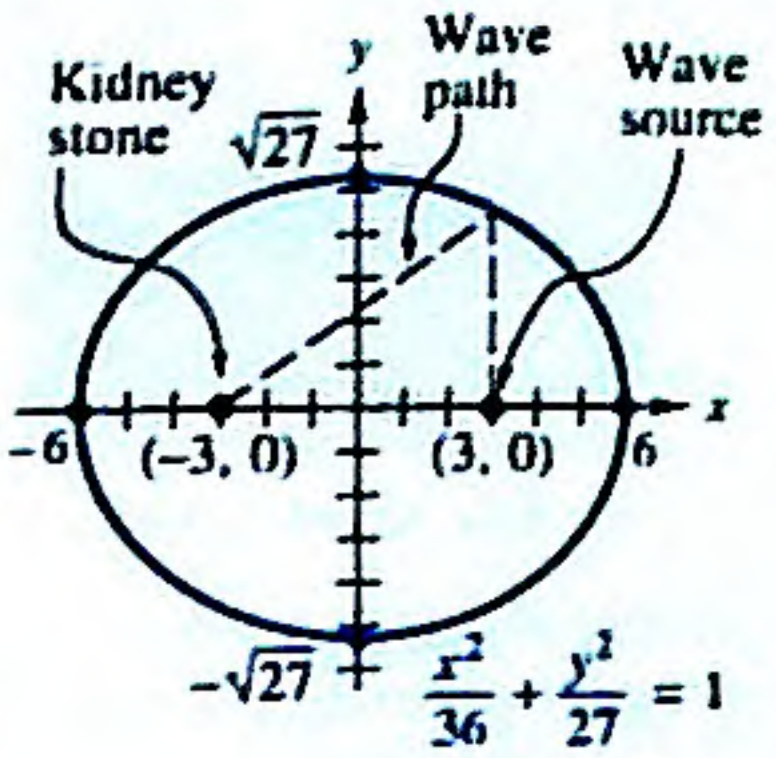


Figure 27

✔ *Now Try Exercise 49.*

6.2 Exercises

1. *Concept Check* Match each equation of an ellipse in Column I with the appropriate description in Column II.

I	II
(a) $36x^2 + 9y^2 = 324$	A. x-intercepts ± 3; y-intercepts ± 6
(b) $9x^2 + 36y^2 = 324$	B. x-intercepts ± 4; y-intercepts ± 5
(c) $\frac{x^2}{25}+\frac{y^2}{16}=1$	C. x-intercepts ± 6; y-intercepts ± 3
(d) $\frac{x^2}{16}+\frac{y^2}{25}=1$	D. x-intercepts ± 5; y-intercepts ± 4

2. *Concept Check* Determine whether or not each equation is that of an ellipse. If it is not, state the kind of graph the equation has.

(a) $x^2 + 4y^2 = 4$ (b) $x^2 + y^2 = 4$ (c) $x^2 + y = 4$ (d) $\frac{x}{4}+\frac{y}{25}=1$

Graph each ellipse. Identify the domain, range, center, vertices, endpoints of the minor axis, and foci in each figure. See Examples 1 and 4.

3. $\frac{x^2}{25}+\frac{y^2}{9}=1$ 4. $\frac{x^2}{16}+\frac{y^2}{25}=1$ 5. $\frac{x^2}{9}+y^2=1$

6. $\frac{x^2}{36}+\frac{y^2}{16}=1$ 7. $9x^2 + y^2 = 81$ 8. $4x^2 + 16y^2 = 64$

9. $4x^2 = 100 - 25y^2$ 10. $4x^2 = 16 - y^2$

11. $\frac{(x-2)^2}{25}+\frac{(y-1)^2}{4}=1$ 12. $\frac{(x+2)^2}{16}+\frac{(y+1)^2}{9}=1$

13. $\frac{(x+3)^2}{16}+\frac{(y-2)^2}{36}=1$ 14. $\frac{(x-1)^2}{9}+\frac{(y+3)^2}{25}=1$

Write an equation for each ellipse. See Example 2.

15. x-intercepts ± 5; y-intercepts ± 4

16. x-intercepts $\pm \sqrt{15}$; y-intercepts ± 4

17. major axis with length 6; foci at $(0, 2)$, $(0, -2)$

18. minor axis with length 4; foci at $(-5, 0)$, $(5, 0)$

19. center at $(3, 1)$; minor axis vertical, with length 8; $c = 3$

20. center at $(-2, 7)$; major axis vertical, with length 10; $c = 2$

21. foci at $(0, 4)$, $(0, -4)$; sum of distances from foci to point on ellipse is 10 (*Hint:* Consider one of the vertices.)

22. foci at $(-4, 0)$, $(4, 0)$; sum of distances from foci to point on ellipse is 9

23. foci at $(0, -3)$, $(0, 3)$; the point $(8, 3)$ on ellipse

24. foci at $(-3, -3)$, $(7, -3)$; the point $(2, -7)$ on ellipse

25. eccentricity $\frac{4}{5}$; vertices at $(-5, 0)$, $(5, 0)$

26. eccentricity $\frac{1}{2}$; vertices at $(-4, 0)$, $(4, 0)$

27. eccentricity $\frac{3}{4}$; foci at $(0, -2)$, $(0, 2)$

28. eccentricity $\frac{2}{3}$; foci at $(0, -9)$, $(0, 9)$

Graph each equation. Give the domain and range. Identify any that are graphs of functions. See Example 3.

29. $\dfrac{y}{2} = \sqrt{1 - \dfrac{x^2}{25}}$

30. $\dfrac{x}{4} = \sqrt{1 - \dfrac{y^2}{9}}$

31. $x = -\sqrt{1 - \dfrac{y^2}{64}}$

32. $y = -\sqrt{1 - \dfrac{x^2}{100}}$

Determine the two equations necessary to graph each ellipse with a graphing calculator, and graph it in the viewing window indicated. See **Figure 18.**

33. $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$;
$[-4.7, 4.7]$ by $[-3.1, 3.1]$

34. $\dfrac{x^2}{4} + \dfrac{y^2}{25} = 1$;
$[-9.4, 9.4]$ by $[-6.2, 6.2]$

35. $\dfrac{(x-3)^2}{25} + \dfrac{y^2}{9} = 1$;
$[-9.4, 9.4]$ by $[-6.2, 6.2]$

36. $\dfrac{x^2}{36} + \dfrac{(y+4)^2}{4} = 1$;
$[-9.4, 9.4]$ by $[-6.2, 6.2]$

Find the eccentricity of each ellipse. If applicable, round to the nearest hundredth. See Example 5.

37. $\dfrac{x^2}{3} + \dfrac{y^2}{4} = 1$

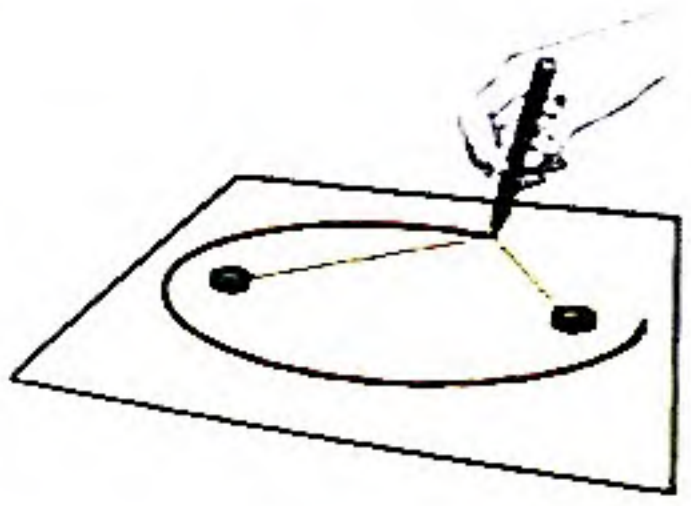
38. $\dfrac{x^2}{8} + \dfrac{y^2}{4} = 1$

39. $4x^2 + 7y^2 = 28$

40. $x^2 + 25y^2 = 25$

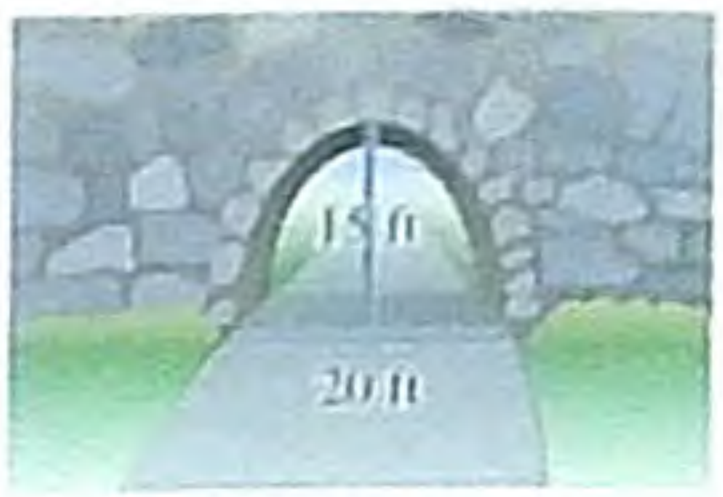
41. Draftspeople often use the method shown in the sketch to draw an ellipse. Explain why the method works.

42. Explain how the method of Exercise 41 can be modified to draw a circle.



Solve each problem. See Examples 6 and 7.

43. *Height of an Overpass* A one-way road passes under an overpass in the shape of half an ellipse, 15 ft high at the center and 20 ft wide. Assuming a truck is 12 ft wide, what is the tallest truck that can pass under the overpass?

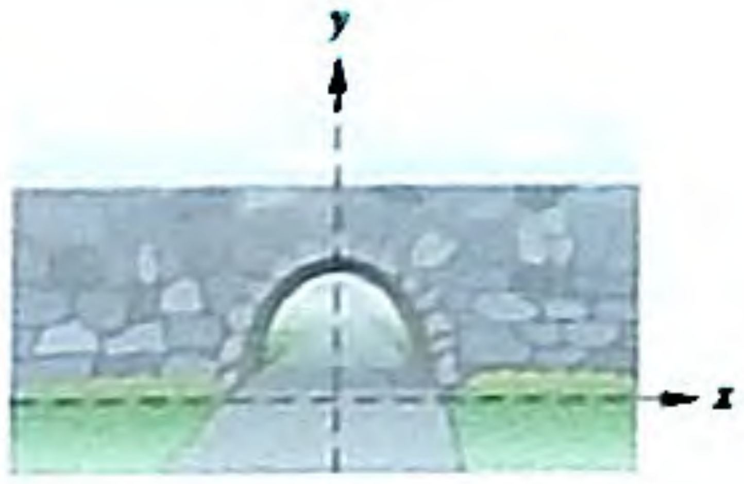


NOT TO SCALE

44. *Height and Width of an Overpass* An arch has the shape of half an ellipse. The equation of the ellipse is $100x^2 + 324y^2 = 32{,}400$, where x and y are in meters.

(a) How high is the center of the arch?

(b) How wide is the arch across the bottom?



NOT TO SCALE

45. *Orbit of Halley's Comet* The famous Halley's comet last passed by Earth in February 1986 and will next return in 2062. It has an elliptical orbit of eccentricity 0.9673 with the sun at one focus. The greatest distance of the comet from the sun is 3281 million mi. Find the least distance between Halley's comet and the sun. (*Source: World Almanac and Book of Facts.*)

46. *(Modeling) Orbit of a Satellite* The coordinates in miles for the orbit of the artificial satellite Explorer VII can be modeled by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a = 4465$ and $b = 4462$. Earth's center is located at one focus of the elliptical orbit. (*Source:* Loh, W., *Dynamics and Thermodynamics of Planetary Entry,* Prentice-Hall; Thomson, W., *Introduction to Space Dynamics,* John Wiley and Sons.)

(a) Graph both the orbit of Explorer VII and the Earth's surface on the same coordinate axes if the average radius of Earth is 3960 mi. Use the window $[-6750, 6750]$ by $[-4500, 4500]$.

(b) Find the maximum and minimum heights of the satellite above Earth's surface.

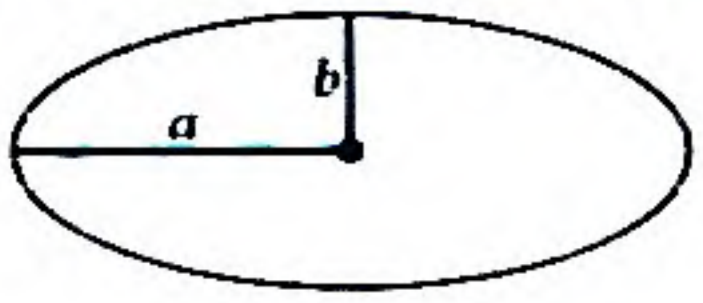
47. *(Modeling) Orbits of Satellites* Neptune and Pluto both have elliptical orbits with the sun at one focus. Neptune's orbit has $a = 30.1$ astronomical units (AU) with an eccentricity of $e = 0.009$, whereas Pluto's orbit has $a = 39.4$ and $e = 0.249$. (*Source:* Zeilik, M., S. Gregory, and E. Smith, *Introductory Astronomy and Astrophysics,* Fourth Edition, Saunders College Publishers.)

(a) Position the sun at the origin and determine equations that model each orbit.

(b) Graph both equations on the same coordinate axes. Use the window $[-60, 60]$ by $[-40, 40]$.

48. *(Modeling) The Roman Colosseum*

(a) The Roman Colosseum is an ellipse with major axis 620 ft and minor axis 513 ft. Find the distance between the foci of this ellipse.



(b) A formula for the approximate perimeter of an ellipse is

$$P \approx 2\pi\sqrt{\frac{a^2 + b^2}{2}},$$

where a and b are the lengths shown in the figure. Use this formula to find the approximate perimeter of the Roman Colosseum.

49. *Design of a Lithotripter* Suppose a lithotripter is based on the ellipse with equation

$$\frac{x^2}{36}+\frac{y^2}{9}=1.$$

How far from the center of the ellipse must the kidney stone and the source of the beam be placed? Give the exact answer.

50. *Design of a Lithotripter* Rework Exercise 49 if the equation of the ellipse is $9x^2+4y^2=36$.

Chapter 6 Quiz (Sections 6.1–6.2)

1. *Concept Check* Match each equation of a conic section in Column I with the appropriate description or descriptions in Column II.

I	II
(a) $x+3=4(y-1)^2$	A. circle; center $(-3, 1)$
(b) $(x+3)^2+(y-1)^2=81$	B. parabola; opens right
(c) $25(x-2)^2+(y-1)^2=100$	C. ellipse; major axis horizontal
(d) $\dfrac{(x-2)^2}{16}+\dfrac{(y-1)^2}{9}=1$	D. parabola; vertex $(-3, 1)$
(e) $-2(x+3)^2+1=y$	E. ellipse; center $(2, 1)$

Write an equation for each conic section.

2. parabola with vertex $(-1, 2)$ and focus $(2, 2)$
3. parabola with vertex at the origin; through $(\sqrt{10}, -5)$; opening downward
4. ellipse with center $(3, -2)$; $a=5$; $c=3$; major axis vertical
5. ellipse with foci at $(-3, 3)$ and $(-3, 11)$; major axis of length 10

Identify and then graph each conic section. If it is a parabola, give the vertex, focus, directrix, and axis of symmetry. If it is an ellipse, give the center, vertices, and foci.

6. $y+4=(x+3)^2$
7. $4x^2+9y^2=36$
8. $8(x+1)=(y+3)^2$
9. $\dfrac{(x+3)^2}{25}+\dfrac{(y+2)^2}{36}=1$
10. $x=-4y^2-4y-3$

6.3 Hyperbolas

- Equations and Graphs of Hyperbolas
- Translated Hyperbolas
- Eccentricity

Equations and Graphs of Hyperbolas An ellipse was defined as the set of all points in a plane the sum of whose distances from two fixed points is a constant. A *hyperbola* is defined similarly.

Hyperbola

A **hyperbola** is the set of all points in a plane such that the absolute value of the difference of the distances from two fixed points is constant. The two fixed points are the **foci** of the hyperbola.

Suppose a hyperbola has center at the origin and foci at $F'(-c, 0)$ and $F(c, 0)$. See **Figure 28.** The midpoint of the segment $F'F$ is the center of the hyperbola and the points $V'(-a, 0)$ and $V(a, 0)$ are the **vertices** of the hyperbola. The line segment $V'V$ is the **transverse axis** of the hyperbola.

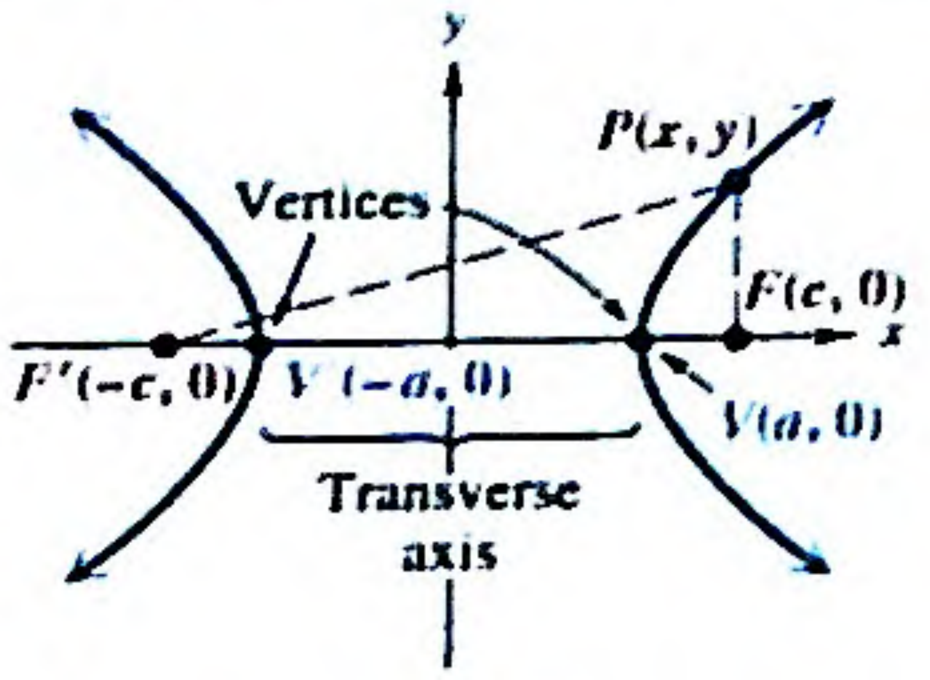


Figure 28

For hyperbolas,

$$d(V, F') - d(V, F) = (c + a) - (c - a) = 2a,$$

so the constant in the definition is $2a$, and

$$|d(P, F') - d(P, F)| = 2a$$

for any point $P(x, y)$ on the hyperbola. The distance formula and algebraic manipulation similar to that used for finding an equation for an ellipse (**Exercise 62**) produce the following result.

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$$

Replacing $c^2 - a^2$ with b^2 gives

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

as an equation of the hyperbola in **Figure 28.** Letting $y = 0$ shows that the x-intercepts are $\pm a$. If $x = 0$, the equation becomes $y^2 = -b^2$, which has no real number solutions, showing that this hyperbola has no y-intercepts. Again, see **Figure 28.**

Start with the equation for a hyperbola and solve for y.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{a^2} - 1 = \frac{y^2}{b^2}$$

Subtract 1 and add $\frac{y^2}{b^2}$.

$$\frac{x^2 - a^2}{a^2} = \frac{y^2}{b^2}$$

Write the left side as a single fraction.

Remember both the positive and negative square roots.

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$$

Take the square root on each side (Section 1.6); multiply by b.

If x^2 is very large in comparison to a^2, the difference $x^2 - a^2$ is very close to x^2. If this happens, then the points satisfying the final equation above are very close to one of the lines

$$y = \pm\frac{b}{a}x.$$

Thus, as $|x|$ increases without bound, the points of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ approach the lines $y = \pm\frac{b}{a}x$. These lines are **asymptotes** of the hyperbola and are useful when sketching the graph.

The following chart summarizes our discussion of eccentricity in this chapter.

Conic	Eccentricity
Parabola	$e = 1$
Circle	$e = 0$
Ellipse	$e = \frac{c}{a}$ and $0 < e < 1$
Hyperbola	$e = \frac{c}{a}$ and $e > 1$

6.3 Exercises

Concept Check *Match each equation with the correct graph.*

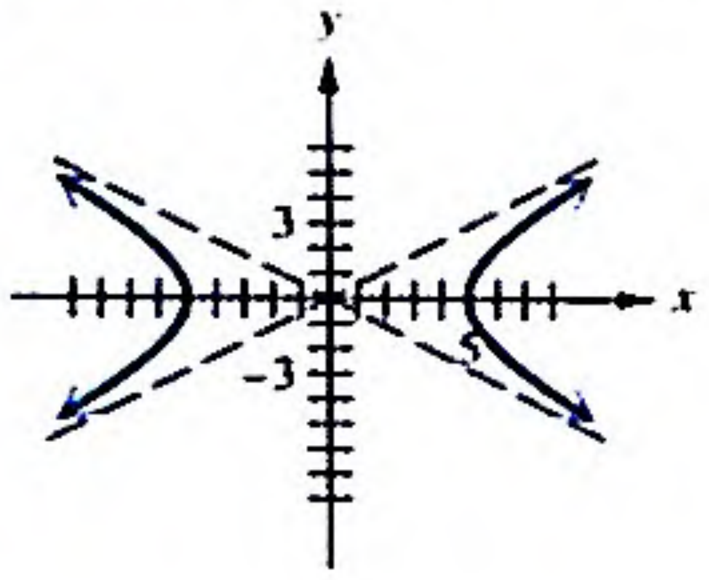
1. $\frac{x^2}{25} + \frac{y^2}{9} = 1$

2. $\frac{x^2}{9} + \frac{y^2}{25} = 1$

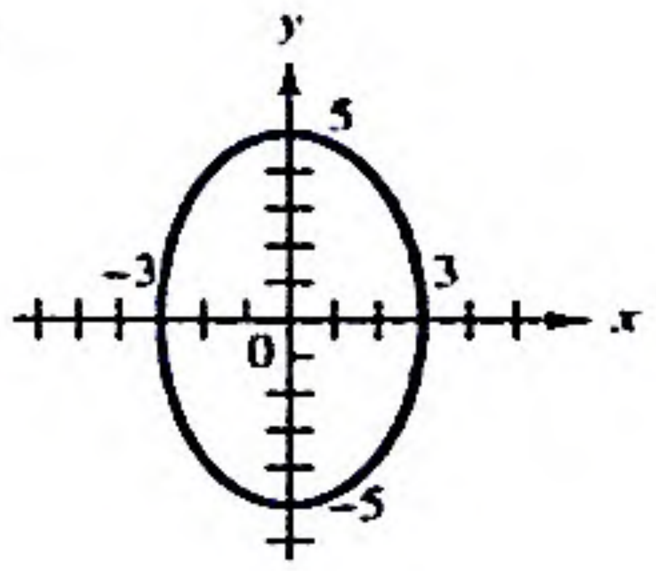
3. $\frac{x^2}{9} - \frac{y^2}{25} = 1$

4. $\frac{x^2}{25} - \frac{y^2}{9} = 1$

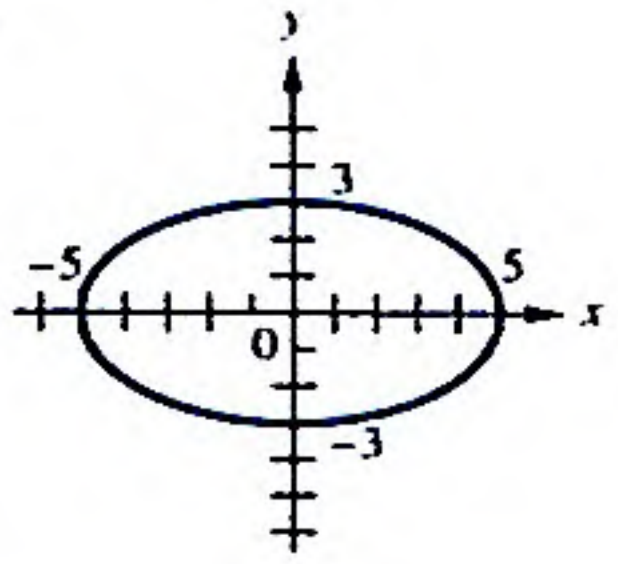
A.



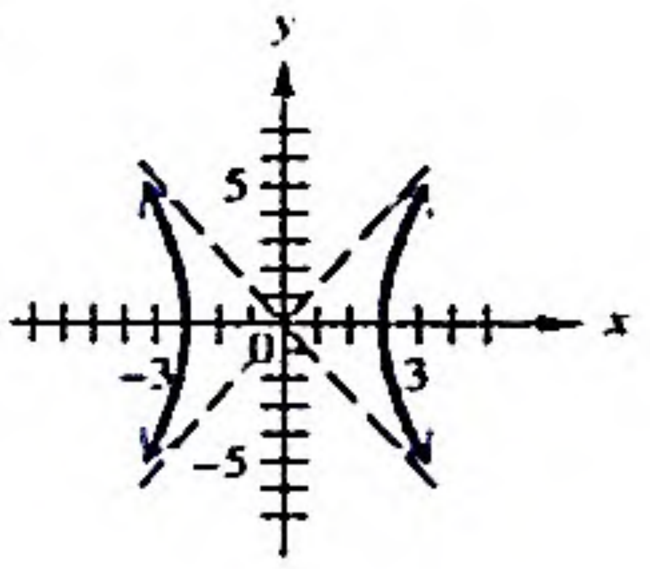
B.



C.



D.



Graph each hyperbola. Give the domain, range, center, vertices, foci, and equations of the asymptotes for each figure. See Examples 1–3.

5. $\frac{x^2}{16} - \frac{y^2}{9} = 1$

6. $\frac{x^2}{25} - \frac{y^2}{144} = 1$

7. $\frac{y^2}{25} - \frac{x^2}{49} = 1$

8. $\frac{y^2}{64} - \frac{x^2}{4} = 1$

9. $x^2 - y^2 = 9$

10. $x^2 - 4y^2 = 64$

11. $9x^2 - 25y^2 = 225$

12. $4y^2 - 16x^2 = 64$

13. $4y^2 - 25x^2 = 100$

14. $x^2 - 4y^2 = 16$

15. $9x^2 - 4y^2 = 1$

16. $25y^2 - 9x^2 = 1$

17. $\frac{(y-7)^2}{36} - \frac{(x-4)^2}{64} = 1$

18. $\frac{(x+6)^2}{144} - \frac{(y+4)^2}{81} = 1$

19. $\frac{(x+3)^2}{16} - \frac{(y-2)^2}{9} = 1$

20. $\frac{(y+5)^2}{4} - \frac{(x-1)^2}{16} = 1$

21. $16(x+5)^2 - (y-3)^2 = 1$

22. $4(x+9)^2 - 25(y+6)^2 = 100$

Graph each equation. Give the domain and range. Identify any that are graphs of functions. See Example 3 in the previous section.

23. $\frac{y}{3} = \sqrt{1 + \frac{x^2}{16}}$

24. $\frac{x}{3} = -\sqrt{1 + \frac{y^2}{25}}$

25. $5x = -\sqrt{1 + 4y^2}$

26. $3y = \sqrt{4x^2 - 16}$

Find the eccentricity of each hyperbola to the nearest tenth. See Example 4.

27. $\frac{x^2}{8} - \frac{y^2}{8} = 1$

28. $\frac{x^2}{2} - \frac{y^2}{18} = 1$

29. $16y^2 - 8x^2 = 16$

30. $8y^2 - 2x^2 = 16$

Write an equation for each hyperbola. See Examples 4 and 5.

31. x-intercepts ± 3; foci at $(-5, 0)$, $(5, 0)$

32. y-intercepts ± 12; foci at $(0, -15)$, $(0, 15)$

33. vertices at $(0, 6)$, $(0, -6)$; asymptotes $y = \pm\frac{1}{2}x$

34. vertices at $(-10, 0)$, $(10, 0)$; asymptotes $y = \pm 5x$

35. vertices at $(-3, 0)$, $(3, 0)$; passing through $(-6, -1)$

36. vertices at $(0, 5)$, $(0, -5)$; passing through $(-3, 10)$

37. foci at $(0, \sqrt{13})$, $(0, -\sqrt{13})$; asymptotes $y = \pm 5x$

38. foci at $(-3\sqrt{5}, 0)$, $(3\sqrt{5}, 0)$; asymptotes $y = \pm 2x$

39. vertices at $(4, 5)$, $(4, 1)$, asymptotes $y = \pm 7(x - 4) + 3$

40. vertices at $(5, -2)$, $(1, -2)$; asymptotes $y = \pm\frac{3}{2}(x - 3) - 2$

41. center at $(1, -2)$; focus at $(-2, -2)$; vertex at $(-1, -2)$

42. center at $(9, -7)$; focus at $(9, -17)$; vertex at $(9, -13)$

43. eccentricity 3; center at $(0, 0)$; vertex at $(0, 7)$

44. eccentricity $\frac{5}{3}$; center at $(8, 7)$; focus at $(3, 7)$

45. eccentricity $\frac{25}{9}$; foci at $(9, -1)$, $(-11, -1)$

46. eccentricity $\frac{5}{4}$; vertices at $(2, 10)$, $(2, 2)$

Determine the two equations necessary to graph each hyperbola with a graphing calculator, and graph it in the viewing window indicated. See Example 1.

47. $\frac{x^2}{4} - \frac{y^2}{16} = 1$; $[-9.4, 9.4]$ by $[-10, 10]$

48. $\frac{x^2}{25} - \frac{y^2}{49} = 1$; $[-9.4, 9.4]$ by $[-10, 10]$

49. $4y^2 - 36x^2 = 144$; $[-10, 10]$ by $[-15, 15]$

50. $y^2 - 9x^2 = 9$; $[-10, 10]$ by $[-10, 10]$

Relating Concepts

For individual or collaborative investigation *(Exercises 51–56)*

The graph of $\frac{x^2}{4} - y^2 = 1$ *is a hyperbola. We know that the graph of this hyperbola approaches its asymptotes as* $|x|$ *increases without bound.* ***Work Exercises 51–56 in order,*** *to see the relationship between the hyperbola and one of its asymptotes.*

51. Solve $\frac{x^2}{4} - y^2 = 1$ for y, and choose the positive square root.

52. Find the equation of the asymptote with positive slope.

53. Use a calculator to evaluate the y-coordinate of the point where $x = 50$ on the graph of the portion of the hyperbola represented by the equation obtained in **Exercise 51.** Round your answer to the nearest hundredth.

54. Find the y-coordinate of the point where $x = 50$ on the graph of the asymptote found in **Exercise 52.**

55. Compare your results in **Exercises 53 and 54.** How do they support the following statement?

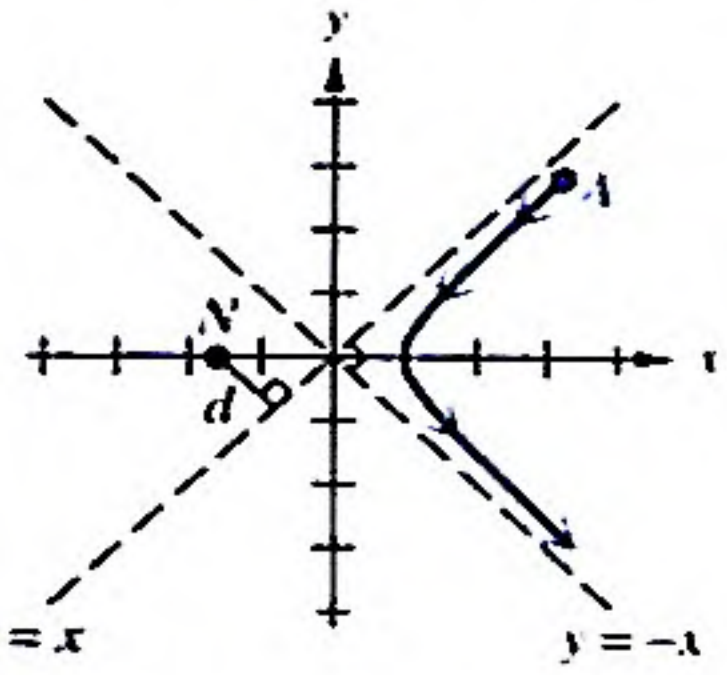
> When $x = 50$, the graph of the function defined by the equation found in **Exercise 51** lies *below* the graph of the asymptote found in **Exercise 52.**

56. What do you think will happen if we choose x-values greater than 50?

Solve each problem.

57. ***(Modeling) Atomic Structure*** In 1911, Ernest Rutherford discovered the basic structure of the atom by "shooting" positively charged alpha particles with a speed of 10^7 m per sec at a piece of gold foil 6×10^{-7} m thick. Only a small percentage of the alpha particles struck a gold nucleus head-on and were deflected directly back toward their source. The rest of the particles often followed a hyperbolic trajectory because they were repelled by positively charged gold nuclei. As a result of this famous experiment, Rutherford proposed that the atom was composed mostly of empty space with a small and dense nucleus.

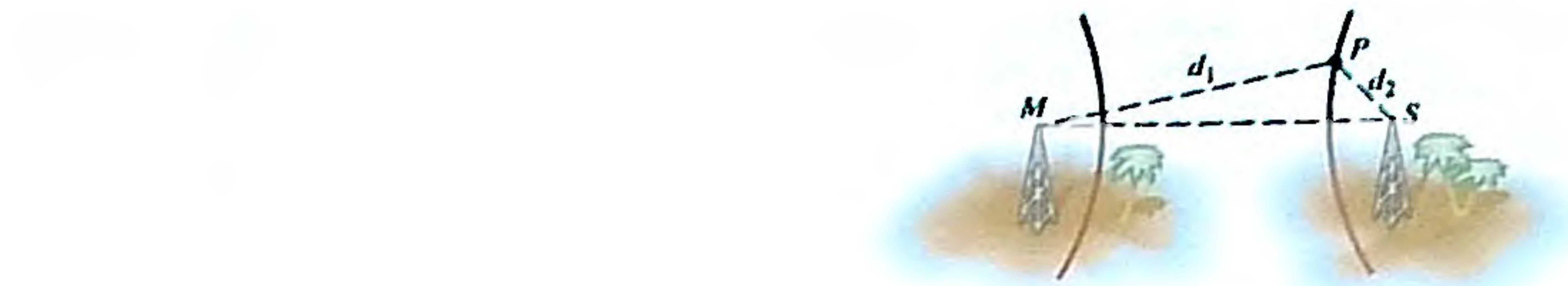
The figure shows an alpha particle A initially approaching a gold nucleus N and being deflected at an angle $\theta = 90°$. N is located at a focus of the hyperbola, and the trajectory of A passes through a vertex of the hyperbola. (*Source:* Semat, H., and J. Albright, *Introduction to Atomic and Nuclear Physics,* Fifth Edition, International Thomson Publishing.)



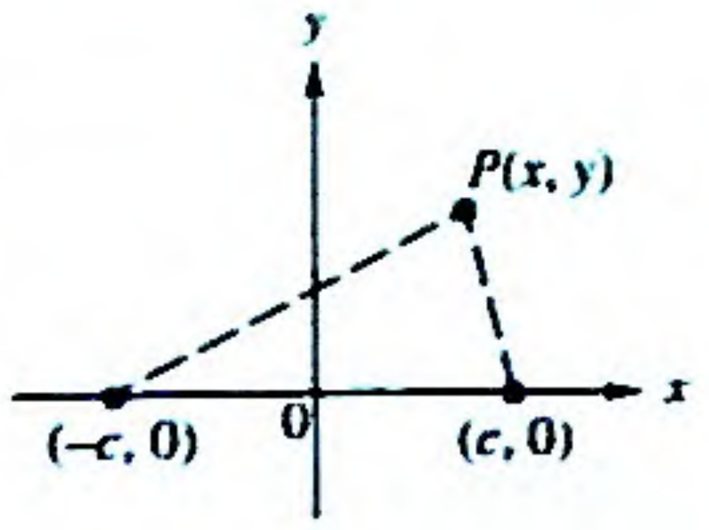
(a) Determine the equation of the trajectory of the alpha particle if $d = 5 \times 10^{-14}$ m.

(b) What was the minimum distance between the centers of the alpha particle and the gold nucleus?

58. ***LORAN System*** Ships and planes often use a location-finding system called LORAN. With this system, a radio transmitter at M in the figure on the next page sends out a series of pulses. When each pulse is received at transmitter S, it then sends out a pulse. A ship at P receives pulses from both M and S. A receiver on the ship measures the difference in the arrival times of the pulses. The navigator then consults a special map showing hyperbolas that correspond to the differences in arrival times (which give the distances d_1 and d_2 in the figure). In this way the ship can be located as lying on a branch of a particular hyperbola.



Suppose that in the figure, $d_1 = 80$ mi, $d_2 = 30$ mi, and the distance MS between the transmitters is 100 mi. Use the definition of a hyperbola to find an equation of the hyperbola on which the ship is located.



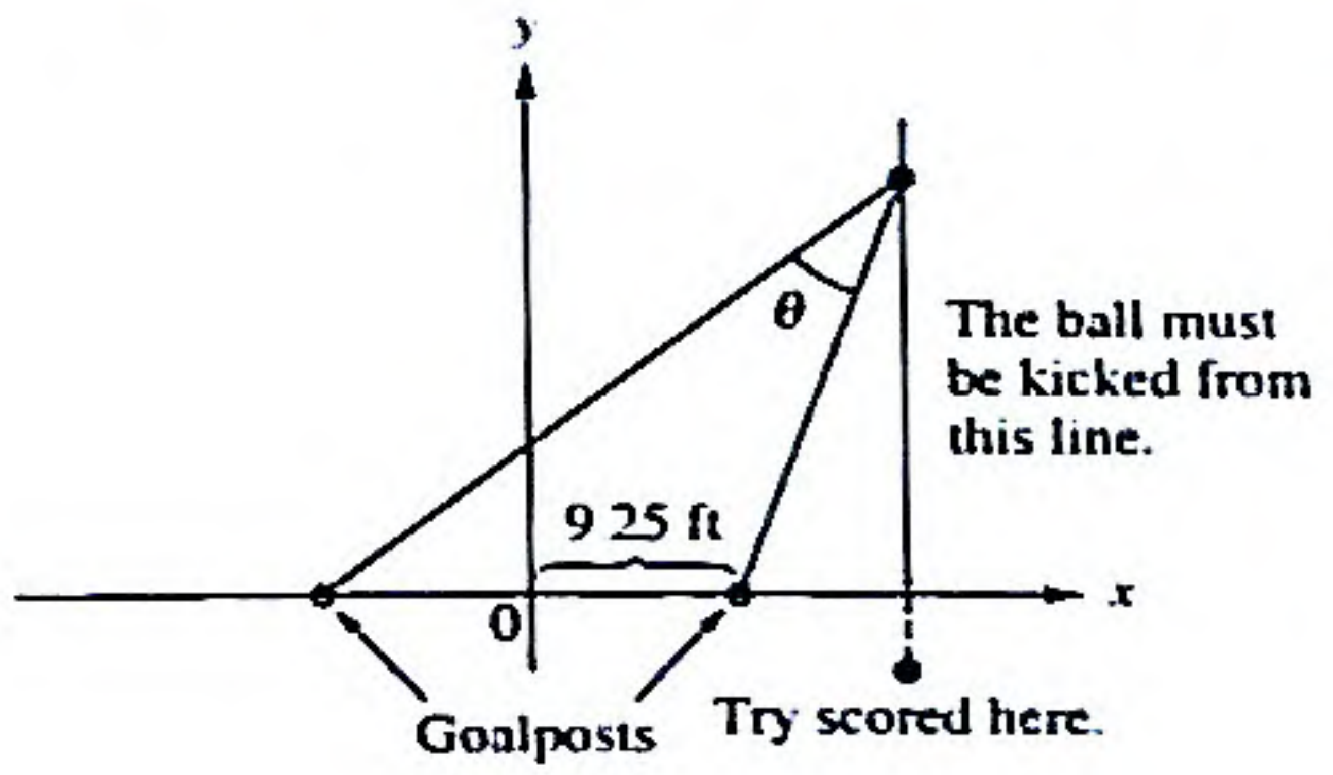
59. ***Sound Detection*** Microphones are placed at points $(-c, 0)$ and $(c, 0)$. An explosion occurs at point $P(x, y)$ having positive x-coordinate. See the figure. The sound is detected at the closer microphone t seconds before being detected at the farther microphone. Assume that sound travels at a speed of 330 m per sec, and show that P must be on the hyperbola

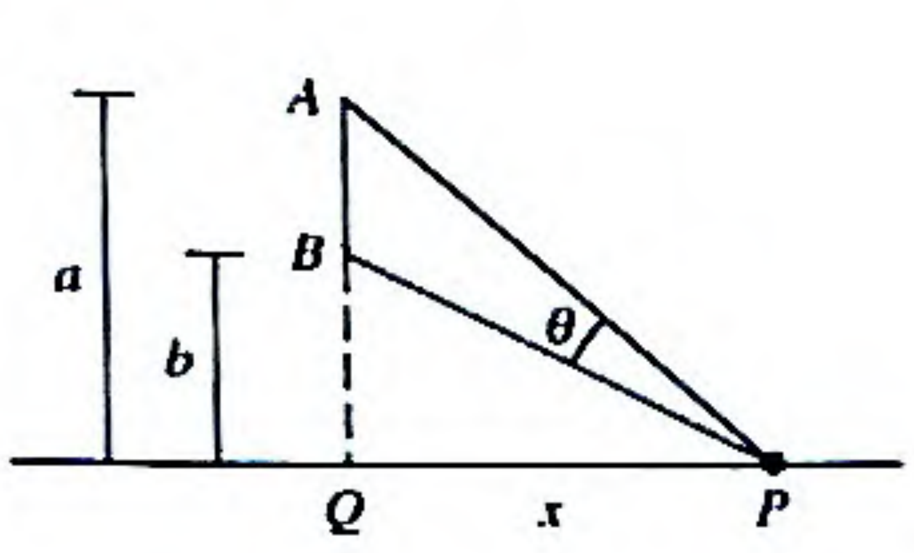
$$\frac{x^2}{330^2t^2} - \frac{y^2}{4c^2 - 330^2t^2} = \frac{1}{4}.$$

60. ***Rugby Algebra*** A rugby field is similar to a modern football field except that the goalpost, which is 18.5 ft wide, is located on the goal line instead of at the back of the endzone. The rugby equivalent of a touchdown, called a **try**, is scored by touching the ball down beyond the goal line. After a try is scored, the scoring team can earn extra points by kicking the ball through the goalposts. The ball must be placed somewhere on the line perpendicular to the goal line and passing through the point where the try was scored. See the figure below on the left.

If that line passes through the goalposts, then the kicker should place the ball at whatever distance is most comfortable. If the line passes outside the goalposts, then the player might choose the point on the line where angle θ in the figure on the left is as large as possible. The problem of determining this optimal point is similar to a problem posed in 1471 by the astronomer Regiomontanus. (*Source:* Maor, E., *Trigonometric Delights*, Princeton, NJ: Princeton University Press.)

The figure on the right shows a vertical line segment AB, where A and B are a and b units above the horizontal axis, respectively. If point P is located on the axis at a distance of x units from point Q, then angle θ is greatest when $x = \sqrt{ab}$.



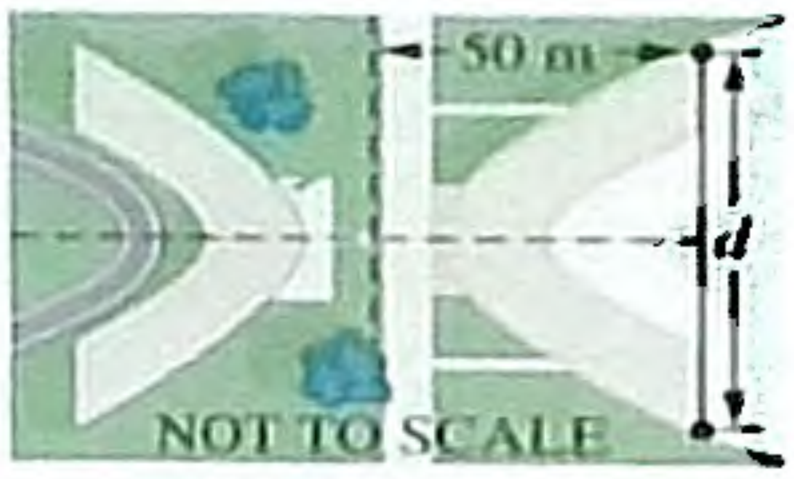


(a) Use the result from Regiomontanus' problem to show that when the line is outside the goalposts, the optimal location to kick the rugby ball lies on the hyperbola

$$x^2 - y^2 = 9.25^2.$$

(b) If the line on which the ball must be kicked is 10 ft to the right of the goalpost, how far from the goal line should the ball be placed to maximize angle θ?

(c) Rugby players find it easier to kick the ball from the hyperbola's asymptote. When the line on which the ball must be kicked is 10 ft to the right of the goalpost, how far will this point differ from the exact optimal location?



61. *(Modeling) Design of a Sports Complex* Two buildings in a sports complex are shaped and positioned like a portion of the branches of the hyperbola

$$400x^2 - 625y^2 = 250{,}000,$$

where x and y are in meters.

(a) How far apart are the buildings at their closest point?
(b) Find the distance d in the figure.

62. Suppose a hyperbola has center at the origin, foci at $F'(-c, 0)$ and $F(c, 0)$, and

$$|d(P, F') - d(P, F)| = 2a.$$

Let $b^2 = c^2 - a^2$, and show that an equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

6.4 Summary of the Conic Sections

- Characteristics
- Identifying Conic Sections
- Geometric Definition of Conic Sections

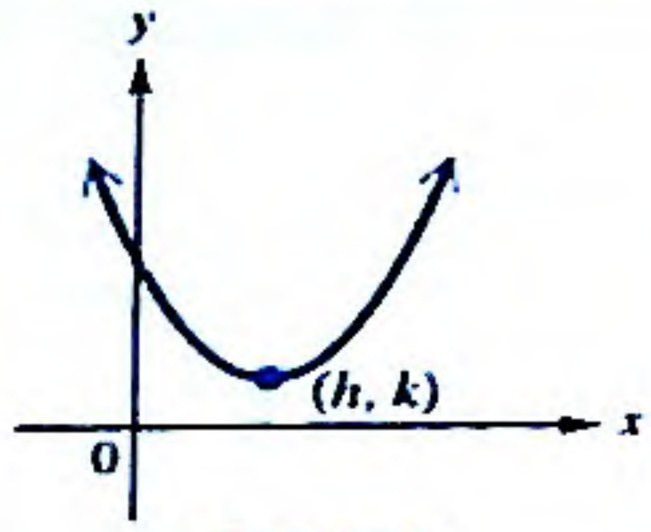
Characteristics The graphs of parabolas, circles, ellipses, and hyperbolas are called conic sections since each graph can be obtained by cutting a cone with a plane, as suggested by **Figure 1** at the beginning of the chapter. All conic sections of the types presented in this chapter have equations of the general form

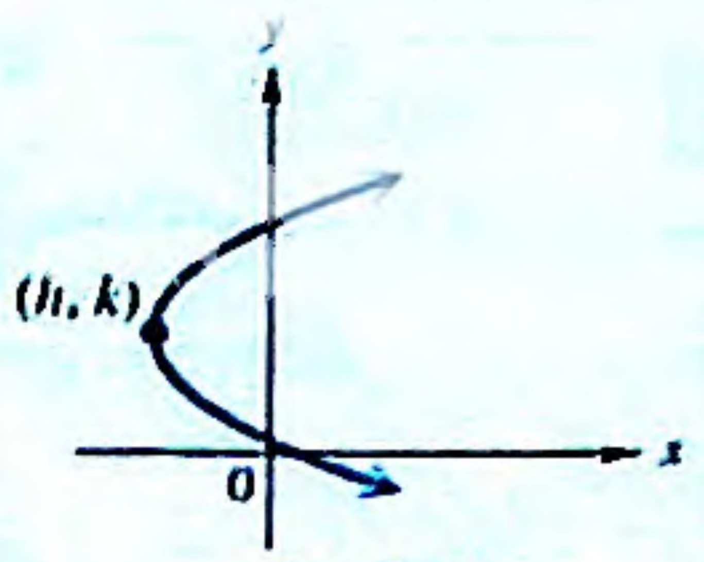
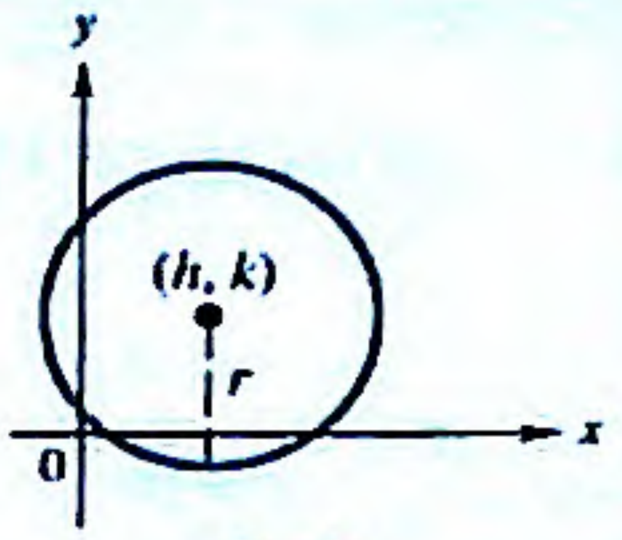
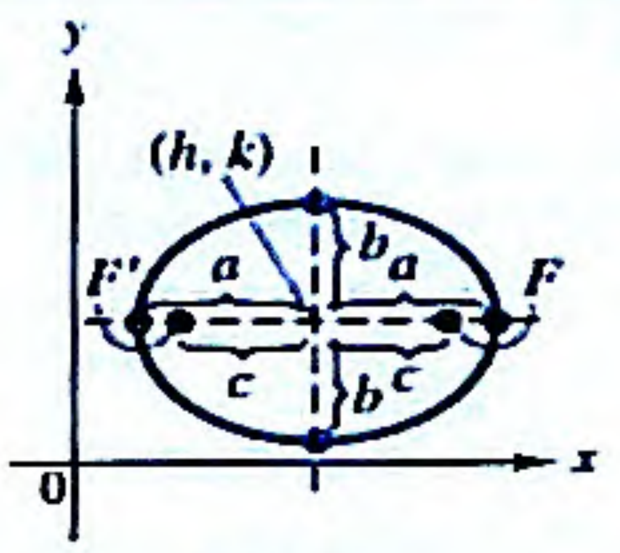
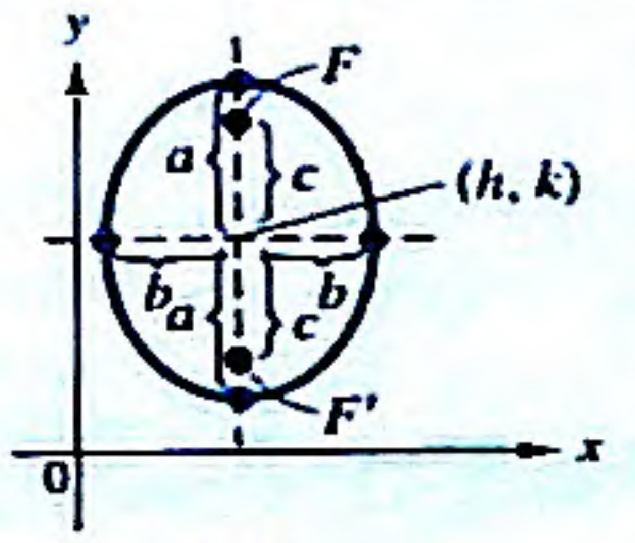
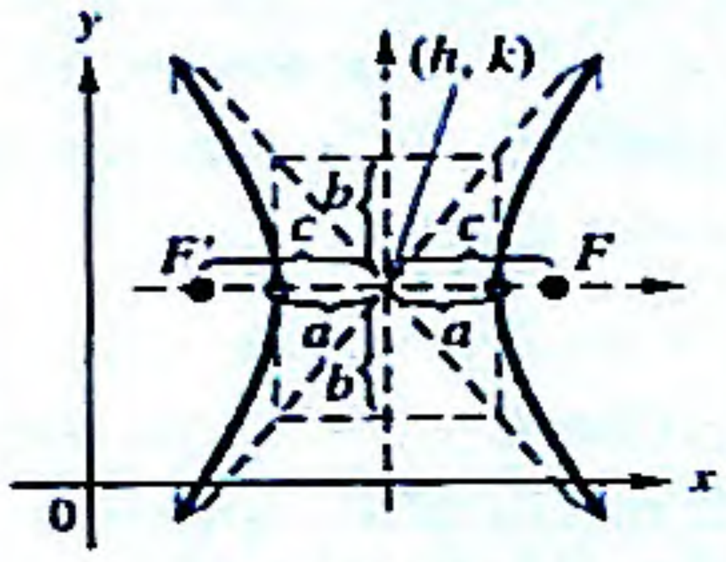
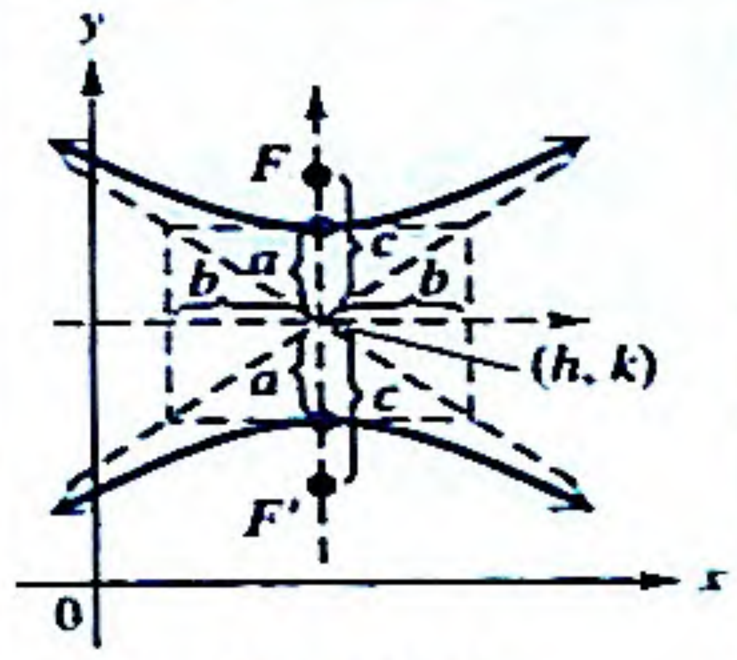
$$Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

where either A or C must be nonzero. The special characteristics of the general equations of the conic sections presented earlier are summarized below.

Conic Section	Characteristic	Example
Parabola	Either $A = 0$ or $C = 0$, but not both.	$x^2 - y - 4 = 0$ $y^2 - x - 4y = 0$
Circle	$A = C \neq 0$	$x^2 + y^2 - 16 = 0$
Ellipse	$A \neq C, AC > 0$	$25x^2 + 16y^2 - 400 = 0$
Hyperbola	$AC < 0$	$x^2 - y^2 - 1 = 0$

The following chart summarizes our work with conic sections.

Equation	Graph	Description	Identification
$(x - h)^2 = 4p(y - k)$ or $y - k = a(x - h)^2$	 Parabola	Graph opens up if $p > 0$ (or $a > 0$), down if $p < 0$ (or $a < 0$) Vertex is (h, k). Axis of symmetry is $x = h$	There is an x^2-term. y is not squared.

Equation	Graph	Description	Identification
$(y-k)^2 = 4p(x-h)$ or $x - h = a(y-k)^2$	 Parabola	Graph opens to the right if $p > 0$ (or $a > 0$), to the left if $p < 0$ (or $a < 0$). Vertex is (h, k). Axis of symmetry is $y = k$.	There is a y^2-term. x is not squared.
$(x-h)^2 + (y-k)^2 = r^2$	 Circle	Center is (h, k), radius is r.	x^2- and y^2-terms have the same positive coefficient.
$\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1 \quad (a > b)$	 y (h, k) F' a b a F c b c 0 x Ellipse	Horizontal major axis, length $= 2a$. $c^2 = a^2 - b^2$ Center is (h, k).	x^2- and y^2-terms have different positive coefficients.
$\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1 \quad (a > b)$	 y F a c (h, k) b a b c F' 0 x Ellipse	Vertical major axis, length $= 2a$. $c^2 = a^2 - b^2$ Center is (h, k).	x^2- and y^2-terms have different positive coefficients.
$\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$	 y (h, k) b c c F' F a a b 0 x Hyperbola	Graph has horizontal transverse axis. $c^2 = a^2 + b^2$ Asymptotes are $y = \pm\frac{b}{a}(x-h) + k$. Center is (h, k).	x^2-term has a positive coefficient. y^2-term has a negative coefficient.
$\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$	 y F b a c b a c (h, k) F' 0 x Hyperbola	Graph has vertical transverse axis. $c^2 = a^2 + b^2$ Asymptotes are $y = \pm\frac{a}{b}(x-h) + k$. Center is (h, k).	y^2-term has a positive coefficient. x^2-term has a negative coefficient.

Geometric Definition of Conic Sections In **Section 6.1**, a parabola was defined as the set of points in a plane equidistant from a fixed point (focus) and a fixed line (directrix). A parabola has eccentricity 1. This definition can be generalized to apply to the ellipse and the hyperbola. **Figure 39** shows an ellipse with $a = 4$, $c = 2$, and $e = \frac{1}{2}$. The line $x = 8$ is shown also. For any point P on the ellipse,

$$\text{distance of } P \text{ from the focus} = \frac{1}{2}[\text{distance of } P \text{ from the line}].$$

Figure 40 shows a hyperbola with $a = 2$, $c = 4$, and $e = 2$, along with the line $x = 1$. For any point P on the hyperbola,

$$\text{distance of } P \text{ from the focus} = 2[\text{distance of } P \text{ from the line}].$$

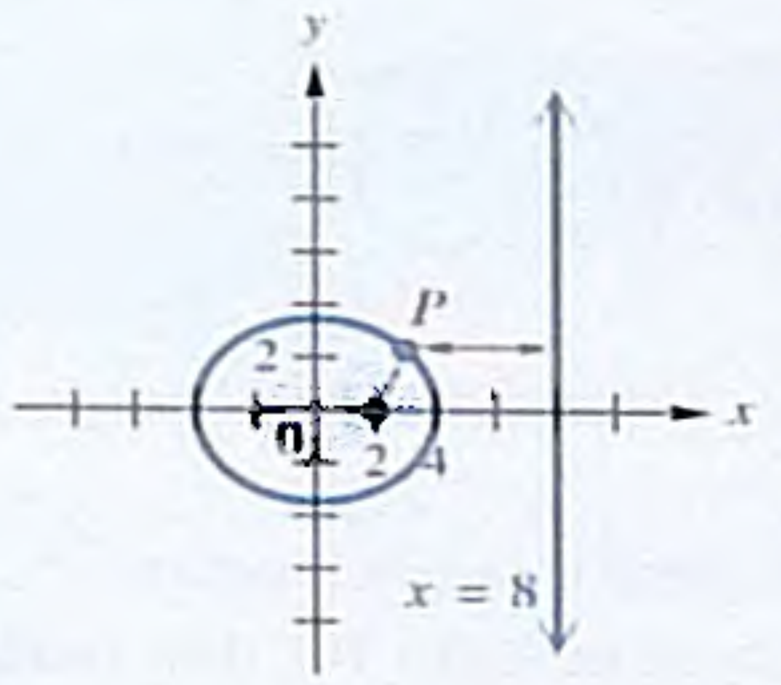


Figure 39

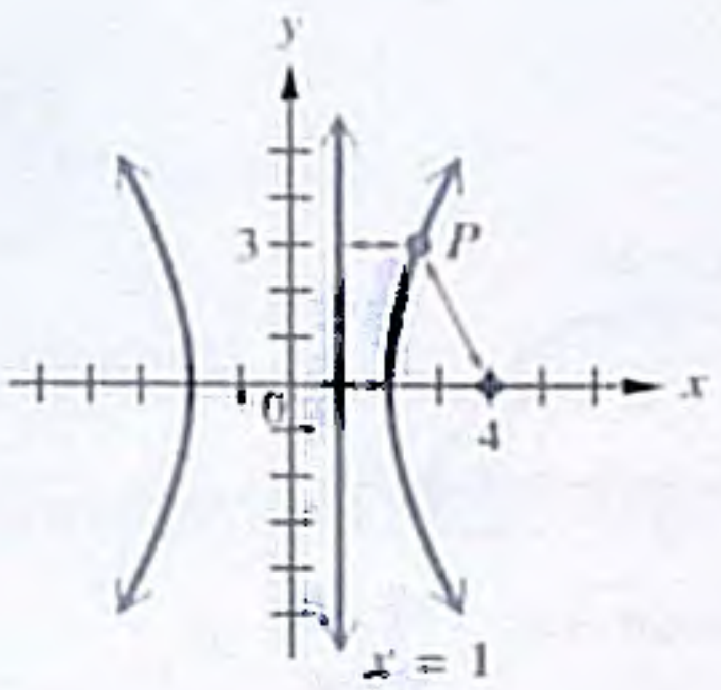


Figure 40

The following geometric definition applies to all conic sections except circles, which have $e = 0$.

Geometric Definition of a Conic Section

Given a fixed point F (focus), a fixed line L (directrix), and a positive number e, the set of all points P in the plane such that

$$\textbf{distance of } P \textbf{ from } F = e \cdot [\textbf{distance of } P \textbf{ from } L]$$

is a conic section of eccentricity e. *The conic section is a parabola when* $e = 1$, *an ellipse when* $0 < e < 1$, *and a hyperbola when* $e > 1$.

6.4 Exercises

The equation of a conic section is given in a familiar form. Identify the type of graph that each equation has, without actually graphing. See Examples 1 and 2.

1. $x^2 + y^2 = 144$

2. $(x - 2)^2 + (y + 3)^2 = 25$

3. $y = 2x^2 + 3x - 4$

4. $x = 3y^2 + 5y - 6$

5. $x - 1 = -3(y - 4)^2$

6. $\frac{x^2}{25} + \frac{y^2}{36} = 1$

7. $\frac{x^2}{49} + \frac{y^2}{100} = 1$

8. $x^2 - y^2 = 1$

9. $\frac{x^2}{4} - \frac{y^2}{16} = 1$

10. $\frac{(x+2)^2}{9} + \frac{(y-4)^2}{16} = 1$

11. $\frac{x^2}{25} - \frac{y^2}{25} = 1$

12. $y + 7 = 4(x+3)^2$

13. $\frac{x^2}{4} = 1 - \frac{y^2}{9}$

14. $\frac{x^2}{4} = 1 + \frac{y^2}{9}$

15. $\frac{(x+3)^2}{16} + \frac{(y-2)^2}{16} = 1$

16. $x^2 = 25 - y^2$

17. $x^2 - 6x + y = 0$

18. $11 - 3x = 2y^2 - 8y$

19. $4(x-3)^2 + 3(y+4)^2 = 0$

20. $2x^2 - 8x + 2y^2 + 20y = 12$

21. $x - 4y^2 - 8y = 0$

22. $x^2 + 2x = -4y$

23. $4x^2 - 24x + 5y^2 + 10y + 41 = 0$

24. $6x^2 - 12x + 6y^2 - 18y + 25 = 0$

Determine the type of conic section represented by each equation, and graph it. See Examples 1 and 2.

25. $\frac{x^2}{4} + \frac{y^2}{4} = -1$

26. $\frac{x^2}{4} + \frac{y^2}{4} = 1$

27. $x^2 = 25 + y^2$

28. $9x^2 + 36y^2 = 36$

29. $x^2 = 4y - 8$

30. $\frac{(x-4)^2}{8} + \frac{(y+1)^2}{2} = 0$

31. $y^2 - 4y = x + 4$

32. $(x+7)^2 + (y-5)^2 + 4 = 0$

33. $3x^2 + 6x + 3y^2 - 12y = 12$

34. $-4x^2 + 8x + y^2 + 6y = -6$

35. $4x^2 - 8x + 9y^2 - 36y = -4$

36. $3x^2 + 12x + 3y^2 = 0$

In Exercises 37–40, identify which type of conic section is described.

37. The conic section that consists of the set of all points in the plane for which the sum of the distances from the points $(5, 0)$ and $(-5, 0)$ is 14

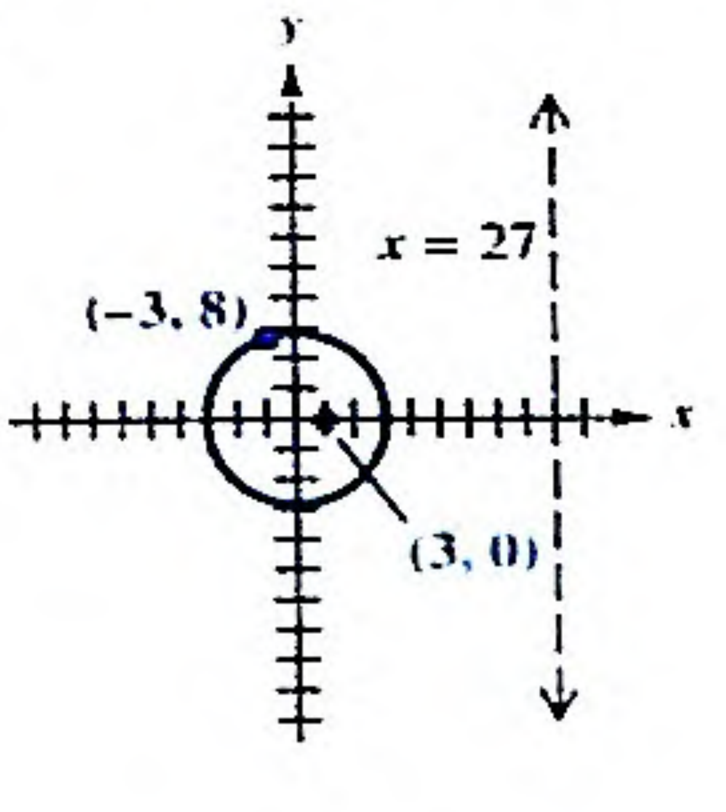
38. The conic section that consists of the set of all points in the plane for which the absolute value of the difference of the distances from the points $(3, 0)$ and $(-3, 0)$ is 2

39. The conic section that consists of the set of all points in the plane for which the distance from the point $(3, 0)$ is one and one-half times the distance from the line $x = \frac{4}{3}$

40. The conic section that consists of the set of all points in the plane for which the distance from the point $(2, 0)$ is one-third of the distance from the line $x = 10$

Find the eccentricity of each conic section. The point shown on the x-axis is a focus, and the line shown is a directrix.

41.



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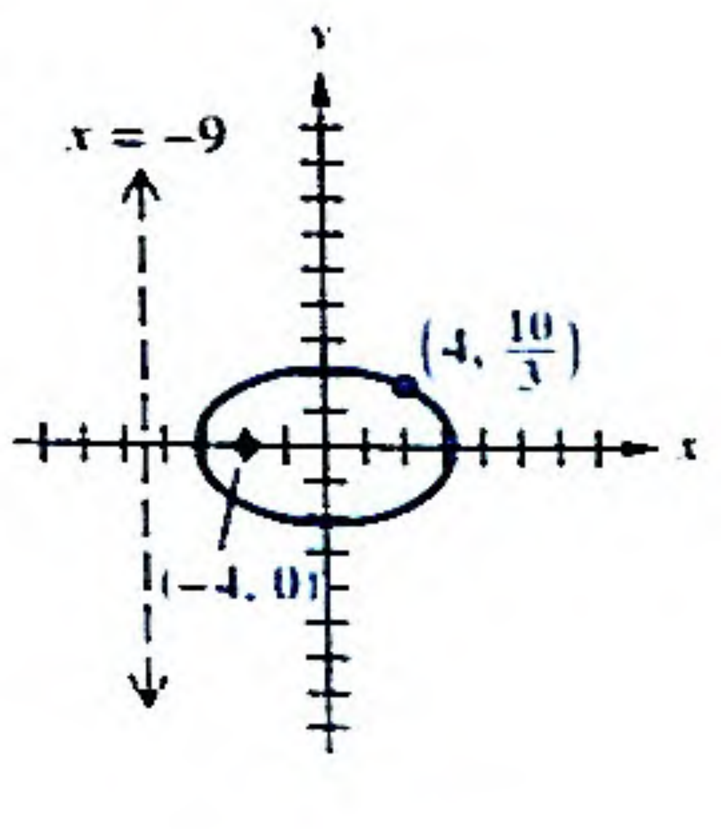

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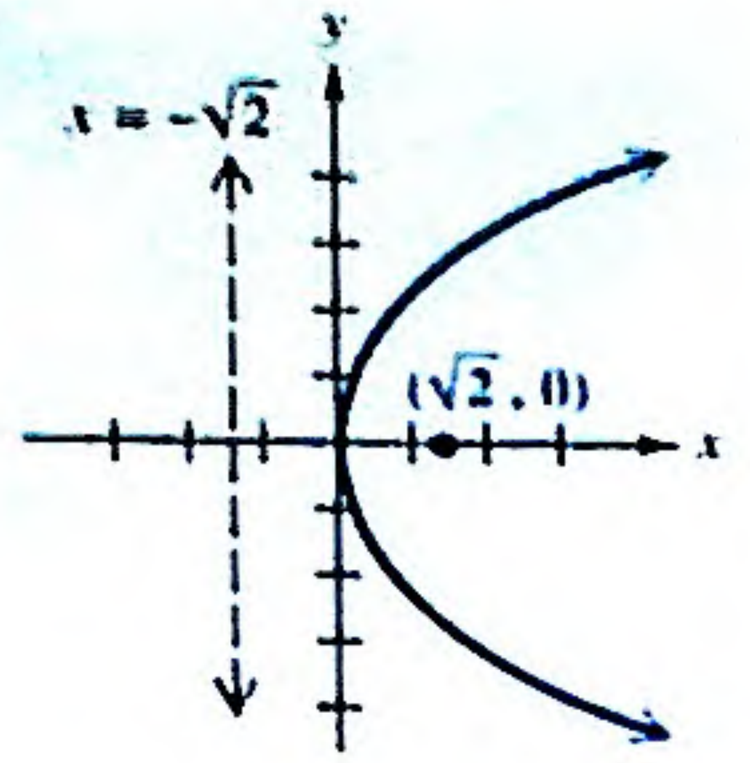

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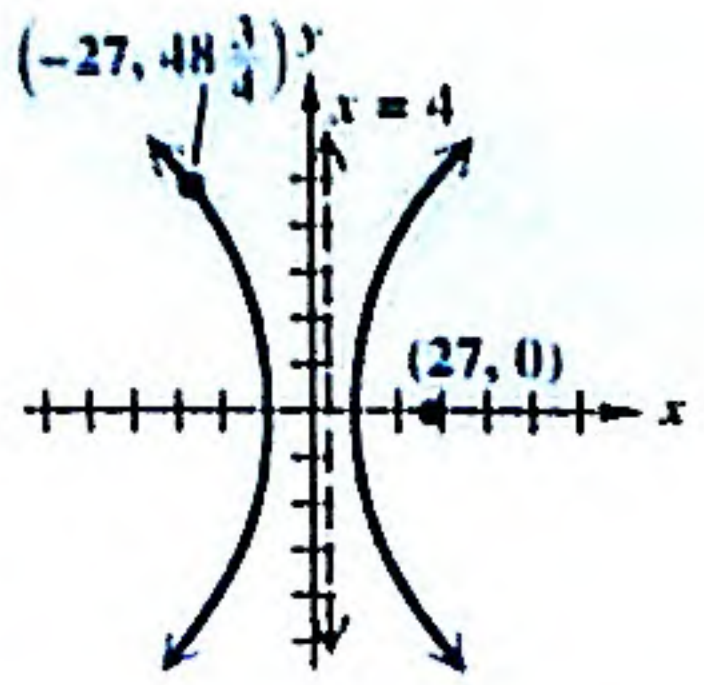

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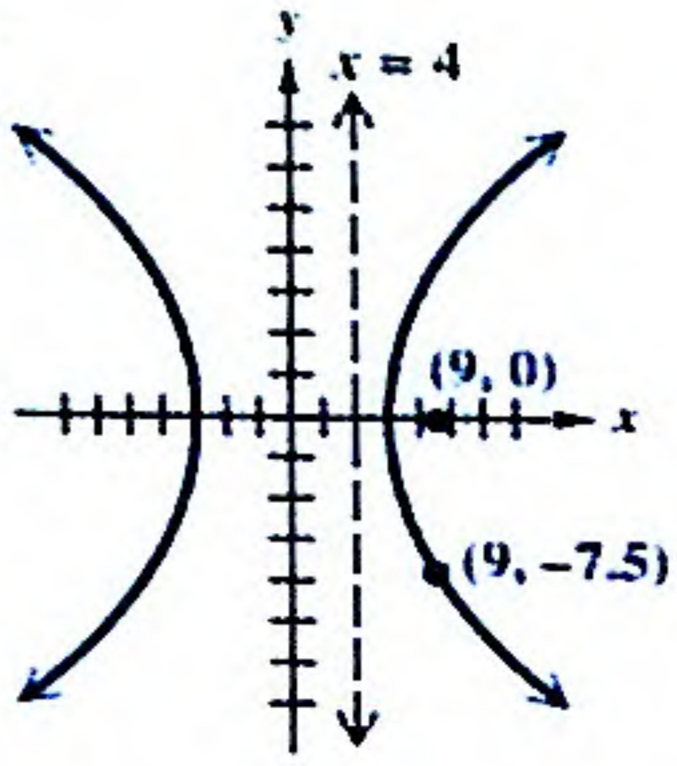

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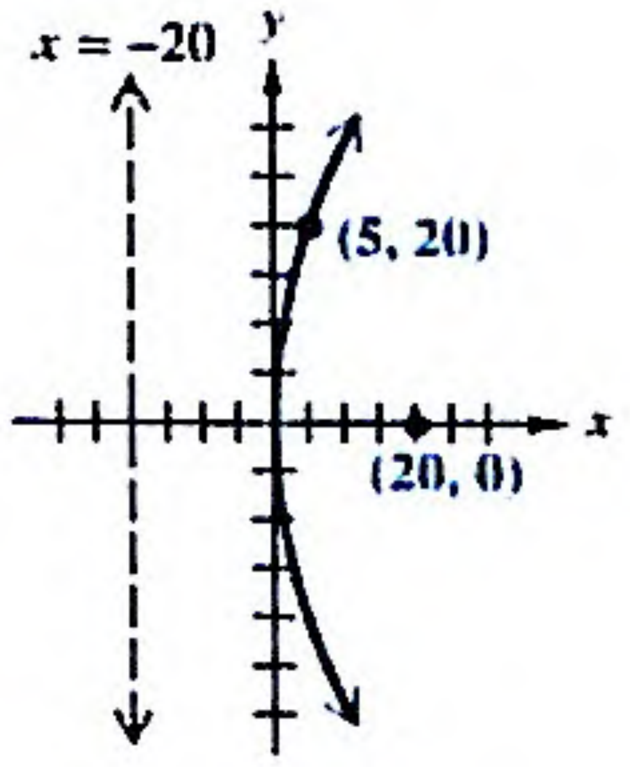


Satellite Trajectory *When a satellite is near Earth, its orbital trajectory may trace out a hyperbola, a parabola, or an ellipse. The type of trajectory depends on the satellite's velocity V in meters per second. It will be hyperbolic if* $V > \frac{k}{\sqrt{D}}$, *parabolic if* $V = \frac{k}{\sqrt{D}}$, *and elliptical if* $V < \frac{k}{\sqrt{D}}$, *where* $k = 2.82 \times 10^7$ *is a constant and D is the distance in meters from the satellite to the center of Earth. Use this information in Exercises 47–49.* (*Source:* Loh, W., *Dynamics and Thermodynamics of Planetary Entry,* Prentice-Hall, *and* Thomson, W., *Introduction to Space Dynamics,* John Wiley and Sons.)

47. When the artificial satellite Explorer IV was at a maximum distance D of 42.5×10^6 m from Earth's center, it had a velocity V of 2090 m per sec. Determine the shape of its trajectory.

48. If a satellite is scheduled to leave Earth's gravitational influence, its velocity must be increased so that its trajectory changes from elliptical to hyperbolic. Determine the minimum increase in velocity necessary for Explorer IV to escape Earth's gravitational influence when $D = 42.5 \times 10^6$ m.

49. Explain why it is easier to change a satellite's trajectory from an ellipse to a hyperbola when D is maximum rather than minimum.

Solve each problem.

50. If $Ax^2 + Cy^2 + Dx + Ey + F = 0$ is the general equation of an ellipse, find its center point by completing the square

51. Graph the ellipse $\frac{x^2}{16} + \frac{y^2}{12} = 1$ with a graphing calculator Trace to find the coordinates of several points on the ellipse. For each of these points P, verify that

$$\text{distance of } P \text{ from } (2, 0) = \frac{1}{2}\left[\text{distance of } P \text{ from the line } x = 8\right].$$

52. Graph the hyperbola $\frac{x^2}{4} - \frac{y^2}{12} = 1$ with a graphing calculator. Trace to find the coordinates of several points on the hyperbola. For each of these points P, verify that

$$\text{distance of } P \text{ from } (4, 0) = 2\left[\text{distance of } P \text{ from the line } x = 1\right]$$

Chapter 6 Test Prep

Key Terms

6.1 conic sections
parabola
focus
directrix

6.2 ellipse
foci
major axis
minor axis
center
vertices
conic
eccentricity

6.3 hyperbola
transverse axis
asymptotes
fundamental rectangle

New Symbols

e eccentricity

Quick Review

Concepts

Examples

6.1 Parabolas

Parabola with Vertical Axis and Vertex $(0, 0)$
The parabola with focus $(0, p)$ and directrix $y = -p$ has equation $x^2 = 4py$. The parabola has vertical axis of symmetry $x = 0$ and opens up if $p > 0$ or down if $p < 0$.

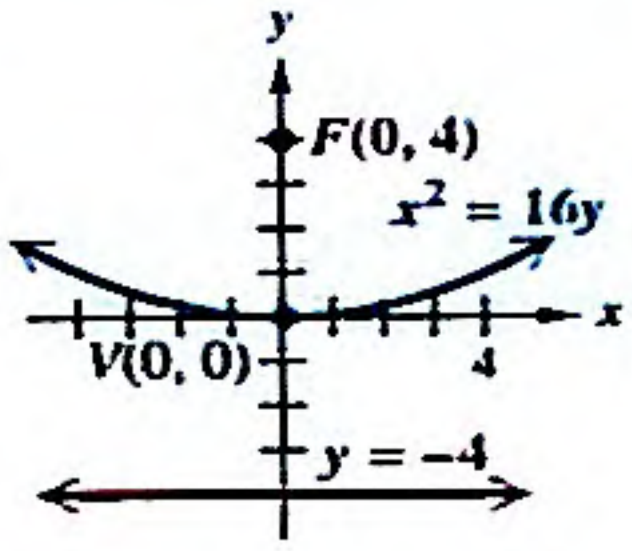
Parabola with Horizontal Axis and Vertex $(0, 0)$
The parabola with focus $(p, 0)$ and directrix $x = -p$ has equation $y^2 = 4px$. The parabola has horizontal axis of symmetry $y = 0$ and opens to the right if $p > 0$ or to the left if $p < 0$.

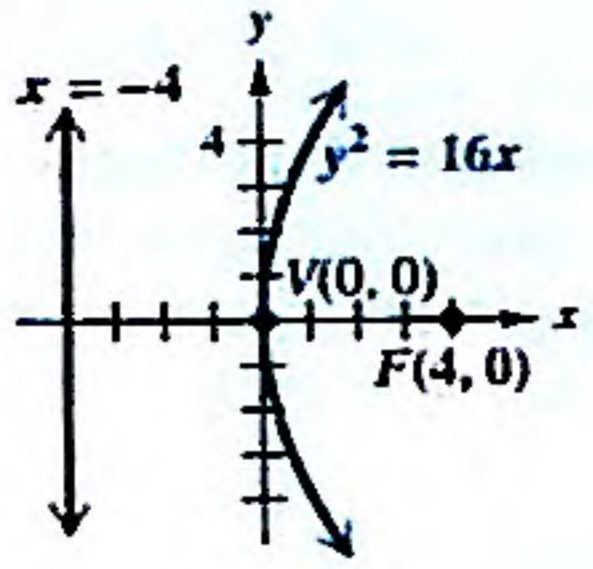
Equation Forms for Translated Parabolas
A parabola with vertex (h, k) has an equation of the form

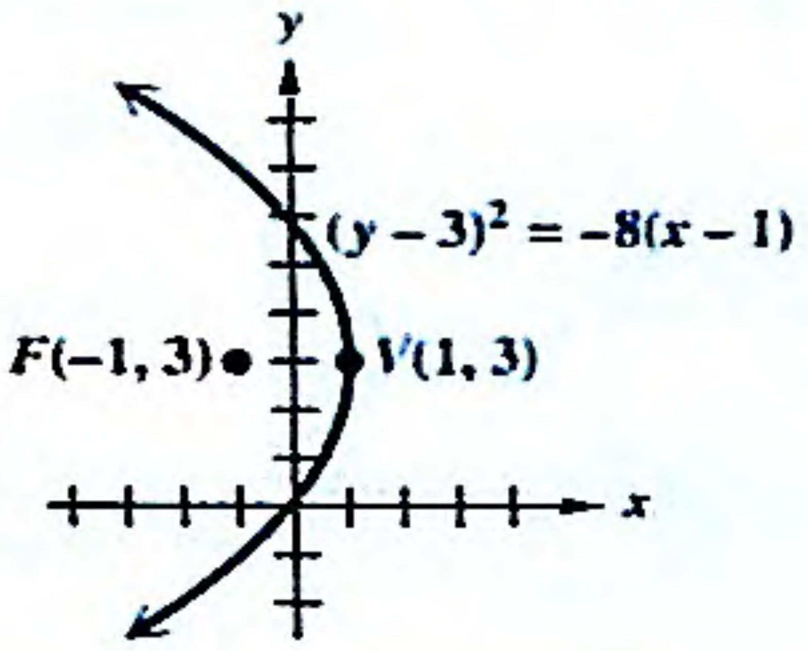
$$(x - h)^2 = 4p(y - k) \quad \text{Vertical axis of symmetry}$$

or $$(y - k)^2 = 4p(x - h), \quad \text{Horizontal axis of symmetry}$$

where p is the directed distance from the vertex to the focus.







6.2 Ellipses

Standard Forms of Equations for Ellipses
The ellipse with center at the origin and equation

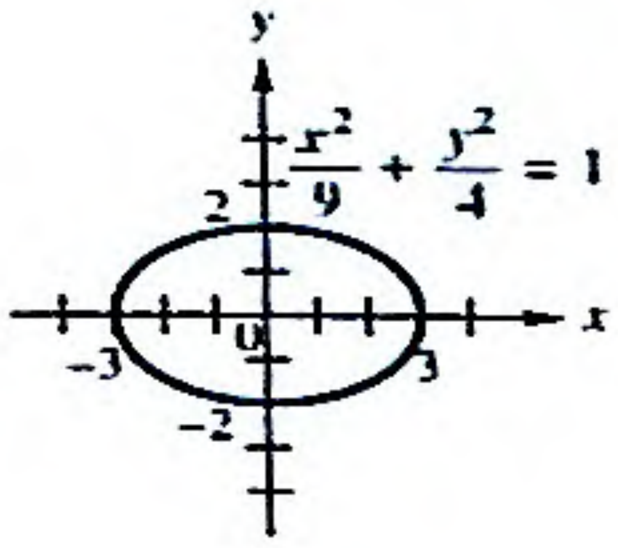
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (\text{where } a > b)$$

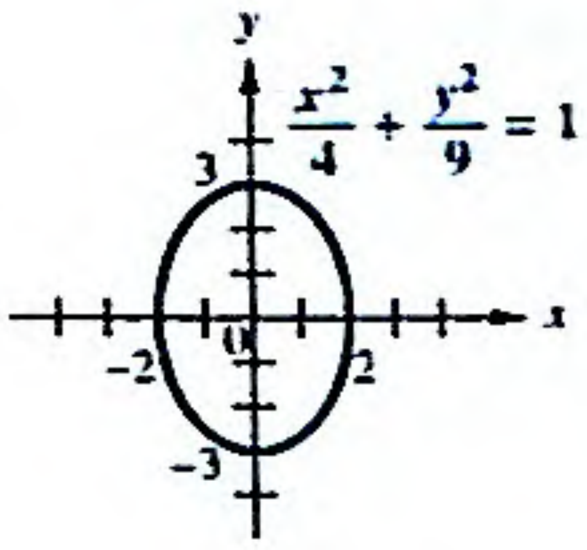
has vertices $(\pm a, 0)$, endpoints of the minor axis $(0, \pm b)$, and foci $(\pm c, 0)$, where $c^2 = a^2 - b^2$.

The ellipse with center at the origin and equation

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad (\text{where } a > b)$$

has vertices $(0, \pm a)$, endpoints of the minor axis $(\pm b, 0)$, and foci $(0, \pm c)$, where $c^2 = a^2 - b^2$.





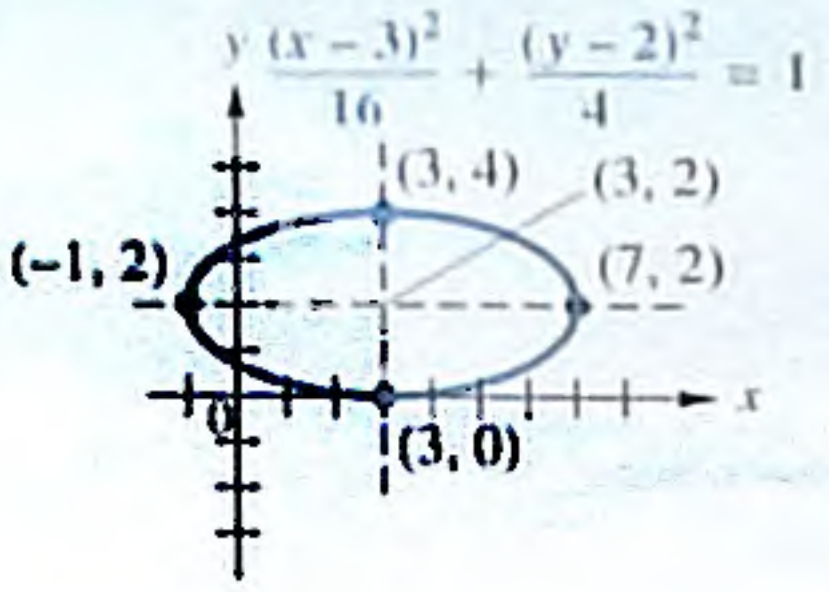
(continued)

Concepts

Examples

Translated Ellipses

The preceding equations can be extended to ellipses having center (h, k) by replacing x and y with $x - h$ and $y - k$, respectively.



6.3 Hyperbolas

Standard Forms of Equations for Hyperbolas

The hyperbola with center at the origin and equation

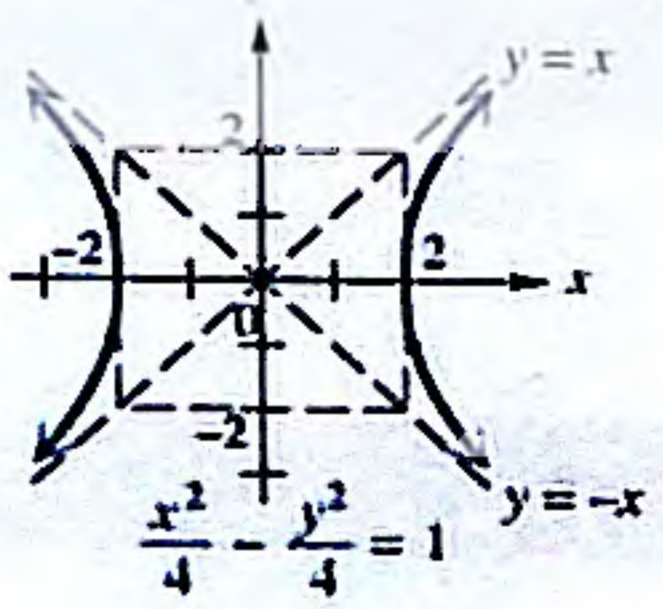
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

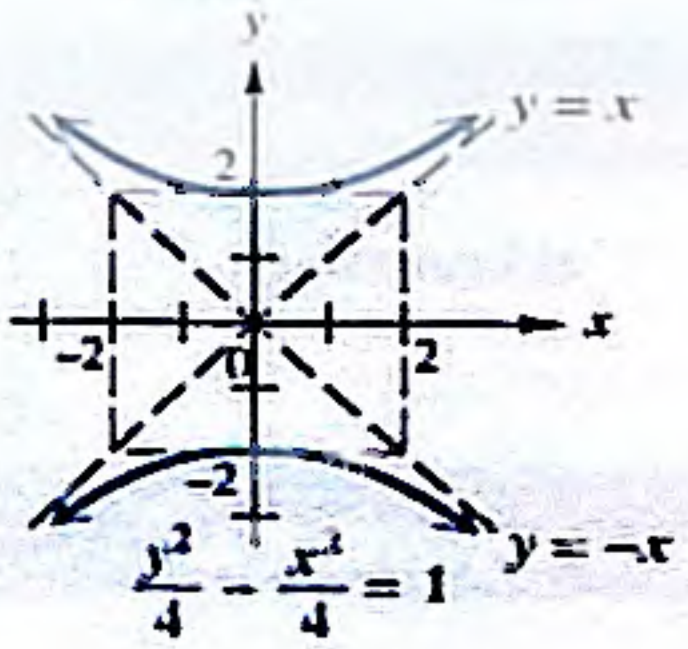
has vertices $(\pm a, 0)$, asymptotes $y = \pm\frac{b}{a}x$, and foci $(\pm c, 0)$, where $c^2 = a^2 + b^2$.

The hyperbola with center at the origin and equation

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

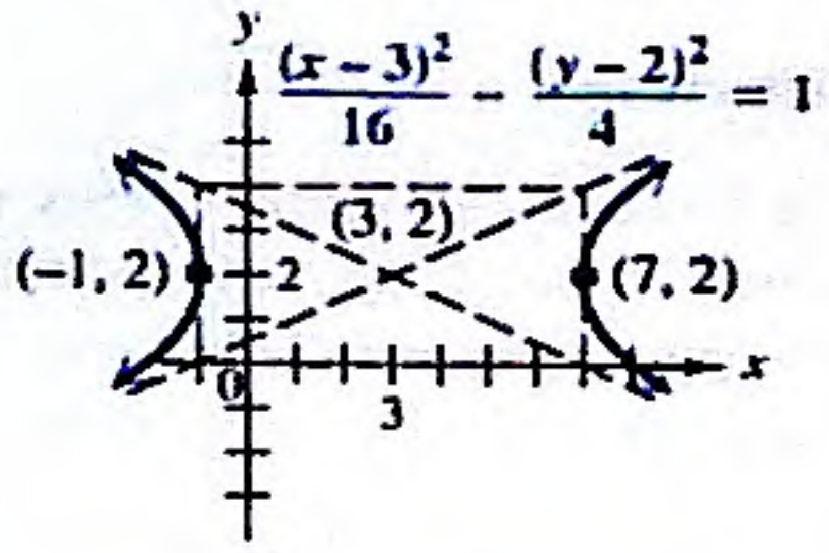
has vertices $(0, \pm a)$, asymptotes $y = \pm\frac{a}{b}x$, and foci $(0, \pm c)$, where $c^2 = a^2 + b^2$.





Translated Hyperbolas

The preceding equations can be extended to hyperbolas having center (h, k) by replacing x and y with $x - h$ and $y - k$, respectively.



6.4 Summary of the Conic Sections

Conic sections in this chapter have equations that can be written in the following form.

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

Conic Section	Characteristic	Example
Parabola	Either $A = 0$ or $C = 0$, but not both.	$x^2 - y - 4 = 0$ $y^2 - x - 4y = 0$
Circle	$A = C \neq 0$	$x^2 + y^2 - 16 = 0$
Ellipse	$A \neq C, AC > 0$	$25x^2 + 16y^2 - 400 = 0$
Hyperbola	$AC < 0$	$x^2 - y^2 - 1 = 0$

See the summary chart in **Section 6.4.**

$y^2 - 4x - 10y + 21 = 0$, or $(y - 5)^2 = 4(x + 1)$, is a parabola with vertex $(-1, 5)$, opening to the right.

$$x^2 - 4x + y^2 + 2y - 4 = 0,$$

or

$$(x - 2)^2 + (y + 1)^2 = 9,$$

is a circle centered at $(2, -1)$ with radius 3.

$$4x^2 + y^2 - 16 = 0, \quad \text{or} \quad \frac{x^2}{4} + \frac{y^2}{16} = 1,$$

is an ellipse centered at the origin with vertical major axis.

$$4x^2 - y^2 - 8x - 4y - 16 = 0,$$

or

$$\frac{(x - 1)^2}{4} - \frac{(y + 2)^2}{16} = 1,$$

is a hyperbola centered at $(1, -2)$ with horizontal transverse axis.

Chapter 6 Review Exercises

Graph each parabola. In Exercises 1–4, give the domain, range, vertex, and axis of symmetry. In Exercises 5–8, give the domain, range, focus, directrix, and axis of symmetry.

1. $x = 4(y - 5)^2 + 2$

2. $x = -(y + 1)^2 - 7$

3. $x = 5y^2 - 5y + 3$

4. $x = 2y^2 - 4y + 1$

5. $y^2 = -\frac{2}{3}x$

6. $y^2 = 2x$

7. $3x^2 = y$

8. $x^2 + 2y = 0$

Write an equation for each parabola with vertex at the origin.

9. focus $(4, 0)$

10. focus $(0, -3)$

11. through $(-3, 4)$, opening up

12. through $(2, 5)$, opening right

An equation of a conic section is given. Identify the type of conic section. It may be necessary to transform the equation into a more familiar form.

13. $y^2 + 9x^2 = 9$

14. $9x^2 - 16y^2 = 144$

15. $3y^2 - 5x^2 = 30$

16. $y^2 + x = 4$

17. $4x^2 - y = 0$

18. $x^2 + y^2 = 25$

19. $4x^2 - 8x + 9y^2 + 36y = -4$

20. $9x^2 - 18x - 4y^2 - 16y - 43 = 0$

Concept Check Match each equation in Exercises 21–26 with its calculator graph in choices A–F. In all cases except choice B, Xscl = Yscl = 1.

21. $4x^2 + y^2 = 36$

22. $x = 2y^2 + 3$

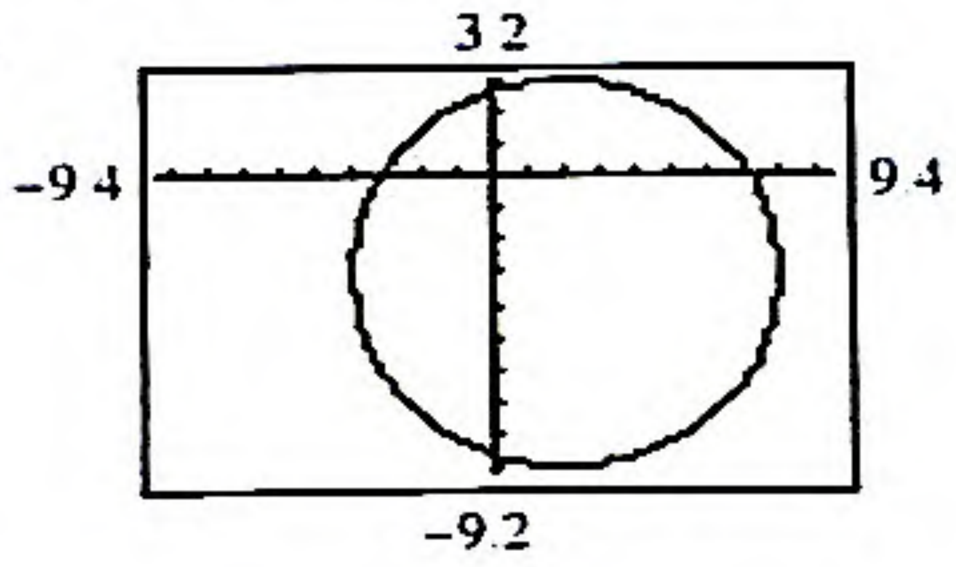
23. $(x - 2)^2 + (y + 3)^2 = 36$

24. $\frac{x^2}{36} + \frac{y^2}{9} = 1$

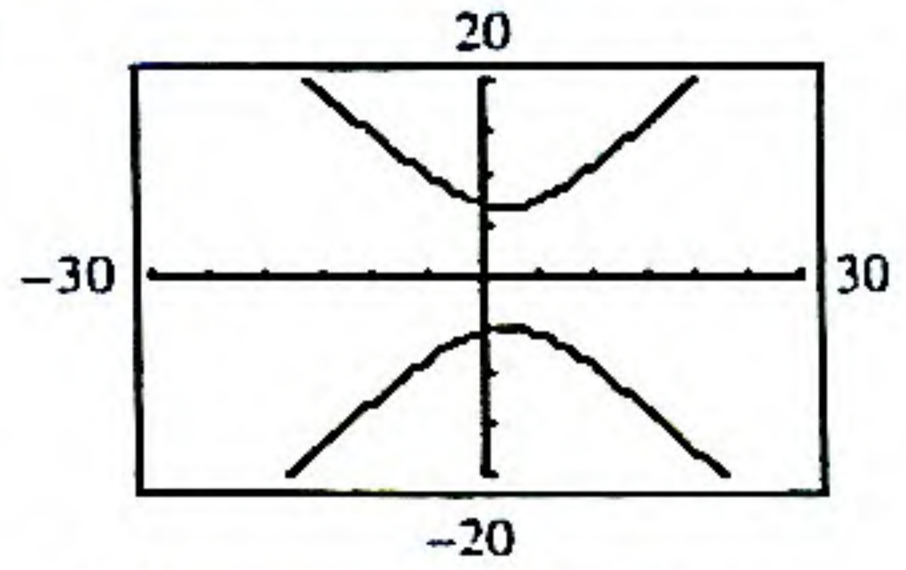
25. $(y - 1)^2 - (x - 2)^2 = 36$

26. $y^2 = 36 + 4x^2$

A.

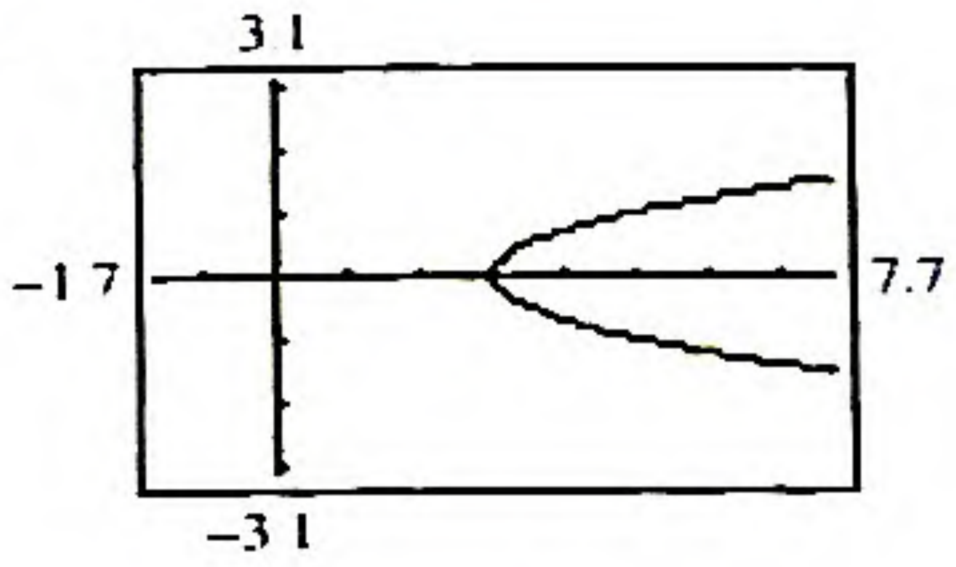


B.

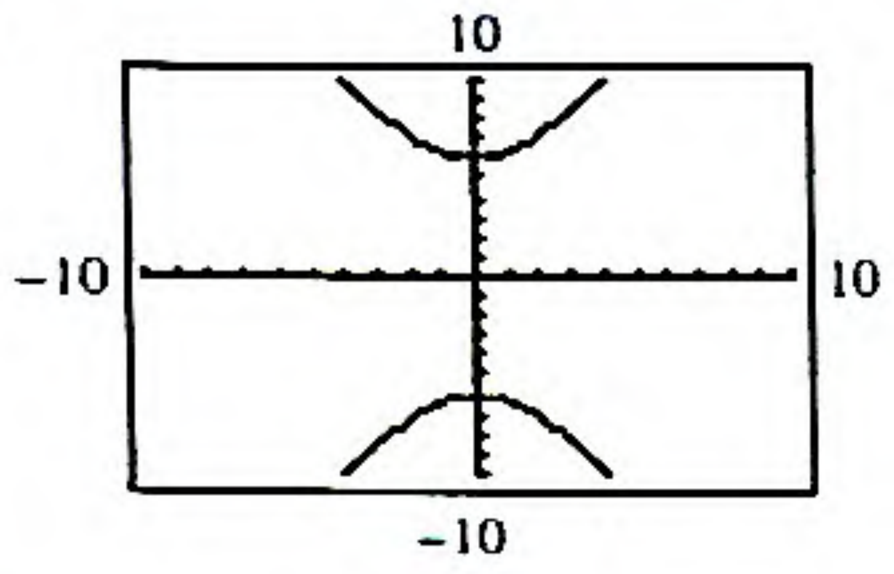


In this screen, Xscl = Yscl = 5.

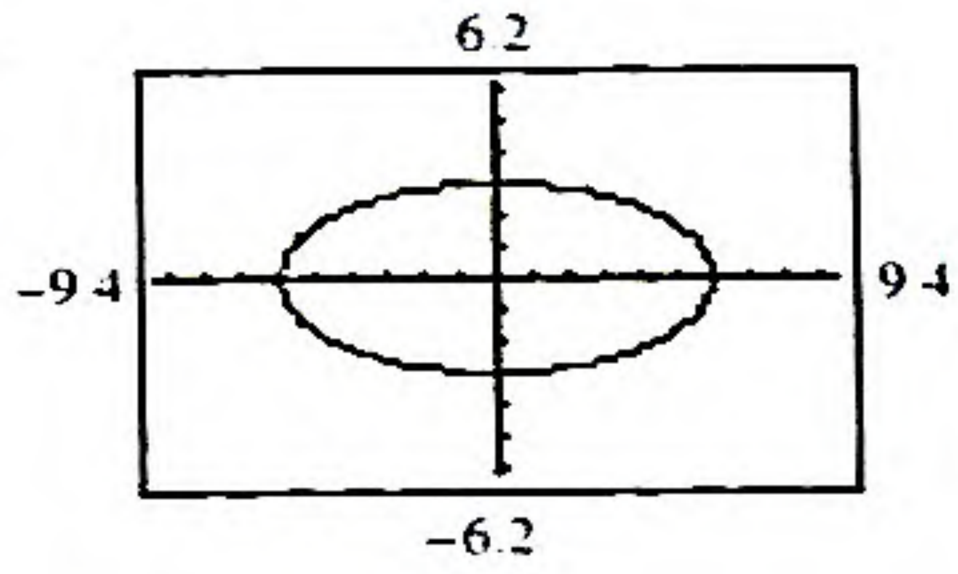
C.



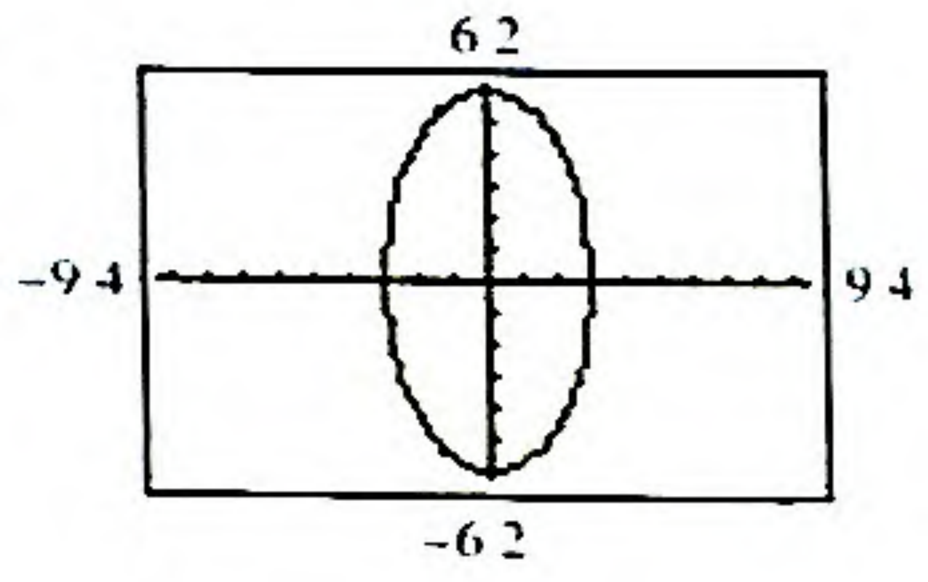
D.



E.



F.



Graph each relation and identify the graph. Give the domain, range, coordinates of the vertices for each ellipse or hyperbola, and equations of the asymptotes for each hyperbola. Give the domain and range for each circle.

27. $\frac{x^2}{4} + \frac{y^2}{9} = 1$

28. $\frac{x^2}{16} + \frac{y^2}{4} = 1$

29. $\frac{x^2}{64} - \frac{y^2}{36} = 1$

30. $\frac{y^2}{25} - \frac{x^2}{9} = 1$

31. $\frac{(x + 1)^2}{16} + \frac{(y - 1)^2}{16} = 1$

32. $(x - 3)^2 + (y + 2)^2 = 9$

33. $4x^2 + 9y^2 = 36$

34. $x^2 = 16 + y^2$

35. $\frac{(x - 3)^2}{4} + (y + 1)^2 = 1$

36. $\frac{(x - 2)^2}{9} + \frac{(y + 3)^2}{4} = 1$

37. $\frac{(y + 2)^2}{4} - \frac{(x + 3)^2}{9} = 1$

38. $\frac{(x + 1)^2}{16} - \frac{(y - 2)^2}{4} = 1$

39. $x^2 - 4x + y^2 + 6y = -12$

40. $4x^2 + 8x + 25y^2 - 250y = -529$

41. $5x^2 + 20x + 2y^2 - 8y = -18$

42. $-4x^2 + 8x + 4y^2 + 8y = 16$

Graph each relation. Give the domain and range, and identify any that are graphs of functions.

43. $\frac{x}{3} = -\sqrt{1 - \frac{y^2}{16}}$

44. $x = -\sqrt{1 - \frac{y^2}{36}}$

45. $y = -\sqrt{1 + x^2}$

46. $y = -\sqrt{1 - \frac{x^2}{25}}$

Write an equation for each conic section with center at the origin.

47. ellipse; vertex at $(0, -4)$, focus at $(0, -2)$

48. ellipse; x-intercept 6, focus at $(2, 0)$

49. hyperbola; focus at $(0, 5)$, transverse axis of length 8

50. hyperbola; y-intercept -2, passing through $(2, 3)$

Write an equation for each conic section satisfying the given conditions.

51. parabola with focus at $(3, 2)$ and directrix $x = -3$

52. parabola with vertex at $(-3, 2)$ and y-intercepts 5 and -1

53. ellipse with foci at $(-2, 0)$ and $(2, 0)$ and major axis of length 10

54. ellipse with foci at $(0, 3)$ and $(0, -3)$ and vertex at $(0, -7)$

55. hyperbola with x-intercepts ± 3 and foci at $(-5, 0)$ and $(5, 0)$

56. hyperbola with foci at $(0, 12)$ and $(0, -12)$ and asymptotes $y = \pm x$

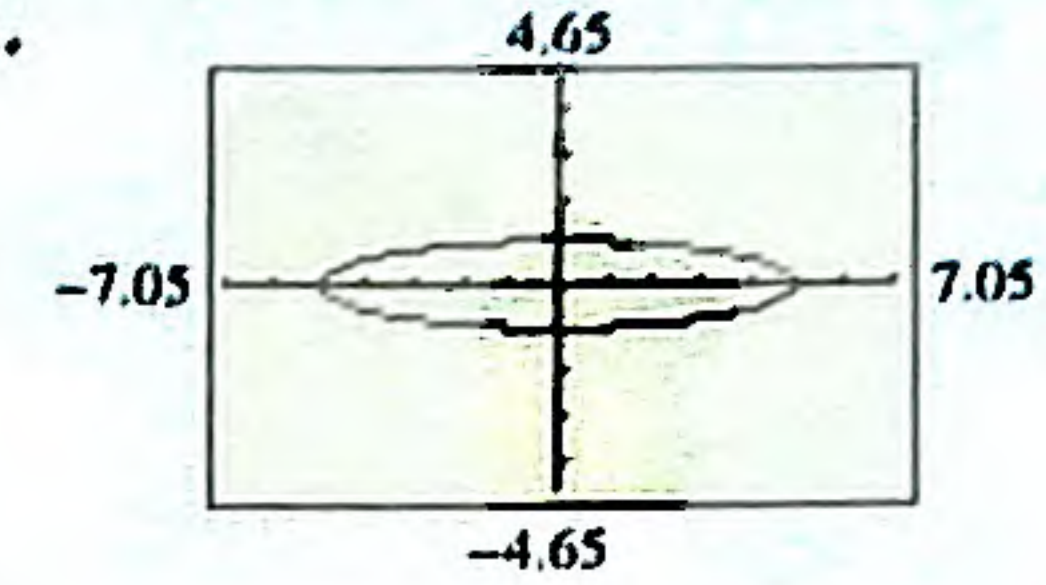
Solve each problem.

57. Find the equation of the ellipse consisting of all points in the plane the sum of whose distances from $(0, 0)$ and $(4, 0)$ is 8

58. Find the equation of the hyperbola consisting of all points in the plane for which the absolute value of the difference of the distances from $(0, 0)$ and $(0, 4)$ is 2

59. Calculator graphs are shown in choices A–D. Arrange the graphs so that their eccentricities are in increasing order.

A.



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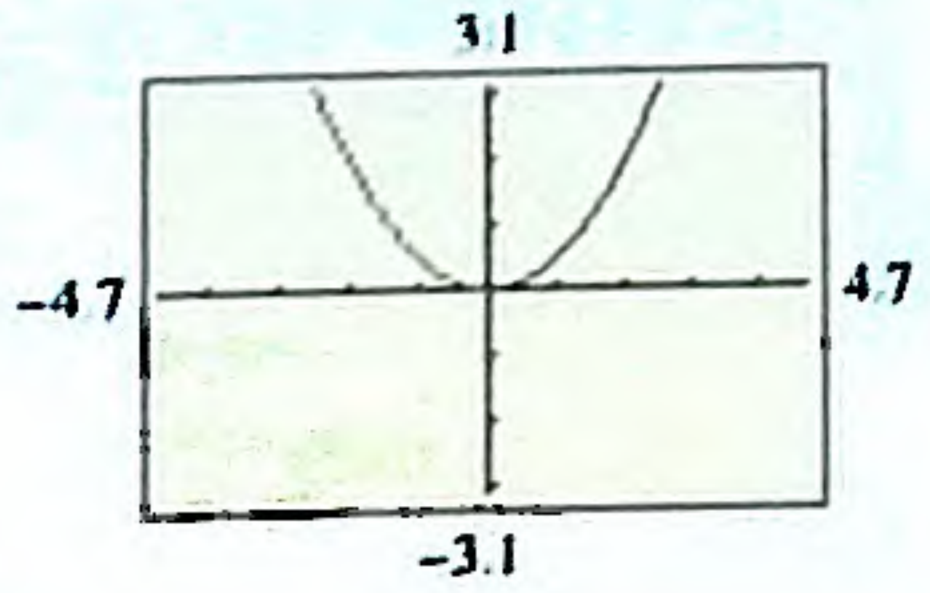

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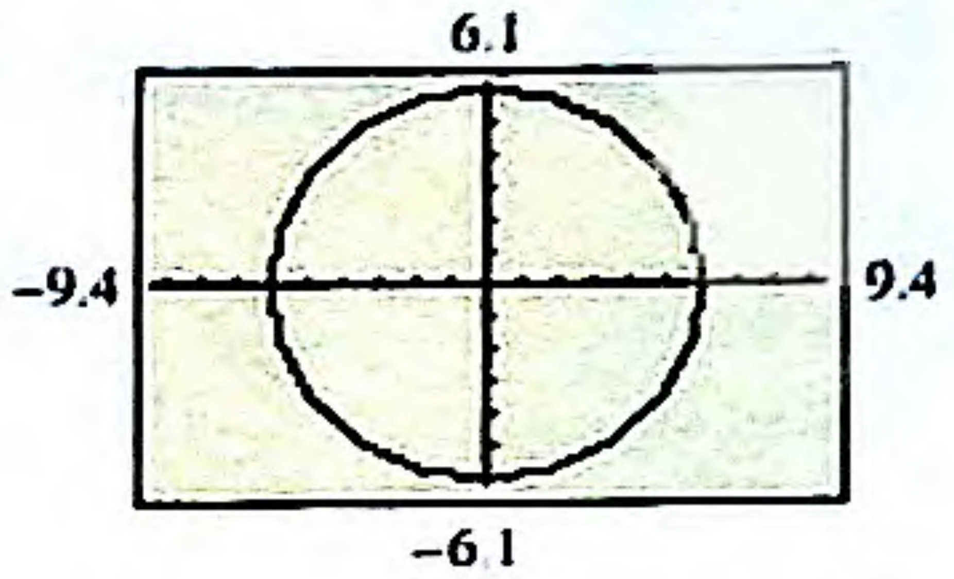

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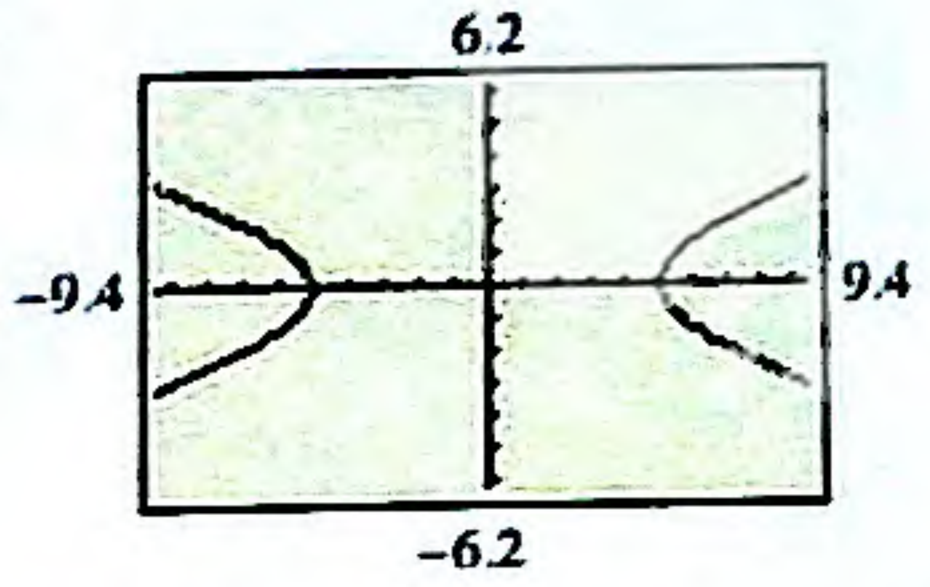


60. ***Orbit of Venus*** The orbit of Venus is an ellipse with the sun at one of the foci. The eccentricity of the orbit is $e = 0.006775$, and the major axis has length 134.5 million mi. (*Source: World Almanac and Book of Facts.*) Find the least and greatest distances of Venus from the sun.

61. ***Orbit of a Comet*** The comet Swift-Tuttle has an elliptical orbit of eccentricity $e = 0.964$, with the sun at one of the foci. Find the equation of the comet given that the closest it comes to the sun is 89 million mi.

62. Find the equation of the hyperbola consisting of all points P in the plane for which the absolute value of the difference of the distances of P from $(-5, 0)$ and $(5, 0)$ is 8. Then graph the hyperbola with a graphing calculator and trace to find the coordinates of several points on the graph of the hyperbola. For each of these points, verify that the absolute value of the differences of the distances is indeed 8.

Chapter 6 Test

Graph each parabola. Give the domain, range, vertex, and axis of symmetry.

1. $y = -x^2 + 6x$

2. $x = 4y^2 + 8y$

3. Give the coordinates of the focus and the equation of the directrix for the parabola with equation $x = 8y^2$.

4. Write an equation for the parabola with vertex $(2, 3)$, passing through the point $(-18, 1)$, and opening to the left.

5. Explain how to determine just by looking at the equation whether a parabola has a vertical or a horizontal axis of symmetry, and whether it opens up, down, to the left, or to the right.

Graph each ellipse. Give the domain and range.

6. $\dfrac{(x-8)^2}{100} + \dfrac{(y-5)^2}{49} = 1$

7. $16x^2 + 4y^2 = 64$

8. Graph $y = -\sqrt{1 - \dfrac{x^2}{36}}$. Tell whether the graph is that of a function.

9. Write an equation for the ellipse centered at the origin having horizontal major axis with length 6 and minor axis with length 4.

10. ***Height of the Arch of a Bridge*** An arch of a bridge has the shape of the top half of an ellipse. The arch is 40 ft wide and 12 ft high at the center. Find the equation of the complete ellipse. Find the height of the arch 10 ft from the center of the bottom.

Graph each hyperbola. Give the domain, range, and equations of the asymptotes.

11. $\frac{x^2}{4} - \frac{y^2}{4} = 1$

12. $9x^2 - 4y^2 = 36$

13. Find the equation of the hyperbola with y-intercepts ± 5 and foci at $(0, -6)$ and $(0, 6)$.

Identify the type of graph, if any, defined by each equation.

14. $x^2 + 8x + y^2 - 4y + 2 = 0$

15. $5x^2 + 10x - 2y^2 - 12y - 23 = 0$

16. $3x^2 + 10y^2 - 30 = 0$

17. $x^2 - 4y = 0$

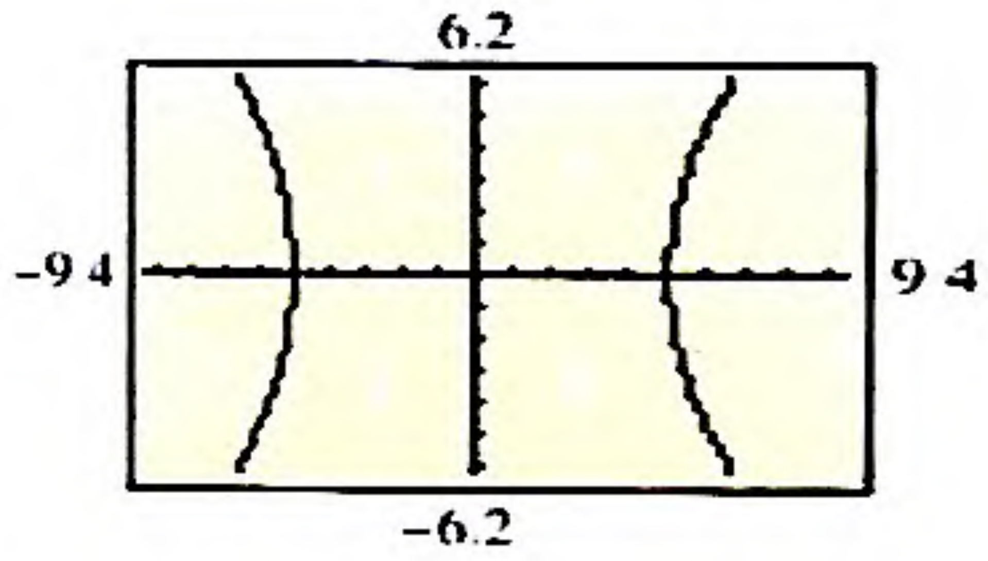
18. $(x + 9)^2 + (y - 3)^2 = 0$

19. $x^2 + 4x + y^2 - 6y + 30 = 0$

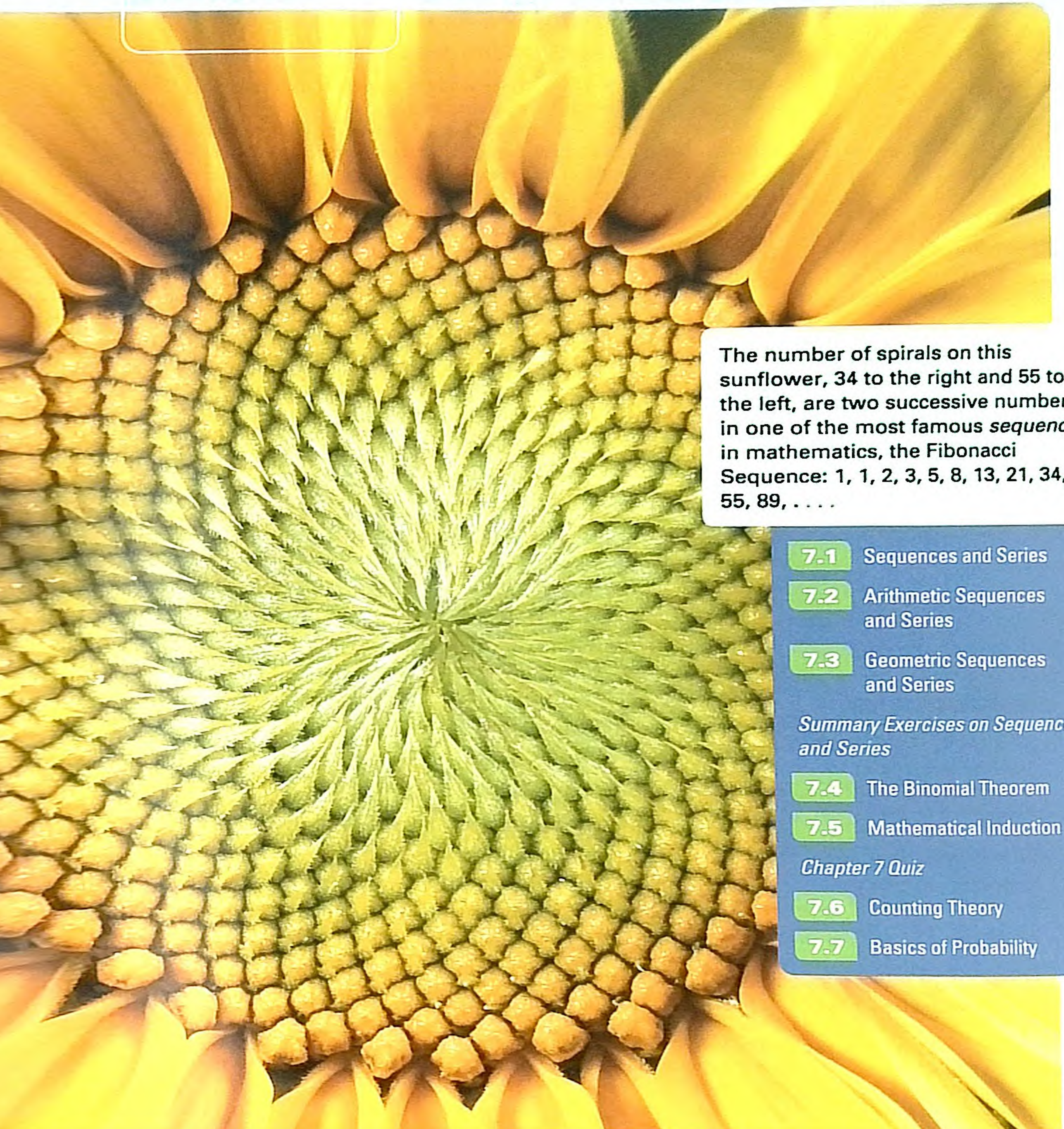
20. The screen shown here gives the graph of

$$\frac{x^2}{25} - \frac{y^2}{49} = 1$$

as generated by a graphing calculator. What two functions y_1 and y_2 were used to obtain the graph?



7 Further Topics in Algebra



The number of spirals on this sunflower, 34 to the right and 55 to the left, are two successive numbers in one of the most famous *sequences* in mathematics, the Fibonacci Sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

7.1 Sequences and Series

7.2 Arithmetic Sequences and Series

7.3 Geometric Sequences and Series

Summary Exercises on Sequences and Series

7.4 The Binomial Theorem

7.5 Mathematical Induction

Chapter 7 Quiz

7.6 Counting Theory

7.7 Basics of Probability

EXAMPLE 6 Using the Summation Properties

Use the summation properties to find each sum.

(a) $\sum_{i=1}^{40} 5$ (b) $\sum_{i=1}^{22} 2i$ (c) $\sum_{i=1}^{14} (2i^2 - 3)$

SOLUTION

(a)
$$\sum_{i=1}^{40} 5 = 40(5) = 200 \qquad \text{Property (a) with } n = 40 \text{ and } c = 5$$

(b)
$$\begin{aligned} \sum_{i=1}^{22} 2i &= 2\sum_{i=1}^{22} i && \text{Property (b) with } c = 2 \text{ and } a_i = i \\ &= 2 \cdot \frac{22(22+1)}{2} && \text{Summation rules} \\ &= 506 && \text{Evaluate} \end{aligned}$$

(c)
$$\begin{aligned} \sum_{i=1}^{14} (2i^2 - 3) &= \sum_{i=1}^{14} 2i^2 - \sum_{i=1}^{14} 3 && \text{Property (d) with } a_i = 2i^2 \text{ and } b_i = 3 \\ &= 2\sum_{i=1}^{14} i^2 - \sum_{i=1}^{14} 3 && \text{Property (b) with } c = 2 \text{ and } a_i = i^2 \\ &= 2 \cdot \frac{14(14+1)(2 \cdot 14 + 1)}{6} - 14(3) && \text{Summation rules and Property (a)} \\ &= 2030 - 42 && \text{Simplify.} \\ &= 1988 && \text{Subtract.} \end{aligned}$$

✓ *Now Try Exercises 67, 69, and 71.*

EXAMPLE 7 Using the Summation Properties

Evaluate $\sum_{i=1}^{6} (i^2 + 3i + 5)$.

SOLUTION

$$\begin{aligned} \sum_{i=1}^{6} (i^2 + 3i + 5) &= \sum_{i=1}^{6} i^2 + \sum_{i=1}^{6} 3i + \sum_{i=1}^{6} 5 && \text{Property (c)} \\ &= \sum_{i=1}^{6} i^2 + 3\sum_{i=1}^{6} i + \sum_{i=1}^{6} 5 && \text{Property (b)} \\ &= \sum_{i=1}^{6} i^2 + 3\sum_{i=1}^{6} i + 6(5) && \text{Property (a)} \\ &= \frac{6(6+1)(2 \cdot 6 + 1)}{6} + 3\left[\frac{6(6+1)}{2}\right] + 6(5) && \text{Summation rules} \\ &= 91 + 63 + 30 && \text{Simplify.} \\ &= 184 && \text{Add.} \end{aligned}$$

✓ *Now Try Exercise 73.*

NOTE It is possible to evaluate the sums in **Examples 6 and 7** without using the summation properties and rules; however, this can be tedious

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7.1 Exercises

Write the first five terms of each sequence. See Example 1.

1. $a_n = 4n + 10$

2. $a_n = 6n - 3$

3. $a_n = \dfrac{n+5}{n+4}$

4. $a_n = \dfrac{n-7}{n-6}$

5. $a_n = \left(\dfrac{1}{3}\right)^n(n-1)$

6. $a_n = \left(\dfrac{1}{2}\right)^n(n)$

7. $a_n = (-1)^n(2n)$

8. $a_n = (-1)^{n-1}(n+1)$

9. $a_n = \dfrac{4n-1}{n^2+2}$

10. $a_n = \dfrac{n^2-1}{n^2+1}$

11. $a_n = \dfrac{n^3+8}{n+2}$

12. $a_n = \dfrac{n^3+27}{n+3}$

13. Your friend does not understand what is meant by the nth term, or general term, of a sequence. How would you explain this idea?

14. *Concept Check* How are sequences related to functions?

Concept Check Decide whether each sequence is finite *or* infinite.

15. The sequence of days of the week

16. The sequence of pages in a book

17. 1, 2, 3, 4, 5

18. −1, −2, −3, −4, −5

19. 1, 2, 3, 4, 5, ...

20. −1, −2, −3, −4, −5, ...

21. $a_1 = 4$
$a_n = 4 \cdot a_{n-1}$, if $n \ge 2$

22. $a_1 = 2$
$a_2 = 5$
$a_n = a_{n-1} + a_{n-2}$, if $n \ge 3$

Find the first four terms of each sequence. See Example 2.

23. $a_1 = -2$
$a_n = a_{n-1} + 3$, if $n > 1$

24. $a_1 = -1$
$a_n = a_{n-1} - 4$, if $n > 1$

25. $a_1 = 1$
$a_2 = 1$
$a_n = a_{n-1} + a_{n-2}$, if $n \ge 3$
(This is the Fibonacci sequence.)

26. $a_1 = 1$
$a_2 = 3$
$a_n = a_{n-1} + a_{n-2}$, if $n \ge 3$
(This is the Lucas sequence.)

27. $a_1 = 2$
$a_n = n \cdot a_{n-1}$, if $n > 1$

28. $a_1 = -3$
$a_n = 2n \cdot a_{n-1}$, if $n > 1$

Evaluate each series. See Example 4.

29. $\sum_{i=1}^{5}(2i+1)$

30. $\sum_{i=1}^{6}(3i-2)$

31. $\sum_{j=1}^{4}j^{-1}$

32. $\sum_{i=1}^{5}(i+1)^{-1}$

33. $\sum_{i=1}^{4}i^{i}$

34. $\sum_{k=1}^{4}(k+1)^{k}$

35. $\sum_{k=1}^{6}(-1)^k \cdot k$

36. $\sum_{i=1}^{7}(-1)^{i+1} \cdot i^2$

37. $\sum_{i=2}^{5}(6-3i)$

38. $\sum_{i=3}^{7}(5i+2)$

39. $\sum_{i=-2}^{3}2(3)^{i}$

40. $\sum_{i=-1}^{2}5(2)^{i}$

41. $\sum_{i=-1}^{5}(i^2-2i)$

42. $\sum_{i=3}^{6}(2i^2+1)$

43. $\sum_{i=1}^{5}(3^i-4)$

44. $\sum_{i=1}^{4}\left[(-2)^i-3\right]$

45. $\sum_{i=1}^{3}(i^3-i)$

46. $\sum_{i=1}^{4}(i^4-i^3)$

Use a graphing calculator to evaluate each series. See Example 4.

47. $\sum_{i=1}^{10} (4i^2 - 5)$

48. $\sum_{i=1}^{10} (i^3 - 6)$

49. $\sum_{j=3}^{9} (3j - j^2)$

50. $\sum_{k=3}^{10} (k^2 - 4k + 7)$

Write the terms for each series. Evaluate the sum, given that $x_1 = -2$, $x_2 = -1$, $x_3 = 0$, $x_4 = 1$, and $x_5 = 2$. See Examples 5(a) and 5(b).

51. $\sum_{i=1}^{5} x_i$

52. $\sum_{i=1}^{5} -x_i$

53. $\sum_{i=1}^{5} (2x_i + 3)$

54. $\sum_{i=1}^{4} (-3x_i - 2)$

55. $\sum_{i=1}^{3} (3x_i - x_i^2)$

56. $\sum_{i=1}^{3} (x_i^2 + x_i)$

57. $\sum_{i=2}^{5} \frac{x_i + 1}{x_i + 2}$

58. $\sum_{i=1}^{5} \frac{x_i}{x_i + 3}$

59. $\sum_{i=1}^{4} \frac{x_i^3 + 1000}{x_i + 10}$

60. Explain how factoring can make the work in Exercises 11, 12, and 59 easier.

Write the terms of $\sum_{i=1}^{4} f(x_i)\Delta x$, with $x_1 = 0$, $x_2 = 2$, $x_3 = 4$, $x_4 = 6$, and $\Delta x = 0.5$, for each function. Evaluate the sum. See Example 5(c).

61. $f(x) = 4x - 7$

62. $f(x) = 6 + 2x$

63. $f(x) = 2x^2$

64. $f(x) = x^2 - 1$

65. $f(x) = \frac{-2}{x + 1}$

66. $f(x) = \frac{5}{2x - 1}$

Use the summation properties and rules to evaluate each series. See Examples 6 and 7.

67. $\sum_{i=1}^{100} 6$

68. $\sum_{i=1}^{20} 5$

69. $\sum_{i=1}^{15} i^2$

70. $\sum_{i=1}^{50} 2i^3$

71. $\sum_{i=1}^{5} (5i + 3)$

72. $\sum_{i=1}^{5} (8i - 1)$

73. $\sum_{i=1}^{5} (4i^2 - 2i + 6)$

74. $\sum_{i=1}^{6} (2 + i - i^2)$

75. $\sum_{i=1}^{4} (3i^3 + 2i - 4)$

76. $\sum_{i=1}^{6} (i^2 + 2i^3)$

*Concept Check Use summation notation to write each series.**

77. $\frac{1}{3(1)} + \frac{1}{3(2)} + \frac{1}{3(3)} + \cdots + \frac{1}{3(9)}$

78. $\frac{5}{1 + 1} + \frac{5}{1 + 2} + \frac{5}{1 + 3} + \cdots + \frac{5}{1 + 15}$

79. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots - \frac{1}{128}$

80. $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots - \frac{1}{400}$

Use the sequence feature of a graphing calculator to graph the first ten terms of each sequence as defined. Use the graph to make a conjecture as to whether the sequence converges or diverges. If you think it converges, determine the number to which it converges.

81. $a_n = \frac{n + 4}{2n}$

82. $a_n = \frac{1 + 4n}{2n}$

83. $a_n = 2e^n$

84. $a_n = n(n + 2)$

85. $a_n = \left(1 + \frac{1}{n}\right)^n$

86. $a_n = (1 + n)^{1/n}$

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*These exercises were suggested by Joe Lloyd Harris, Gulf Coast Community College.

Solve each problem involving sequences and series. See Example 3.

87. ***(Modeling) Insect Population*** Suppose an insect population density, in thousands per acre, during year n can be modeled by the recursively defined sequence

$$a_1 = 8$$

$$a_n = 2.9a_{n-1} - 0.2a_{n-1}^2, \quad \text{for } n > 1.$$

(a) Find the population for $n = 1, 2, 3$.

(b) Graph the sequence for $n = 1, 2, 3, \ldots, 20$. Use the window $[0, 21]$ by $[0, 14]$. Interpret the graph.

88. ***Male Bee Ancestors*** One of the most famous sequences in mathematics is the Fibonacci sequence,

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots.$$

(Also see Exercise 25.) Male honeybees hatch from eggs that have not been fertilized, so a male bee has only one parent, a female. On the other hand, female honeybees hatch from fertilized eggs, so a female has two parents, one male and one female. The number of ancestors in consecutive generations of bees follows the Fibonacci sequence. Draw a tree showing the number of ancestors of a male bee in each generation following the description given above.

89. ***(Modeling) Bacteria Growth*** If certain bacteria are cultured in a medium with sufficient nutrients, they will double in size and then divide every 40 minutes. Let N_1 be the initial number of bacteria cells, N_2 the number after 40 minutes, N_3 the number after 80 minutes, and N_j the number after $40(j - 1)$ minutes. (*Source:* Hoppensteadt, F. and C. Peskin, *Mathematics in Medicine and the Life Sciences*, Springer-Verlag.)

(a) Write N_{j+1} in terms of N_j for $j \geq 1$.

(b) Determine the number of bacteria after 2 hr if $N_1 = 230$.

(c) Graph the sequence N_j for $j = 1, 2, 3, \ldots, 7$, where $N_1 = 230$. Use the window $[0, 10]$ by $[0, 15{,}000]$.

(d) Describe the growth of these bacteria when there are unlimited nutrients.

90. ***(Modeling) Verhulst's Model for Bacteria Growth*** Refer to Exercise 89. If the bacteria are not cultured in a medium with sufficient nutrients, competition will ensue and growth will slow. According to Verhulst's model, the number of bacteria N_j at time $40(j - 1)$ in minutes can be determined by the sequence

$$N_{j+1} = \left[\frac{2}{1 + \frac{N_j}{K}}\right] N_j,$$

where K is a constant and $j \geq 1$. (*Source:* Hoppensteadt, F. and C. Peskin, *Mathematics in Medicine and the Life Sciences*, Springer-Verlag.)

(a) If $N_1 = 230$ and $K = 5000$, make a table of N_j for $j = 1, 2, 3, \ldots, 20$. Round values in the table to the nearest integer.

(b) Graph the sequence N_j for $j = 1, 2, 3, \ldots, 20$. Use the window $[0, 20]$ by $[0, 6000]$.

(c) Describe the growth of these bacteria when there are limited nutrients.

(d) Make a conjecture as to why K is called the **saturation constant.** Test your conjecture by changing the value of K in the given formula.

91. *Approximating ln(1 + x)* The series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

can be used to approximate the value of $\ln(1 + x)$ for values of x in $(-1, 1]$. Use the first six terms of this series to approximate each expression. Compare this approximation with the value obtained on a calculator.

(a) $\ln 1.02$ $(x = 0.02)$ (b) $\ln 0.97$ $(x = -0.03)$

92. *Approximating π* Find the sum of the first six terms of the series

$$\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \cdots + \frac{1}{n^4} + \cdots.$$

Multiply this result by 90, and take the fourth root to obtain an approximation of π. Compare your answer to the actual decimal approximation of π.

93. *Approximating Powers of e* The series

$$e^a \approx 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \cdots + \frac{a^n}{n!},$$

where $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot n$, can be used to approximate the value of e^a for any real number a. Use the first eight terms of this series to approximate each expression. Compare this approximation with the value obtained on a calculator.

(a) e (b) e^{-1}

94. *Approximating Square Roots* The recursively defined sequence

$$a_1 = k$$

$$a_n = \frac{1}{2}\left(a_{n-1} + \frac{k}{a_{n-1}}\right), \quad \text{if } n > 1$$

can be used to compute $\sqrt{k}$ for any positive number k. This sequence was known to Sumerian mathematicians 4000 years ago, and it is still used today. Use this sequence to approximate the given square root by finding a_6. Compare your result with the actual value. (*Source:* Heinz-Otto, P., *Chaos and Fractals*, Springer-Verlag.)

(a) $\sqrt{2}$ (b) $\sqrt{11}$

7.2 Arithmetic Sequences and Series

- Arithmetic Sequences
- Arithmetic Series

Arithmetic Sequences An **arithmetic sequence** (or **arithmetic progression**) is a sequence in which each term after the first differs from the preceding term by a fixed constant, called the **common difference**. The sequence

$$5, 9, 13, 17, 21, \ldots$$

is an arithmetic sequence since each term after the first is obtained by adding 4 to the previous term. That is,

$$9 = 5 + 4$$

$$13 = 9 + 4$$

$$17 = 13 + 4$$

$$21 = 17 + 4, \quad \text{and so on.}$$

The common difference is 4

If the common difference of an arithmetic sequence is d, then by the definition of an arithmetic sequence,

$$d = a_{n+1} - a_n, \qquad \text{Common difference } d$$

for every positive integer n in the domain of the sequence.

EXAMPLE 1 Finding the Common Difference

Find the common difference, d, for the arithmetic sequence

$$-9, -7, -5, -3, -1, \ldots.$$

SOLUTION We find d by choosing any two adjacent terms and subtracting the first from the second. Choosing -7 and -5 gives

$$d = -5 - (-7) = 2.$$

Be careful when subtracting a negative number

Choosing -9 and -7 would give $d = -7 - (-9) = 2$, the same result.

✔ *Now Try Exercise 1.*

EXAMPLE 2 Finding Terms Given a_1 and d

Find the first five terms for each arithmetic sequence.

(a) The first term is 7, and the common difference is -3.

(b) $a_1 = -12, d = 5$

SOLUTION

(a)
$$\begin{aligned}
a_1 &= 7 && \text{Start with } a_1 = 7\\
a_2 &= 7 + (-3) = 4 && \text{Add } d = -3\\
a_3 &= 4 + (-3) = 1 && \text{Add } -3.\\
a_4 &= 1 + (-3) = -2 && \text{Add } -3\\
a_5 &= -2 + (-3) = -5 && \text{Add } -3.
\end{aligned}$$

(b)
$$\begin{aligned}
a_1 &= -12 && \text{Start with } a_1 = -12.\\
a_2 &= -12 + 5 = -7 && \text{Add } d = 5.\\
a_3 &= -7 + 5 = -2 && \text{Add } 5\\
a_4 &= -2 + 5 = 3 && \text{Add } 5.\\
a_5 &= 3 + 5 = 8 && \text{Add } 5
\end{aligned}$$

✔ *Now Try Exercises 7 and 9.*

If a_1 is the first term of an arithmetic sequence and d is the common difference, then the terms of the sequence are given as follows.

$$\begin{aligned}
a_1 &= a_1\\
a_2 &= a_1 + d\\
a_3 &= a_2 + d = a_1 + d + d = a_1 + 2d\\
a_4 &= a_3 + d = a_1 + 2d + d = a_1 + 3d\\
a_5 &= a_1 + 4d\\
a_6 &= a_1 + 5d
\end{aligned}$$

By this pattern, $a_n = a_1 + (n - 1)d$

EXAMPLE 8 Using the Sum Formulas

The sum of the first 17 terms of an arithmetic sequence is 187. If $a_{17} = -13$, find a_1 and d.

SOLUTION

$$S_{17} = \frac{17}{2}(a_1 + a_{17}) \qquad \text{Use the first formula for } S_n, \text{ with } n = 17.$$

$$187 = \frac{17}{2}(a_1 - 13) \qquad \text{Let } S_{17} = 187 \text{ and } a_{17} = -13.$$

$$22 = a_1 - 13 \qquad \text{Multiply by } \tfrac{2}{17}.$$

$$a_1 = 35 \qquad \text{Add 13 and interchange sides.}$$

Now find d.

$$a_{17} = a_1 + (17 - 1)d \qquad \text{Formula for the } n\text{th term}$$

$$-13 = 35 + 16d \qquad \text{Let } a_{17} = -13 \text{ and } a_1 = 35.$$

$$-48 = 16d \qquad \text{Subtract 35.}$$

$$d = -3 \qquad \text{Divide by 16 and interchange sides.}$$

✔ ***Now Try Exercise 49.***

Any sum of the form

$$\sum_{i=1}^{n}(di + c), \quad \text{where } d \text{ and } c \text{ are real numbers,}$$

represents the sum of the terms of an arithmetic sequence having first term $a_1 = d(1) + c = d + c$ and common difference d. These sums can be evaluated using the formulas in this section.

EXAMPLE 9 Using Summation Notation

Evaluate each sum.

(a) $\sum_{i=1}^{10}(4i + 8)$ **(b)** $\sum_{k=3}^{9}(4 - 3k)$

SOLUTION

(a) This sum contains the first 10 terms of the arithmetic sequence having

$$a_1 = 4 \cdot 1 + 8 = 12, \qquad \text{First term}$$

and

$$a_{10} = 4 \cdot 10 + 8 = 48. \qquad \text{Last term}$$

Thus, $$\sum_{i=1}^{10}(4i + 8) = S_{10} = \frac{10}{2}(12 + 48) = 5(60) = 300.$$

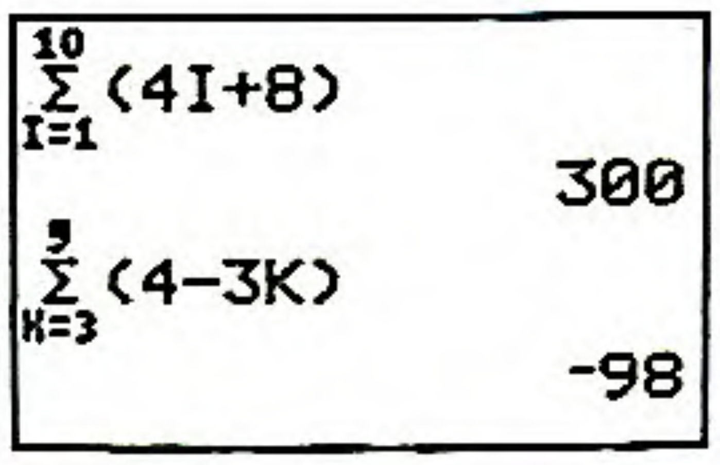
(b) The first few terms are

$$[4 - 3(3)] + [4 - 3(4)] + [4 - 3(5)] + \cdots = -5 + (-8) + (-11) + \cdots.$$

Thus, $a_1 = -5$ and $d = -3$. If the sequence started with $k = 1$, there would be nine terms. Since it starts at 3, two of those terms are missing, so there are seven terms and $n = 7$. Use the second formula for S_n.

$$\sum_{k=3}^{9}(4 - 3k) = \frac{7}{2}[2(-5) + 6(-3)] = -98$$

✔ ***Now Try Exercises 55 and 57***



```
10
Σ(4I+8)
I=1
                300
9
Σ(4-3K)
K=3
                -98
```

As shown in the previous section, the TI-83/84 Plus will give the sum of a sequence without having to first store the sequence. The screen here illustrates this method for the sequences in Example 9.

7.2 Exercises

Find the common difference d for each arithmetic sequence. See Example 1.

1. $2, 5, 8, 11, \ldots$
2. $4, 10, 16, 22, \ldots$
3. $3, -2, -7, -12, \ldots$
4. $-8, -12, -16, -20, \ldots$
5. $x + 3y,\ 2x + 5y,\ 3x + 7y, \ldots$
6. $t^2 + q,\ -4t^2 + 2q,\ -9t^2 + 3q, \ldots$

Write the first five terms of each arithmetic sequence. See Example 2.

7. The first term is 8, and the common difference is 6.
8. The first term is -2, and the common difference is 12.
9. $a_1 = 5,\ d = -2$
10. $a_1 = 4,\ d = 3$
11. $a_1 = 10 + \sqrt{7},\ a_2 = 10$
12. $a_1 = 3 - \sqrt{2},\ a_2 = 3$

Find a_8 and a_n for each arithmetic sequence. See Examples 3 and 4.

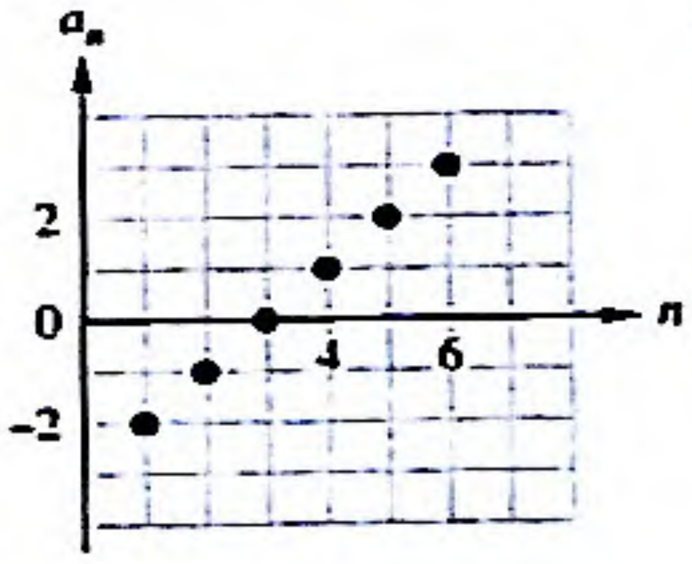
13. $5, 7, 9, \ldots$
14. $-3, -7, -11, \ldots$
15. $a_1 = 5, a_4 = 15$
16. $a_1 = -4, a_5 = 16$
17. $a_{10} = 6, a_{11} = 10.5$
18. $a_{15} = 8, a_{16} = 5$
19. $a_1 = x, a_2 = x + 3$
20. $a_2 = y + 1, d = -3$
21. $a_4 = s + 6p, d = 2p$
22. ***Concept Check*** If a_1, a_2, a_3 represents an arithmetic sequence, express a_2 in terms of a_1 and a_3.

Find a_1 for each arithmetic sequence. See Examples 5 and 8.

23. $a_5 = 27, a_{15} = 87$
24. $a_{12} = 60, a_{20} = 84$
25. $S_{16} = -160, a_{16} = -25$
26. $S_{28} = 2926, a_{28} = 199$

Find a formula for the nth term of the finite arithmetic sequence shown in each graph. Then state the domain and range of the sequence. See Example 6.

27.



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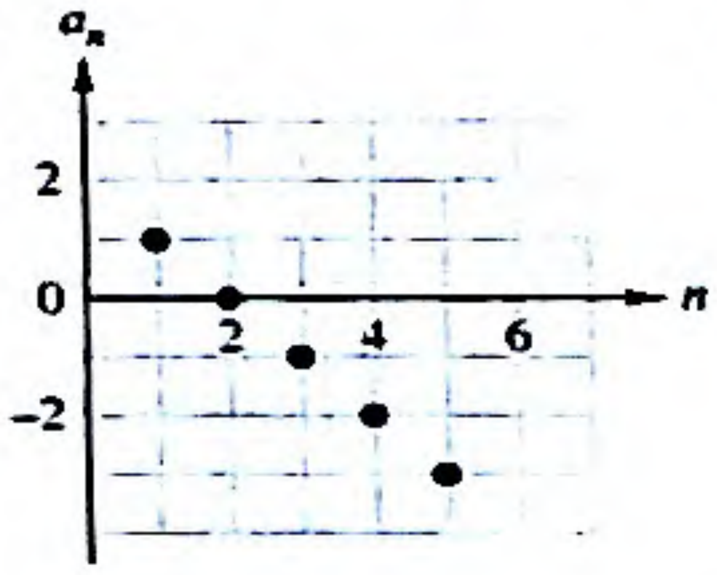

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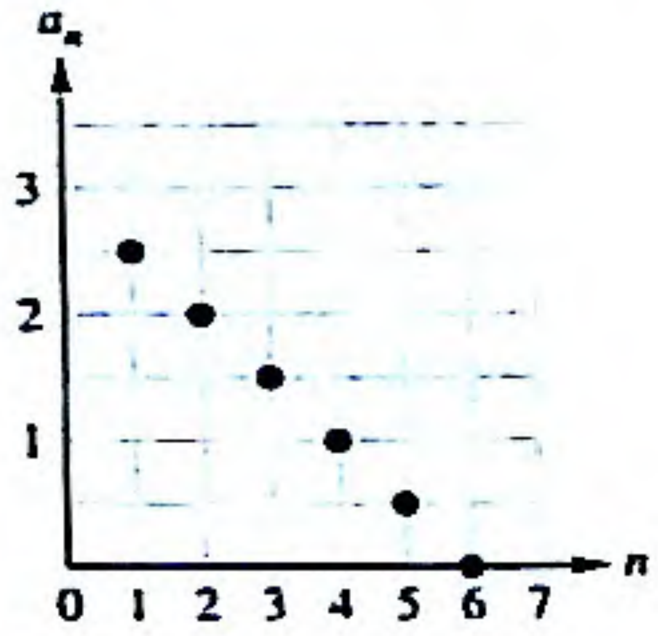

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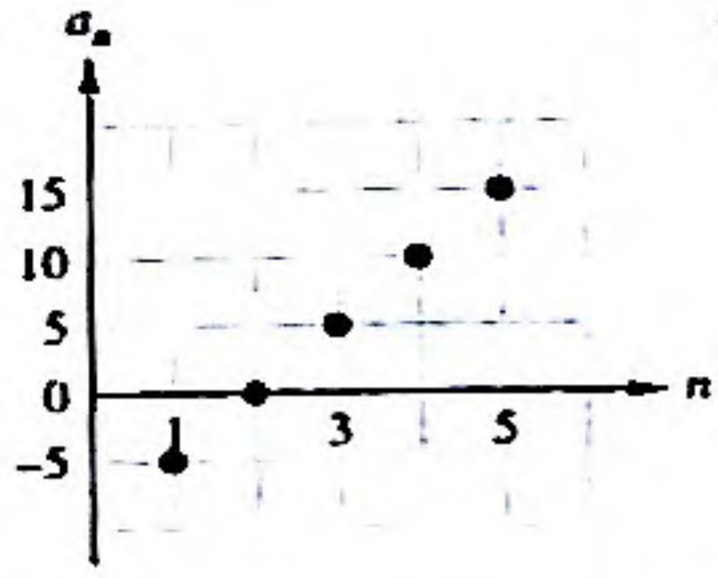
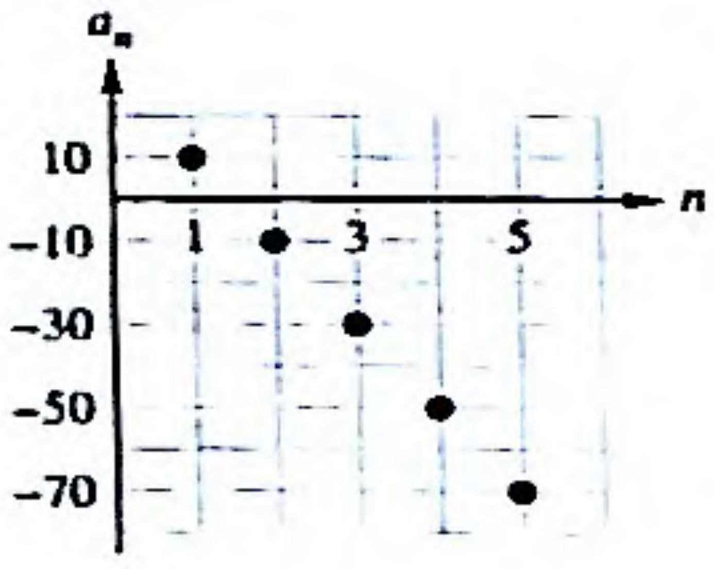
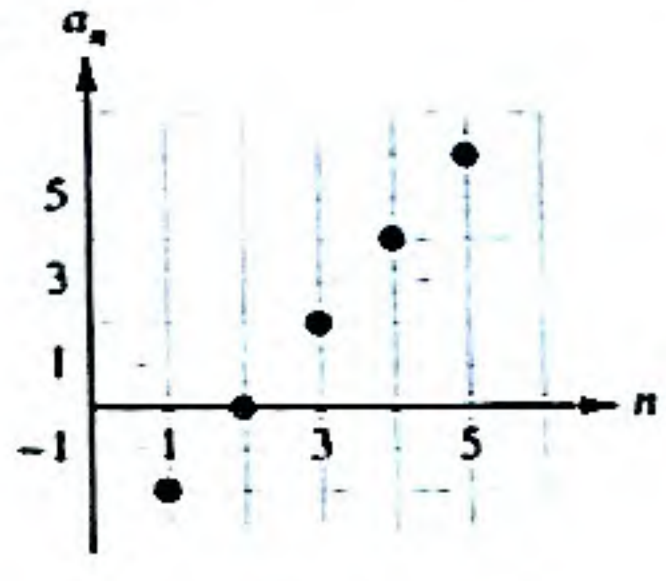


31.



32.



Evaluate S_{10}, the sum of the first ten terms, of each arithmetic sequence. See Example 7(a).

33. $8, 11, 14, \ldots$
34. $-9, -5, -1, \ldots$
35. $5, 9, 13, \ldots$
36. $8, 6, 4, \ldots$
37. $a_2 = 9, a_4 = 13$
38. $a_3 = 5, a_4 = 8$
39. $a_1 = 10, a_{10} = 5.5$
40. $a_1 = -8, a_{10} = -1.25$
41. $a_1 = \pi, a_{10} = 10\pi$

42. *Concept Check* Is this statement accurate? *To find the sum of the first n positive integers, find half the product of n and n + 1.*

Find each sum as described. See Example 7(b).

43. the sum of the first 80 positive integers

44. the sum of the first 120 positive integers

45. the sum of the first 50 positive odd integers

46. the sum of the first 90 positive odd integers

47. the sum of the first 60 positive even integers

48. the sum of the first 70 positive even integers

Find a_1 and d for each arithmetic series. See Example 8.

49. $S_{20} = 1090, a_{20} = 102$

50. $S_{31} = 5580, a_{31} = 360$

51. $S_{12} = -108, a_{12} = -19$

52. $S_{25} = 650, a_{25} = 62$

Evaluate each sum. See Example 9.

53. $\sum_{i=1}^{3} (i + 4)$

54. $\sum_{i=1}^{5} (i - 8)$

55. $\sum_{j=1}^{10} (2j + 3)$

56. $\sum_{j=1}^{15} (5j - 9)$

57. $\sum_{i=4}^{12} (-5 - 8i)$

58. $\sum_{k=5}^{19} (-3 - 4k)$

59. $\sum_{i=1}^{1000} i$

60. $\sum_{k=1}^{2000} k$

61. $\sum_{k=1}^{100} -k$

62. Explain the basic difference between a sequence and a series.

Use the summation feature of a graphing calculator to evaluate the sum of the first ten terms of each arithmetic series with a_n defined as shown. In Exercises 65 and 66, round to the nearest thousandth.

63. $a_n = 4.2n + 9.73$

64. $a_n = 8.42n + 36.18$

65. $a_n = \sqrt{8}n + \sqrt{3}$

66. $a_n = -\sqrt[3]{4}n + \sqrt{7}$

Solve each problem.

67. ***Integer Sum*** Find the sum of all the integers from 51 to 71.

68. ***Integer Sum*** Find the sum of all the integers from −8 to 30.

69. ***Clock Chimes*** If a clock strikes the proper number of chimes each hour on the hour, how many times will it chime in a month of 30 days?

70. ***Telephone Pole Stack*** A stack of telephone poles has 30 in the bottom row, 29 in the next, and so on, with one pole in the top row. How many poles are in the stack?

71. ***Population Growth*** Five years ago, the population of a city was 49,000. Each year the zoning commission permits an increase of 580 in the population. What will the maximum population be 5 yr from now?

72. ***Slide Supports*** A super slide of uniform slope is to be built on a level piece of land. There are to be 20 equally spaced vertical supports, with the longest support 15 m long and the shortest 2 m long. Find the total length of all the supports.

73. ***Rungs of a Ladder*** How much material will be needed for the rungs of a ladder of 31 rungs, if the rungs taper uniformly from 18 in. to 28 in.?

74. ***(Modeling) Children's Growth Pattern*** The normal growth pattern for children aged 3–11 follows that of an arithmetic sequence. An increase in height of about 6 cm per year is expected. Thus, 6 would be the common difference of the sequence. For example, a child who measures 96 cm at age 3 would have his expected height in subsequent years represented by the sequence 102, 108, 114, 120, 126, 132, 138, 144. Each term differs from the adjacent terms by the common difference, 6.

(a) If a child measures 98.2 cm at age 3 and 109.8 cm at age 5, what would be the common difference of the arithmetic sequence describing his yearly height?

(b) What would we expect his height to be at age 8?

75. ***Concept Check*** Suppose that $a_1, a_2, a_3, a_4, a_5, \ldots$ is an arithmetic sequence. Is $a_1, a_3, a_5, \ldots$ an arithmetic sequence?

76. Explain why the sequence log 2, log 4, log 8, log 16, ... is arithmetic.

7.3 Geometric Sequences and Series

- Geometric Sequences
- Geometric Series
- Infinite Geometric Series
- Annuities

Geometric Sequences Suppose you agreed to work for \$0.01 the first day, \$0.02 the second day, \$0.04 the third day, \$0.08 the fourth day, and so on, with your wages doubling each day. How much will you earn on day 20, after working 5 days a week for a month? How much will you have earned altogether in 20 days? These questions will be answered in this section.

A **geometric sequence** (or **geometric progression**) is a sequence in which each term after the first is obtained by multiplying the preceding term by a fixed nonzero real number, called the **common ratio.** The sequence discussed above,

$$1, 2, 4, 8, 16, \ldots, \quad \text{In cents}$$

is a geometric sequence in which the first term is 1 and the common ratio is 2.

Notice that if we divide any term after the first term by the preceding term, we obtain the common ratio $r = 2$.

$$\frac{a_2}{a_1} = \frac{2}{1} = 2; \quad \frac{a_3}{a_2} = \frac{4}{2} = 2; \quad \frac{a_4}{a_3} = \frac{8}{4} = 2; \quad \frac{a_5}{a_4} = \frac{16}{8} = 2$$

If the common ratio of a geometric sequence is r, then by the definition of a geometric sequence,

$$r = \frac{a_{n+1}}{a_n}, \quad \text{Common ratio } r$$

for every positive integer n. ***Therefore, we find the common ratio by choosing any term after the first and dividing it by the preceding term.***

In the geometric sequence 2, 8, 32, 128, ..., $r = 4$. Notice that

$$8 = 2 \cdot 4$$

$$32 = 8 \cdot 4 = (2 \cdot 4) \cdot 4 = 2 \cdot 4^2$$

$$128 = 32 \cdot 4 = (2 \cdot 4^2) \cdot 4 = 2 \cdot 4^3.$$

LOOKING AHEAD TO CALCULUS

In calculus, functions are sometimes defined in terms of infinite series. Here are three functions we studied earlier in the text defined that way.

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

for x in $(-1, 1)$

$$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \cdots$$

for x in $(-1, 1)$

If a geometric series has first term a_1 and common ratio r, then

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \quad (\text{where } r \neq 1),$$

for every positive integer n. If $-1 < r < 1$, then $\lim_{n \to \infty} r^n = 0$, and

$$\lim_{n \to \infty} S_n = \frac{a_1(1 - 0)}{1 - r} = \frac{a_1}{1 - r}.$$

The resulting quotient, $\frac{a_1}{1 - r}$, is **the sum of the terms of an infinite geometric sequence.**

The limit $\lim_{n \to \infty} S_n$ can be expressed as S_∞ or $\sum_{i=1}^{\infty} a_i$.

Sum of the Terms of an Infinite Geometric Sequence

The sum S_∞ of the terms of an infinite geometric sequence with first term a_1 and common ratio r, where $-1 < r < 1$, is given by the following.

$$S_\infty = \frac{a_1}{1 - r}$$

If $|r| > 1$, then the terms get larger and larger in absolute value, so there is no limit as $n \to \infty$.

Hence, if $|r| > 1$, the terms of the sequence will not have a sum.

EXAMPLE 8 Evaluating Infinite Geometric Series

Find each sum.

(a) $\sum_{i=1}^{\infty} \left(-\frac{3}{4}\right)\left(-\frac{1}{2}\right)^{i-1}$ **(b)** $\sum_{i=1}^{\infty} \left(\frac{3}{5}\right)^i$

SOLUTION

(a) Here, $a_1 = -\frac{3}{4}$ and $r = -\frac{1}{2}$. Since $-1 < r < 1$, the preceding formula applies.

$$S_\infty = \frac{a_1}{1 - r} = \frac{-\frac{3}{4}}{1 - \left(-\frac{1}{2}\right)} = \frac{-\frac{3}{4}}{\frac{3}{2}} = -\frac{3}{4} \div \frac{3}{2} = -\frac{3}{4} \cdot \frac{2}{3} = -\frac{1}{2}$$

(b) $\sum_{i=1}^{\infty} \left(\frac{3}{5}\right)^i = \frac{\frac{3}{5}}{1 - \frac{3}{5}} = \frac{\frac{3}{5}}{\frac{2}{5}} = \frac{3}{5} \div \frac{2}{5} = \frac{3}{5} \cdot \frac{5}{2} = \frac{3}{2}$ $a_1 = \frac{3}{5}, r = \frac{3}{5}$

✔ *Now Try Exercises 49 and 51.*

Annuities A sequence of equal payments made after equal periods of time, such as car payments or house payments, is an **annuity.** If the payments are accumulated in an account (with no withdrawals), the sum of the payments and interest on the payments is the **future value** of the annuity.

EXAMPLE 9 **Finding the Future Value of an Annuity**

To save money for a trip, Jacqui Porter deposited \$1000 at the *end* of each year for 4 yr in an account paying 3% interest, compounded annually. Find the future value of this annuity.

SOLUTION We use the formula for interest compounded annually.

$$A = P(1 + r)^t \quad \text{(Section 4.2)}$$

The first payment earns interest for 3 yr, the second payment for 2 yr, and the third payment for 1 yr. The last payment earns no interest.

$$1000(1.03)^3 + 1000(1.03)^2 + 1000(1.03) + 1000 \quad \text{Total amount}$$

This is the sum of the terms of a geometric sequence with first term (starting at the end of the sum as written above) $a_1 = 1000$ and common ratio $r = 1.03$.

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \quad \text{Formula for } S_n$$

$$S_4 = \frac{1000[1 - (1.03)^4]}{1 - 1.03} \quad \text{Substitute.}$$

$$S_4 \approx 4183.63. \quad \text{Evaluate.}$$

The future value of the annuity is \$4183.63.

✓ *Now Try Exercise 71.*

The general formula for the future value of an annuity can be stated as follows. (See Exercise 79.)

Future Value of an Annuity

The formula for the future value of an annuity is given by the following.

$$S = R\left[\frac{(1 + i)^n - 1}{i}\right]$$

Here S is future value, R is payment at the end of each period, i is interest rate per period, and n is number of periods.

7.3 Exercises

Refer to the first sentence of this section, which describes a method of payment for a job. Determine (a) how much you will earn on the day indicated and (b) how much you will have earned altogether after your wages are paid on the day indicated. See Examples 1 and 5.

1. day 10
2. day 12
3. day 15
4. day 18

Find a_5 and a_n for each geometric sequence. See Example 2.

5. $a_1 = 5, r = -2$
6. $a_1 = 8, r = -5$
7. $a_2 = -4, r = 3$
8. $a_3 = -2, r = 4$
9. $a_4 = 243, r = -3$
10. $a_4 = 18, r = 2$
11. $-4, -12, -36, -108, \ldots$
12. $-2, 6, -18, 54, \ldots$

13. $\frac{4}{5}, 2, 5, \frac{25}{2}, \ldots$

14. $\frac{1}{2}, \frac{2}{3}, \frac{8}{9}, \frac{32}{27}, \ldots$

15. $10, -5, \frac{5}{2}, -\frac{5}{4}, \ldots$

16. $3, -\frac{9}{4}, \frac{27}{16}, -\frac{81}{64}, \ldots$

Find r and a_1 for each geometric sequence. ***See Example 3.***

17. $a_2 = -6, a_7 = -192$

18. $a_3 = 5, a_8 = \frac{1}{625}$

19. $a_3 = 50, a_7 = 0.005$

20. $a_4 = -\frac{1}{4}, a_9 = -\frac{1}{128}$

Use the formula for S_n to find the sum of the first five terms of each geometric sequence. In Exercises 25 and 26, round to the nearest hundredth. ***See Example 5.***

21. $2, 8, 32, 128, \ldots$

22. $4, 16, 64, 256, \ldots$

23. $18, -9, \frac{9}{2}, -\frac{9}{4}, \ldots$

24. $12, -4, \frac{4}{3}, -\frac{4}{9}, \ldots$

25. $a_1 = 8.423, r = 2.859$

26. $a_1 = -3.772, r = -1.553$

Find each sum. ***See Example 6.***

27. $\sum_{i=1}^{5} (-3)^i$

28. $\sum_{i=1}^{4} (-2)^i$

29. $\sum_{j=1}^{6} 48\left(\frac{1}{2}\right)^j$

30. $\sum_{j=1}^{5} 243\left(\frac{2}{3}\right)^j$

31. $\sum_{k=4}^{10} 2^k$

32. $\sum_{k=3}^{9} 3^k$

33. *Concept Check* Under what conditions does the sum of an infinite geometric series exist?

34. The number 0.999... can be written as the sum of the terms of an infinite geometric sequence: $0.9 + 0.09 + 0.009 + \cdots$. Here we have $a_1 = 0.9$ and $r = 0.1$. Use the formula for S_∞ to find this sum. Does your intuition indicate that your answer is correct?

Find r for each infinite geometric sequence. Identify any whose sum does not converge.

35. $12, 24, 48, 96, \ldots$

36. $625, 125, 25, 5, \ldots$

37. $-48, -24, -12, -6, \ldots$

38. $2, -10, 50, -250, \ldots$

39. Use $\lim_{n \to \infty} S_n$ to show that $2 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots$ converges to 4. **See Example 7.**

40. In **Example 7,** we determined that $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$ converges to $\frac{3}{2}$ using an argument involving limits. Use the formula for the sum of the terms of an infinite geometric sequence to obtain the same result.

Find each sum that converges. ***See Example 8.***

41. $18 + 6 + 2 + \frac{2}{3} + \cdots$

42. $100 + 10 + 1 + \cdots$

43. $\frac{1}{4} - \frac{1}{6} + \frac{1}{9} - \frac{2}{27} + \cdots$

44. $\frac{4}{3} + \frac{2}{3} + \frac{1}{3} + \cdots$

45. $\sum_{i=1}^{\infty} 3\left(\frac{1}{4}\right)^{i-1}$

46. $\sum_{i=1}^{\infty} 5\left(-\frac{1}{4}\right)^{i-1}$

47. $\sum_{k=1}^{\infty} 3^{-k}$

48. $\sum_{k=1}^{\infty} 10^{-k}$

49. $\sum_{i=1}^{\infty} \left(-\frac{2}{3}\right)\left(-\frac{1}{4}\right)^{i-1}$

50. $\sum_{i=1}^{\infty} \left(-\frac{1}{5}\right)\left(-\frac{2}{5}\right)^{i-1}$

51. $\sum_{i=1}^{\infty} \left(\frac{3}{7}\right)^i$

52. $\sum_{i=1}^{\infty} \left(\frac{5}{9}\right)^i$

Use the summation feature of a graphing calculator to evaluate each sum. Round to the nearest thousandth.

53. $\sum_{i=1}^{10} (1.4)^i$

54. $\sum_{j=1}^{6} -(3.6)^j$

55. $\sum_{j=3}^{8} 2(0.4)^j$

56. $\sum_{i=4}^{9} 3(0.25)^i$

Solve each problem. See Examples 1–8.

57. *(Modeling) Investment for Retirement* According to T. Rowe Price Associates, a person with a moderate investment strategy and n years to retirement should have accumulated savings of a_n percent of his or her annual salary. The geometric sequence

$$a_n = 1276(0.916)^n$$

gives the appropriate percent for each year n.

(a) Find a_1 and r. Round a_1 to the nearest whole number.

(b) Find and interpret the terms a_{10} and a_{20}. Round to the nearest whole number.

58. *(Modeling) Investment for Retirement* Refer to Exercise 57. For someone who has a conservative investment strategy with n years to retirement, the geometric sequence is $a_n = 1278(0.935)^n$. (*Source:* T. Rowe Price Associates.)

(a) Repeat part (a) of Exercise 57.

(b) Repeat part (b) of Exercise 57.

(c) Why are the answers in parts (a) and (b) larger than those in Exercise 57?

59. *(Modeling) Bacterial Growth* The strain of bacteria described in Exercise 89 in Section 7.1 will double in size and then divide every 40 minutes. Let a_1 be the initial number of bacteria cells, a_2 the number after 40 minutes, and a_n the number after $40(n - 1)$ minutes. (*Source:* Hoppensteadt, F. and C. Peskin, *Mathematics in Medicine and the Life Sciences*, Springer-Verlag.)

(a) Write a formula for the nth term a_n of the geometric sequence

$$a_1, a_2, a_3, \ldots, a_n, \ldots.$$

(b) Determine the first value for n where $a_n > 1{,}000{,}000$ if $a_1 = 100$.

(c) How long does it take for the number of bacteria to exceed one million?

60. *Photo Processing* For students taking a course in black-and-white photography, the final step in processing a print is to immerse it in a chemical fixer. The print is then washed in running water. Under certain conditions, 98% of the fixer in a print will be removed with 15 min of washing. How much of the original fixer would be left after 1 hr of washing?

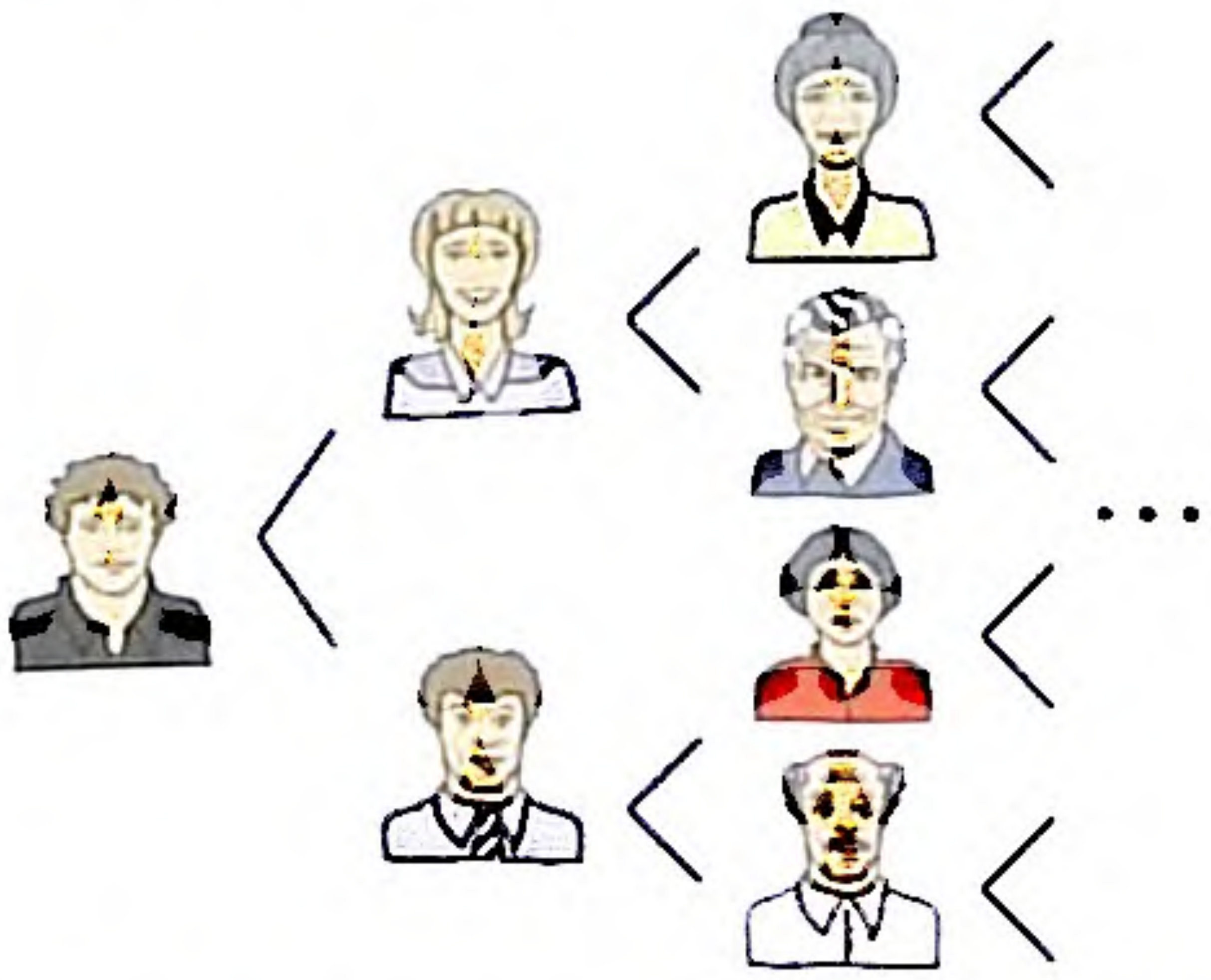
61. *(Modeling) Fruit Flies Population* A population of fruit flies is growing in such a way that each generation is 1.25 times as large as the last generation. Suppose there were 200 insects in the first generation. How many would there be in the fifth generation?

62. *Depreciation* Each year a machine loses 20% of the value it had at the beginning of the year. Find the value of the machine at the end of 6 yr if it cost $100,000 new.

63. *Sugar Processing* A sugar factory receives an order for 1000 units of sugar. The production manager thus orders production of 1000 units of sugar. He forgets, however, that the production of sugar requires some sugar (to prime the machines, for example), and so he ends up with only 900 units of sugar. He then orders an additional 100 units and receives only 90 units. A further order for 10 units produces 9 units. Finally, the manager decides to try mathematics. He views the production process as an infinite geometric progression with $a_1 = 1000$ and $r = 0.1$. Using this, find the number of units of sugar that he should have ordered originally.

64. ***Height of a Dropped Ball*** Alicia drops a ball from a height of 10 m and notices that on each bounce the ball returns to about $\frac{3}{4}$ of its previous height. About how far will the ball travel before it comes to rest? (*Hint:* Consider the sum of two sequences.)

65. ***Number of Ancestors*** Each person has two parents, four grandparents, eight great-grandparents, and so on. What is the total number of ancestors a person has, going back five generations? ten generations?



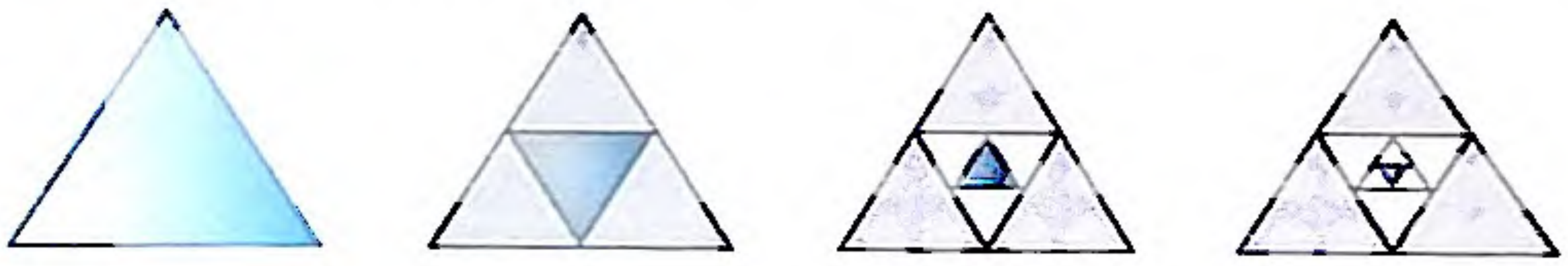
66. ***Drug Dosage*** Certain medical conditions are treated with a fixed dose of a drug administered at regular intervals. Suppose a person is given 2 mg of a drug each day and that during each 24-hr period, the body utilizes 40% of the amount of drug that was present at the beginning of the period.

(a) Show that the amount of the drug present in the body at the end of n days is

$$\sum_{i=1}^{n} 2(0.6)^i.$$

(b) What will be the approximate quantity of the drug in the body at the end of each day after the treatment has been administered for a long period of time?

67. ***Side Length of a Triangle*** A sequence of equilateral triangles is constructed. The first triangle has sides 2 m in length. To get the second triangle, midpoints of the sides of the original triangle are connected. What is the length of each side of the eighth such triangle? See the figure.



68. ***Perimeter and Area of Triangles*** In Exercise 67, if the process could be continued indefinitely, what would be the total perimeter of all the triangles? What would be the total area of all the triangles, disregarding the overlapping?

69. ***Salaries*** You are offered a 6-week summer job and are asked to select one of the following salary options.

Option 1: $5000 for the first day with a $10,000 raise each day for the remaining 29 days (that is, $15,000 for day 2, $25,000 for day 3, and so on)

Option 2: $0.01 for the first day with the pay doubled each day (that is, $0.02 for day 2, $0.04 for day 3, and so on)

Which option would you choose?

70. *Number of Ancestors* Suppose a genealogical Web site allows you to identify all your ancestors that lived during the last 300 yr. Assuming that each generation spans about 25 yr, guess the number of ancestors that would be found during the 12 generations. Then use the formula for a geometric series to find the correct value.

Future Value of an Annuity *Find the future value of each annuity. See Example 9.*

71. There are payments of \$1000 at the end of each year for 9 yr at 3% interest compounded annually.

72. There are payments of \$800 at the end of each year for 12 yr at 2% interest compounded annually.

73. There are payments of \$2430 at the end of each year for 10 yr at 1% interest compounded annually.

74. There are payments of \$1500 at the end of each year for 6 yr at 0.5% interest compounded annually.

75. Refer to **Exercise 73.** Use the answer and recursion to find the balance after 11 yr.

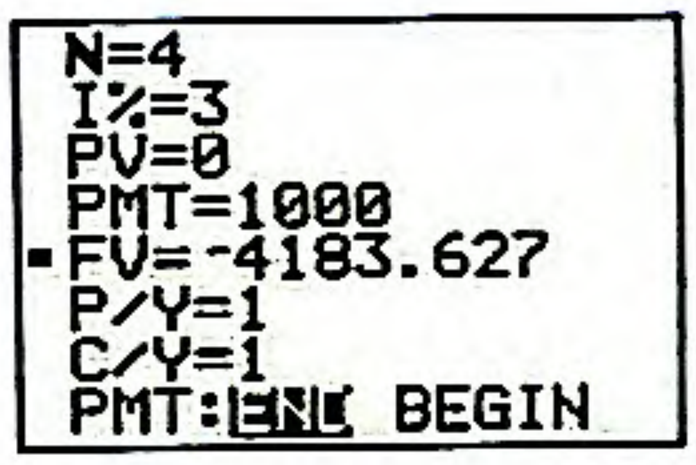
76. Refer to **Exercise 74.** Use the answer and recursion to find the balance after 7 yr.

77. *Individual Retirement Account* Starting on his 40th birthday, Michael Branson deposits \$2000 per year in an Individual Retirement Account until age 65 (last payment at age 64). Find the total amount in the account if he had a guaranteed interest rate of 3% compounded annually.

78. *Retirement Savings* To save for retirement, Mort put \$3000 at the end of each year into an ordinary annuity for 20 yr at 2.5% annual interest. At the end of the 20 yr, what was the amount of the annuity?

79. Show that the formula for future value of an annuity gives the correct answer when compared to the solution in **Example 9.**

80. The screen here shows how the TI-83/84 Plus calculator computes the future value of the annuity described in **Example 9.** Use a calculator with this capability to support your answers in **Exercises 71–78.**



```
N=4
I%=3
PV=0
PMT=1000
▪FV=-4183.627
P/Y=1
C/Y=1
PMT:END BEGIN
```

81. *Concept Check* Suppose that $a_1, a_2, a_3, a_4, a_5, \ldots$ is a geometric sequence. Is $a_1, a_3, a_5, \ldots$ a geometric sequence?

82. Explain why the sequence log 6, log 36, log 1296, log 1,679,616, ... is geometric.

Summary Exercises on Sequences and Series

Use the following guidelines in Exercises 1–16 on the next page.

Given a sequence $a_1, a_2, a_3, a_4, a_5, \ldots,$

- If the differences $a_2 - a_1,\ a_3 - a_2,\ a_4 - a_3,\ a_5 - a_4, \ldots$ are all equal to the same number d, then the sequence is *arithmetic,* and d is the *common difference.*
- If the ratios $\frac{a_2}{a_1}, \frac{a_3}{a_2}, \frac{a_4}{a_3}, \frac{a_5}{a_4}, \ldots$ are all equal to the same number r, then the sequence is *geometric,* and r is the *common ratio.*

In Exercises 1–9, determine whether the sequence is arithmetic, geometric, or neither. If the sequence is arithmetic, give its common difference d. If the sequence is geometric, give its common ratio r.

1. $2, 4, 8, 16, 32, \ldots$ **2.** $1, 4, 7, 10, 13, \ldots$ **3.** $3, \frac{1}{2}, -2, -\frac{9}{2}, -7, \ldots$

4. $1, 1, 2, 3, 5, 8, \ldots$ **5.** $\frac{3}{4}, 1, \frac{4}{3}, \frac{16}{9}, \frac{64}{27}, \ldots$ **6.** $4, -12, 36, -108, 324, \ldots$

7. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$ **8.** $5, 2, -1, -4, -7, \ldots$ **9.** $1, 9, 10, 19, 29, \ldots$

10. *Concept Check* For the sequence $5, 5, 5, \ldots$, find d and r.

In Exercises 11–16, determine whether the given sequence is either arithmetic or geometric. Then find a_n and $\sum_{i=1}^{10} a_i$.

11. $3, 6, 12, 24, 48, \ldots$ **12.** $2, 6, 10, 14, 18, \ldots$ **13.** $4, \frac{5}{2}, 1, -\frac{1}{2}, -2, \ldots$

14. $\frac{3}{2}, 1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \ldots$ **15.** $3, -6, 12, -24, 48, \ldots$ **16.** $-5, -8, -11, -14, -17, \ldots$

Evaluate each sum, where possible. Identify any that diverge.

17. $\sum_{i=1}^{\infty} \frac{1}{3}(-2)^{i-1}$ **18.** $\sum_{j=1}^{4} 2\left(\frac{1}{10}\right)^{j-1}$ **19.** $\sum_{i=1}^{25} (4 - 6i)$

20. $\sum_{i=1}^{6} 3^i$ **21.** $\sum_{i=1}^{\infty} 4\left(-\frac{1}{2}\right)^i$ **22.** $\sum_{i=1}^{\infty} (3i - 2)$

23. $\sum_{j=1}^{12} (2j - 1)$ **24.** $\sum_{k=1}^{\infty} 5^{-k}$ **25.** $\sum_{i=1}^{\infty} 1.0001^i$

26. Write 0.333... as an infinite geometric series. Find the sum.

7.4 The Binomial Theorem

- **A Binomial Expansion Pattern**
- **Pascal's Triangle**
- ***n*-Factorial**
- **Binomial Coefficients**
- **The Binomial Theorem**
- ***k*th Term of a Binomial Expansion**

A Binomial Expansion Pattern In this section, we introduce a method for writing the expansion of expressions of the form $(x + y)^n$, where n is a natural number. Some expansions for various nonnegative integer values of n follow.

$$(x + y)^0 = 1$$
$$(x + y)^1 = x + y$$
$$(x + y)^2 = x^2 + 2xy + y^2$$
$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$
$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

Notice that after the special case $(x + y)^0 = 1$, each expansion begins with x raised to the same power as the binomial itself. That is, the expansion of $(x + y)^1$ has a first term of x^1, $(x + y)^2$ has a first term of x^2, $(x + y)^3$ has a first term of x^3, and so on. Also, the last term in each expansion is y to the same power as the binomial. Thus, the expansion of $(x + y)^n$ should begin with the term x^n and end with the term y^n.

LOOKING AHEAD TO CALCULUS
Students taking calculus study the binomial series, which follows from Isaac Newton's extension to the case where the exponent is no longer a positive integer. His result led to a series for $(1 + x)^k$, where k is a real number and $|x| < 1$.

Notice that the exponent on x decreases by 1 in each term after the first, while the exponent on y, beginning with y in the second term, increases by 1 in each succeeding term. That is, the *variables* in the terms of the expansion of $(x + y)^n$ have the following pattern.

$$x^n,\ x^{n-1}y,\ x^{n-2}y^2,\ x^{n-3}y^3,\ \ldots,\ xy^{n-1},\ y^n$$

This pattern suggests that the sum of the exponents on x and y in each term is n. For example, the third term in the list above is $x^{n-2}y^2$, and the sum of the exponents is $n - 2 + 2 = n$.

Pascal's Triangle Now, examine the *coefficients* in the terms of the expansion of $(x + y)^n$. Writing the coefficients alone gives the following pattern.

Pascal's Triangle

	Row
1	0
1 1	1
1 2 1	2
1 3 3 1	3
1 4 6 4 1	4
1 5 10 10 5 1	5

With the coefficients arranged in this way, each number in the triangle is the sum of the two numbers directly above it (one to the right and one to the left). For example, in row four, 1 is the sum of 1 (the only number above it), 4 is the sum of 1 and 3, 6 is the sum of 3 and 3, and so on. This triangular array of numbers is called **Pascal's triangle**, in honor of the seventeenth-century mathematician Blaise Pascal. It was, however, known long before his time.

To find the coefficients for $(x + y)^6$, we need to include row six in Pascal's triangle. Adding adjacent numbers, we find that row six is

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1.$$

Using these coefficients, we obtain the expansion of $(x + y)^6$.

$$(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

n*-Factorial** Although it is possible to use Pascal's triangle to find the coefficients of $(x + y)^n$ for any positive integer n, this calculation becomes impractical for large values of n because of the need to write all the preceding rows. A more efficient way of finding these coefficients uses **factorial notation.** The number $n!$ (read "n*-factorial**") is defined as follows.

***n*-Factorial**

For any positive integer n,

$$n! = n(n - 1)(n - 2) \cdots (3)(2)(1).$$

By definition, $0! = 1.$

Blaise Pascal (1623–1662)

Pascal, a French mathematician, made mathematical contributions in the areas of calculus, geometry, and probability theory. At age 19, he invented the first adding machine, a precursor to our modern-day calculator.

EXAMPLE 5 Finding a Particular Term of a Binomial Expansion

Find the seventh term of $(a + 2b)^{10}$.

SOLUTION In the seventh term, $2b$ has an exponent of $7 - 1$, or 6, while a has an exponent of $10 - 6$, or 4.

$$\binom{10}{6}a^4(2b)^6 = 210a^4(64b^6) \quad \text{Seventh term of a binomial expansion}$$

$$= 13{,}440a^4b^6 \quad \text{Multiply.}$$

✔ *Now Try Exercise 41.*

7.4 Exercises

Evaluate each expression. In Exercises 11 and 12, leave answers in terms of n. ***See Example 1.***

1. $\frac{6!}{3!3!}$
2. $\frac{5!}{2!3!}$
3. $\frac{7!}{3!4!}$
4. $\frac{8!}{5!3!}$
5. $\binom{8}{5}$
6. $\binom{7}{3}$
7. $\binom{10}{2}$
8. $\binom{9}{3}$
9. $\binom{14}{14}$
10. $\binom{15}{15}$
11. $\binom{n}{n-1}$
12. $\binom{n}{n-2}$
13. $_8C_3$
14. $_9C_7$
15. $_{100}C_{98}$
16. $_{20}C_5$
17. $_9C_0$
18. $_5C_1$
19. $_{12}C_1$
20. $_4C_0$
21. *Concept Check* What are the first and last terms in the expansion of $(2x + 3y)^4$?
22. *Concept Check* Determine the binomial coefficient for the fifth term in the expansion of $(x + y)^9$.

Write the binomial expansion for each expression. ***See Examples 2–4.***

23. $(x + y)^6$
24. $(m + n)^4$
25. $(p - q)^5$
26. $(a - b)^7$
27. $(r^2 + s)^5$
28. $(m + n^2)^4$
29. $(p + 2q)^4$
30. $(3r + s)^6$
31. $(7p - 2q)^4$
32. $(4a - 5b)^5$
33. $(3x - 2y)^6$
34. $(7k - 9j)^4$
35. $\left(\frac{m}{2} - 1\right)^6$
36. $\left(3 - \frac{y}{3}\right)^5$
37. $\left(\sqrt{2}r + \frac{1}{m}\right)^4$
38. $\left(\frac{1}{k} + \sqrt{3}p\right)^3$
39. $\left(\frac{1}{x^4} + x^4\right)^4$
40. $\left(\frac{1}{y^5} + y^5\right)^5$

Write the indicated term of each binomial expansion. ***See Example 5.***

41. Sixth term of $(4h - j)^8$
42. Eighth term of $(2c - 3d)^{14}$
43. Seventeenth term of $(a^2 + b)^{22}$
44. Twelfth term of $(2x + y^2)^{16}$
45. Fifteenth term of $(x - y^3)^{20}$
46. Tenth term of $(a^3 + 3b)^{11}$

Concept Check Work Exercises 47–50.

47. Find the middle term of $(3x^7 + 2y^3)^8$.
48. Find the two middle terms of $(-2m^{-1} + 3n^{-2})^{11}$.

49. Find the value of n for which the coefficients of the fifth and eighth terms in the expansion of $(x + y)^n$ are the same.

50. Find the term(s) in the expansion of $(3 + \sqrt{x})^{11}$ that contain(s) x^4.

Relating Concepts

For individual or collaborative investigation *(Exercises 51–54)*

In this section, we saw how the factorial of a positive integer n can be computed as a product:

$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n.$$

Calculators and computers can evaluate factorials very quickly. Before the days of modern technology, mathematicians developed ***Stirling's formula*** *for approximating large factorials. Interestingly enough, the formula involves the irrational numbers π and e.*

$$n! \approx \sqrt{2\pi n} \cdot n^n \cdot e^{-n}$$

As an example, the exact value of 5! is 120, and Stirling's formula gives the approximation as 118.019168 with a graphing calculator. This is "off" by less than 2, an error of only 1.65%. ***Work Exercises 51–54 in order.***

51. Use a calculator to find the exact value of 10! and its approximation, using Stirling's formula.

52. Subtract the smaller value from the larger value in **Exercise 51.** Divide it by 10! and convert to a percent. What is the percent error?

53. Repeat **Exercises 51 and 52** for $n = 12$.

54. Repeat **Exercises 51 and 52** for $n = 13$. What seems to happen as n gets larger?

It can be shown that

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

is true for any real number n (not just positive integer values) and any real number x, where $|x| < 1$. Use this series to approximate the given number to the nearest thousandth.

55. $(1.02)^{-3}$ **56.** $(1.04)^{-5}$ **57.** $(1.01)^{1.5}$ **58.** $(1.03)^{2}$

7.5 Mathematical Induction

- **Proof by Mathematical Induction**
- **Proving Statements**
- **Generalized Principle of Mathematical Induction**
- **Proof of the Binomial Theorem**

Proof by Mathematical Induction Many results in mathematics are claimed true for every positive integer. Any of these results could be checked for $n = 1$, $n = 2$, $n = 3$, and so on, but since the set of positive integers is infinite, it would be impossible to check every possible case. For example, let S_n represent the statement that the sum of the first n positive integers is $\frac{n(n+1)}{2}$.

$$S_n\text{:}\quad 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$\binom{k+1}{1} = \frac{(k+1)(k)(k-1)(k-2)\cdots 1}{1\cdot(k)(k-1)(k-2)\cdots 1} = k+1$$

Also, $$\frac{k!}{1!(k-1)!} + 1 = \frac{k(k-1)!}{1(k-1)!} + 1 = k+1.$$

The second expression becomes $\binom{k+1}{2}$, the last $\binom{k+1}{k}$, and so on. The result of equation (3) is just equation (2) with every k replaced by $k+1$. Thus, the truth of S_k implies the truth of S_{k+1}, which completes the proof of the theorem by mathematical induction.

7.5 Exercises

Write out in full and verify the statements S_1, S_2, S_3, S_4, and S_5 for the following. Then use mathematical induction to prove that each statement is true for every positive integer n. ***See Example 1.***

1. $1 + 3 + 5 + \cdots + (2n-1) = n^2$

2. $2 + 4 + 6 + \cdots + 2n = n(n+1)$

Assume that n is a positive integer. Use mathematical induction to prove each statement S by following these steps. ***See Example 1.***

(a) *Verify the statement for $n = 1$.*
(b) *Write the statement for $n = k$.*
(c) *Write the statement for $n = k + 1$.*
(d) *Assume the statement is true for $n = k$. Use algebra to change the statement in part (b) to the statement in part (c).*
(e) *Write a conclusion based on Steps (a)–(d).*

3. $3 + 6 + 9 + \cdots + 3n = \dfrac{3n(n+1)}{2}$

4. $5 + 10 + 15 + \cdots + 5n = \dfrac{5n(n+1)}{2}$

5. $2 + 4 + 8 + \cdots + 2^n = 2^{n+1} - 2$

6. $3 + 9 + 27 + \cdots + 3^n = \dfrac{1}{2}(3^{n+1} - 3)$

7. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

8. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$

9. $5\cdot 6 + 5\cdot 6^2 + 5\cdot 6^3 + \cdots + 5\cdot 6^n = 6(6^n - 1)$

10. $7\cdot 8 + 7\cdot 8^2 + 7\cdot 8^3 + \cdots + 7\cdot 8^n = 8(8^n - 1)$

11. $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$

12. $\dfrac{1}{1\cdot 4} + \dfrac{1}{4\cdot 7} + \dfrac{1}{7\cdot 10} + \cdots + \dfrac{1}{(3n-2)(3n+1)} = \dfrac{n}{3n+1}$

13. $\dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \cdots + \dfrac{1}{2^n} = 1 - \dfrac{1}{2^n}$

14. $\dfrac{4}{5} + \dfrac{4}{5^2} + \dfrac{4}{5^3} + \cdots + \dfrac{4}{5^n} = 1 - \dfrac{1}{5^n}$

Find all natural number values for n for which the given statement is false.

15. $2^n > 2n$
16. $3^n > 2n + 1$
17. $2^n > n^2$
18. $n! > 2n$

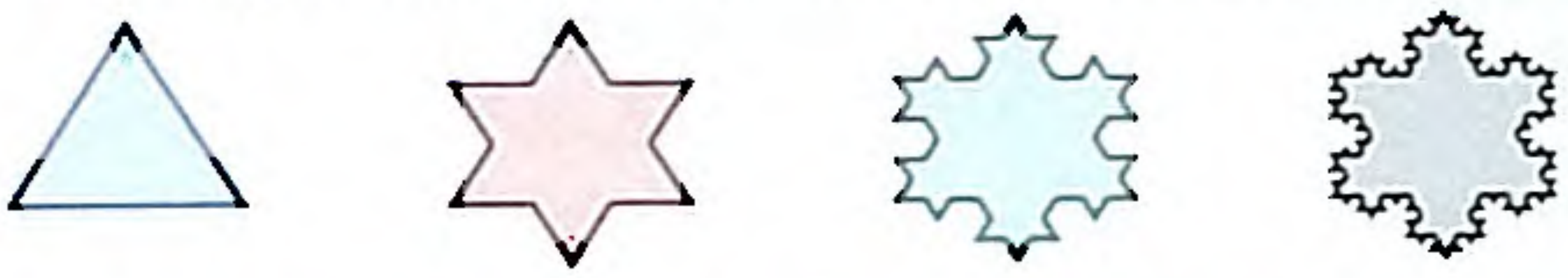
Prove each statement by mathematical induction. See Examples 1–3.

19. $(a^m)^n = a^{mn}$ (Assume a and m are constant.)
20. $(ab)^n = a^nb^n$ (Assume a and b are constant.)
21. $2^n > 2n$, if $n \geq 3$
22. $3^n > 2n + 1$, if $n \geq 2$
23. If $a > 1$, then $a^n > 1$.
24. If $a > 1$, then $a^n > a^{n-1}$.
25. If $0 < a < 1$, then $a^n < a^{n-1}$.
26. $2^n > n^2$, for $n \geq 5$
27. If $n \geq 4$, then $n! > 2^n$, where $n! = n(n-1)(n-2)\cdots(3)(2)(1)$.
28. $4^n > n^4$, for $n \geq 5$

Solve each problem.

29. ***Number of Handshakes*** Suppose that each of the n (for $n \geq 2$) people in a room shakes hands with everyone else, but not with himself or herself. Show that the number of handshakes is $\frac{n^2 - n}{2}$.

30. ***Sides of a Polygon*** The series of sketches below starts with an equilateral triangle having sides of length 1. In the following steps, equilateral triangles are constructed on each side of the preceding figure. The length of the sides of each new triangle is $\frac{1}{3}$ the length of the sides of the preceding triangles. Develop a formula for the number of sides of the nth figure. Use mathematical induction to prove your answer.



31. ***Perimeter*** Find the perimeter of the nth figure in Exercise 30.

32. ***Area*** Show that the area of the nth figure in Exercise 30 is

$$\sqrt{3}\left[\frac{2}{5} - \frac{3}{20}\left(\frac{4}{9}\right)^{n-1}\right].$$

33. ***Tower of Hanoi*** A pile of n rings, each ring smaller than the one below it, is on a peg. Two other pegs are attached to a board with this peg. In the game called the *Tower of Hanoi* puzzle, all the rings must be moved to a different peg, with only one ring moved at a time, and with no ring ever placed on top of a smaller ring. Find the least number of moves that would be required. Prove your result with mathematical induction.

Chapter 7 Quiz (Sections 7.1–7.5)

Write the first five terms of each sequence. State whether the sequence is arithmetic, geometric, or neither.

1. $a_n = -4n + 2$
2. $a_n = -2\left(-\frac{1}{2}\right)^n$
3. $a_1 = 5, a_2 = 3, a_n = a_{n-1} + 3a_{n-2}$, for $n \geq 3$

4. A certain arithmetic sequence has $a_1 = -6$ and $a_9 = 18$. Find a_7.

5. Find the sum of the first ten terms of each series.

(a) arithmetic, $a_1 = -20$, $d = 14$ (b) geometric, $a_1 = -20$, $r = -\frac{1}{2}$

6. Evaluate each sum that exists.

(a) $\sum_{i=1}^{30}(-3i + 6)$ (b) $\sum_{i=1}^{\infty} 2^i$ (c) $\sum_{i=1}^{\infty}\left(\frac{3}{4}\right)^i$

7. Use the binomial theorem to expand $(x - 3y)^5$.

8. Find the fifth term of the expansion of $\left(4x - \frac{1}{2}y\right)^5$.

9. Evaluate each expression.

(a) $9!$ (b) $\binom{10}{4}$

10. Use mathematical induction to prove that for all positive integers n,

$$6 + 12 + 18 + \cdots + 6n = 3n(n + 1).$$

7.6 Counting Theory

- Fundamental Principle of Counting
- Permutations
- Combinations
- Distinguishing between Permutations and Combinations

Fundamental Principle of Counting Consider the following problem.

> *If there are 3 roads from Albany to Baker and 2 roads from Baker to Creswich, in how many ways can one travel from Albany to Creswich by way of Baker?*

For each of the 3 roads from Albany to Baker, there are 2 different roads from Baker to Creswich. Hence, there are $3 \cdot 2 = 6$ different ways to make the trip, as shown in the tree diagram in **Figure 13**.

Here, each choice of road is an example of an *event*. Two events are **independent events** if neither influences the outcome of the other. The opening example illustrates the fundamental principle of counting with independent events.

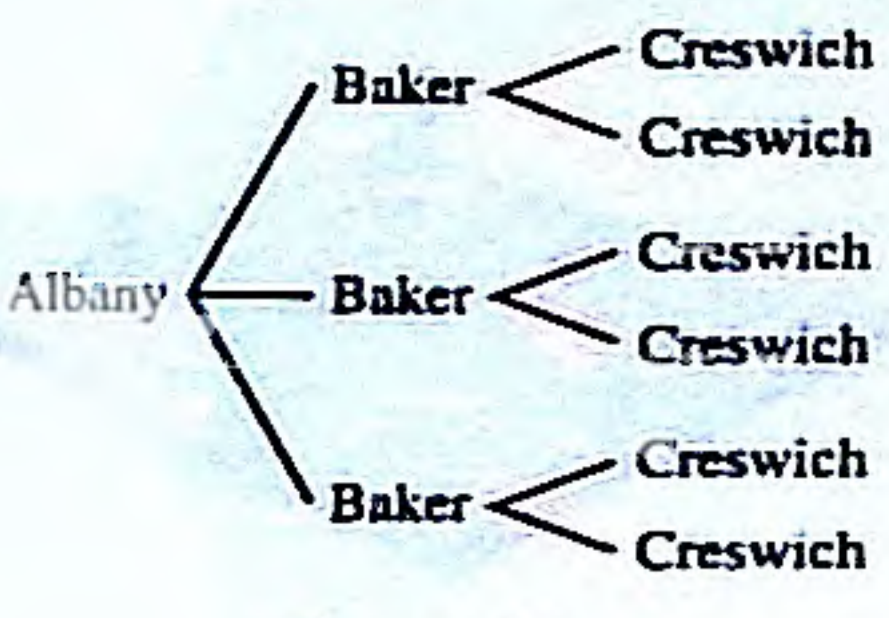


Figure 13

Fundamental Principle of Counting

If n independent events occur, with m_1 ways for event 1 to occur, m_2 ways for event 2 to occur, . . . and m_n ways for event n to occur, then there are

$$m_1 \cdot m_2 \cdot \cdots \cdot m_n \text{ different ways for all } n \text{ events to occur.}$$

EXAMPLE 1 **Using the Fundamental Principle of Counting**

A restaurant offers a choice of 3 salads, 5 main dishes, and 2 desserts. Use the fundamental principle of counting to find the number of different 3-course meals that can be selected.

SOLUTION Three events are involved: selecting a salad, selecting a main dish, and selecting a dessert. The first event can occur in 3 ways, the second event can occur in 5 ways, and the third event can occur in 2 ways. Thus, there are

$$3 \cdot 5 \cdot 2 = 30 \text{ possible meals.}$$

✔ *Now Try Exercise 1.*

EXAMPLE 2 Using the Fundamental Principle of Counting

A teacher has 5 different books that he wishes to arrange in a row. How many different arrangements are possible?

SOLUTION Five events are involved: selecting a book for the first spot, selecting a book for the second spot, and so on. For the first spot the teacher has 5 choices. After a choice has been made, the teacher has 4 choices for the second spot. Continuing in this manner, there are 3 choices for the third spot, 2 for the fourth spot, and 1 for the fifth spot. By the fundamental principle of counting, there are

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \text{ different arrangements.}$$

✓ *Now Try Exercise 5.*

In using the fundamental principle of counting, products such as

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

occur often. We use the symbol $n!$ (read "**n-factorial**"), for any counting number n, as follows.

$$n! = n(n-1)(n-2)\cdots(3)(2)(1) \quad \text{(Section 7.4)}$$

Thus,

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! \quad \text{and} \quad 3 \cdot 2 \cdot 1 = 3!.$$

By the definition of $n!$, $n[(n-1)!] = n!$ for all natural numbers $n \geq 2$. It is convenient to have this relation hold also for $n = 1$, so, by definition,

$$0! = 1. \quad \text{(Section 7.4)}$$

EXAMPLE 3 Arranging r of n Items ($r < n$)

Suppose the teacher in **Example 2** wishes to place only 3 of the 5 books in a row. How many arrangements of 3 books are possible?

SOLUTION The teacher still has 5 ways to fill the first spot, 4 ways to fill the second spot, and 3 ways to fill the third. Since only 3 books will be used, there are only 3 spots to be filled (3 events) instead of 5, with

$$5 \cdot 4 \cdot 3 = 60 \text{ arrangements.}$$

✓ *Now Try Exercise 7.*

Permutations Since each ordering of three books is considered a different *arrangement*, the number 60 in the preceding example is called the number of *permutations* of 5 things taken 3 at a time, written $P(5, 3) = 60$. The number of ways of arranging 5 elements from a set of 5 elements, written $P(5, 5) = 120$, was found in **Example 2.**

A **permutation** of n elements taken r at a time is one of the *arrangements* of r elements from a set of n elements. Generalizing from the examples above, the number of permutations of n elements taken r at a time, denoted by $P(n, r)$, is

$$\begin{aligned} P(n, r) &= n(n-1)(n-2)\cdots(n-r+1) \\ &= \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)(n-r-1)\cdots(2)(1)}{(n-r)(n-r-1)\cdots(2)(1)} \\ &= \frac{n!}{(n-r)!}. \end{aligned}$$

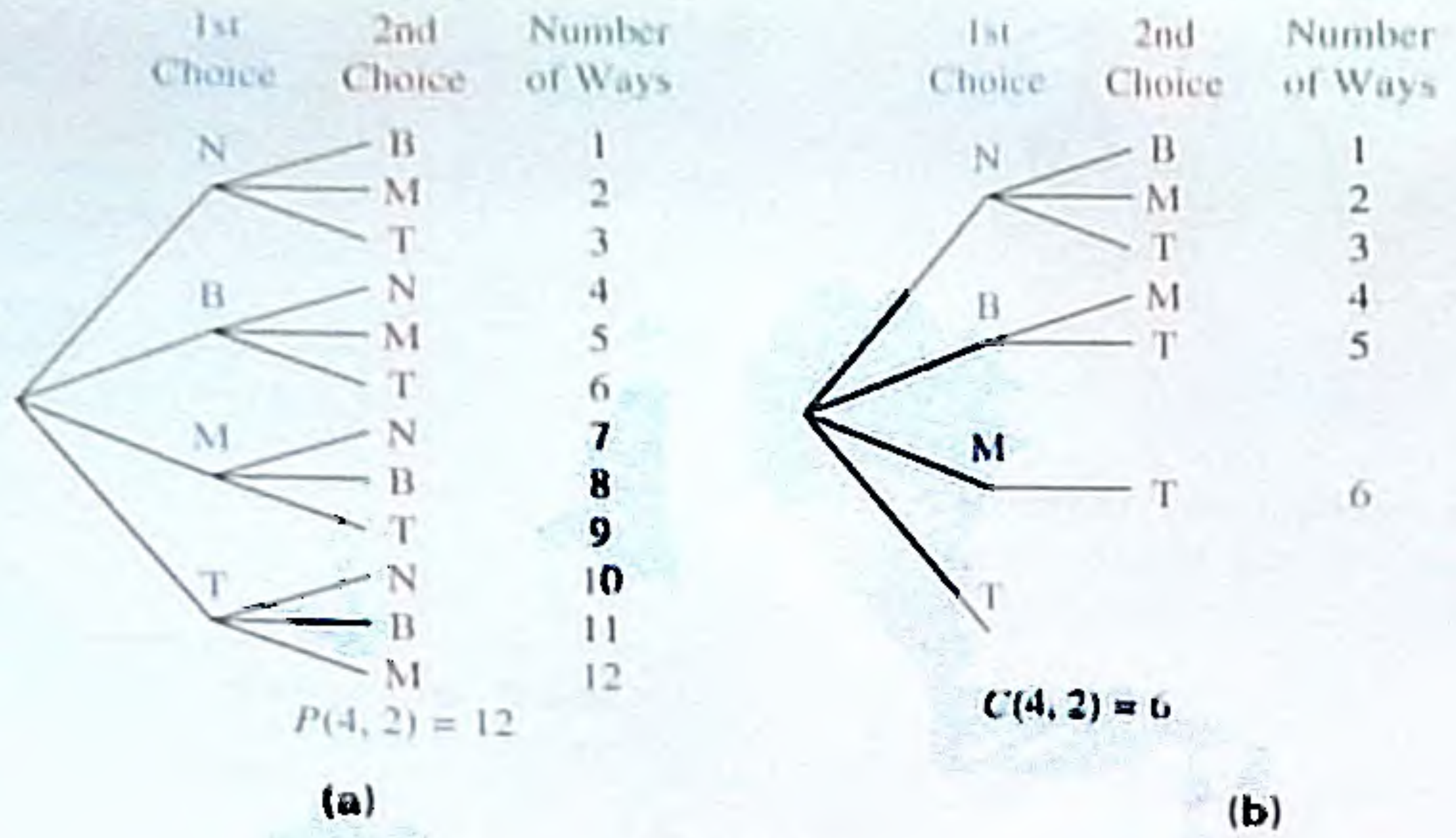


Figure 15

CAUTION Not all counting problems lend themselves to either permutations or combinations. Whenever the fundamental principle of counting or a tree diagram can be used directly, as in the soup example, use it.

7.6 Exercises

Fundamental Counting Principle *Use the fundamental principle of counting to solve each problem. See Examples 1–3.*

1. On a business trip, Terry took 3 pairs of pants, 4 shirts, 1 jacket, and two pairs of shoes. Determine the number of outfits that Terry can choose.

2. When saddling her horse, Judy can choose from 2 saddles, 3 blankets, and 2 cinches. Find the number of possible choices for saddling Judy's horse.

3. A conference schedule offers 2 main sessions, 20 break-out sessions, and 4 mini-courses. In how many ways can an attendee choose 1 of each to attend?

4. A convenience store offers 16 types of soda with 4 options for flavoring and either crushed or cubed ice. Determine the total number of drink options available for selecting 1 soda with 1 flavor and 1 type of ice.

5. A college has 7 portraits of past college presidents to arrange in a row on a wall. How many different arrangements are possible?

6. A telephone messaging system requires a 4-digit security code. How many security codes are possible if numbers may be repeated.

7. In how many ways can judges select a 1st-place winner, a 2nd-place winner, and a 3rd-place winner from 16 desserts entered in a cooking contest?

8. In how many different ways can 4 different boys be selected from a group of 25 boys on a track team to receive 4 different awards?

Evaluate each expression. See Examples 4–8.

9. $P(12, 8)$
10. $P(5, 5)$
11. $P(9, 2)$
12. $P(10, 9)$
13. $P(5, 1)$
14. $P(6, 0)$
15. $C(4, 2)$
16. $C(9, 3)$
17. $C(6, 0)$
18. $C(8, 1)$
19. $\binom{12}{4}$
20. $\binom{16}{3}$

Use a calculator to evaluate each expression. See Examples 4 and 7.

21. ${}_{20}P_5$ **22.** ${}_{100}P_5$ **23.** ${}_{15}P_8$ **24.** ${}_{32}P_4$

25. ${}_{20}C_5$ **26.** ${}_{100}C_5$ **27.** $\binom{15}{8}$ **28.** $\binom{32}{4}$

29. Decide whether the situation described involves a permutation or a combination of objects. See Example 9.

(a) a telephone number
(b) a Social Security number
(c) a hand of cards in poker
(d) a committee of politicians
(e) the "combination" on a padlock
(f) an automobile license plate
(g) a lottery choice of six numbers where order does not matter

30. Explain the difference between a permutation and a combination. What should you look for in a problem to decide which is an appropriate method of solution?

Use the fundamental principle of counting or permutations to solve each problem. See Examples 1–6.

31. ***Home Plan Choices*** How many different types of homes are available if a builder offers a choice of 5 basic plans, 4 roof styles, and 2 exterior finishes?

32. ***Auto Varieties*** An auto manufacturer produces 7 models, each available in 6 different colors, with 4 different upholstery fabrics, and 5 interior colors. How many varieties of the auto are available?

33. ***Radio-Station Call Letters*** How many different 4-letter radio-station call letters can be made

(a) if the first letter must be K or W and no letter may be repeated?
(b) if repetitions are allowed (but the first letter is K or W)?
(c) if the first letter must be K or W, the last letter must be R, and repetitions are not allowed?

34. ***Meal Choices*** A menu offers a choice of 3 salads, 8 main dishes, and 5 desserts. How many different 3-course meals (salad, main dish, dessert) are possible?

35. ***Arranging Blocks*** Baby Finley is arranging 7 blocks in a row. How many different arrangements can he make?

36. ***Names for a Baby*** A couple has narrowed down the choice of a name for their new baby to 5 first names and 3 middle names. How many different first- and middle-name combinations are possible?

37. ***License Plates*** For many years, the state of California used 3 letters followed by 3 digits on its automobile license plates.

(a) How many different license plates are possible with this arrangement?
(b) When the state ran out of new plates, the order was reversed to 3 digits followed by 3 letters. How many additional plates were then possible?
(c) When the plates described in part (b) were also used up, the state then issued plates with 1 letter followed by 3 digits and then 3 letters. How many plates does this scheme provide?

38. *Telephone Numbers* How many 7-digit telephone numbers are possible if the first digit cannot be 0 and

(a) only odd digits may be used?
(b) the telephone number must be a multiple of 10 (that is, it must end in 0)?
(c) the telephone number must be a multiple of 100?
(d) the first 3 digits are 481?
(e) no repetitions are allowed?

39. *Seating People in a Row* In an experiment on social interaction, 9 people will sit in 9 seats in a row. In how many ways can this be done?

40. *Genetics Experiment* In how many ways can 7 of 10 rats be arranged in a row for a genetics experiment?

41. *Course Schedule Arrangement* A business school offers courses in keyboarding, spreadsheets, transcription, business English, technical writing, and accounting. In how many ways can a student arrange a schedule if 3 courses are taken?

42. *Course Schedule Arrangement* If your college offers 400 courses, 20 of which are in mathematics, and your counselor arranges your schedule of 4 courses by random selection, how many schedules are possible that do not include a math course?

43. *Club Officer Choices* In a club with 15 members, in how many ways can a slate of 3 officers consisting of president, vice-president, and secretary/treasurer be chosen?

44. *Batting Orders* A baseball team has 20 players. How many 9-player batting orders are possible?

45. *Letter Arrangement* Consider the word BRUCE.

(a) In how many ways can all the letters of the word BRUCE be arranged?
(b) In how many ways can the first 3 letters of the word BRUCE be arranged?

46. *Basketball Positions* In how many ways can 5 players be assigned to the 5 positions on a basketball team, assuming that any player can play any position? In how many ways can 10 players be assigned to the 5 positions?

Solve each problem involving combinations. See Examples 7 and 8.

47. *Seminar Presenters* A banker's association has 40 members. If 6 members are selected at random to present a seminar, how many different groups of 6 are possible?

48. *Financial Planners* Four financial planners are to be selected from a group of 12 to participate in a special program. In how many ways can this be done? In how many ways can the group that will not participate be selected?

49. *Apple Samples* How many different samples of 4 apples can be drawn from a crate of 25 apples?

50. *Apple Samples* Suppose in Exercise 49 that there are 5 rotten apples in the crate.

(a) How many samples of 3 could be drawn in which all 3 are rotten?
(b) How many samples of 3 could be drawn in which there are 2 good apples and 1 rotten apple?

51. *Hamburger Choices* Howard's Hamburger Heaven sells hamburgers with cheese, relish, lettuce, tomato, mustard, or ketchup.

(a) How many different hamburgers can be made that use any 4 of the extras?
(b) How many different hamburgers can be made if one of the 4 extras must be cheese?

52. *Card Combinations* Five cards are marked with the numbers 1, 2, 3, 4, and 5, shuffled, and 2 cards are then drawn. How many different 2-card hands are possible?

53. *Marble Samples* If a bag contains 15 marbles, how many samples of 2 marbles can be drawn from it? How many samples of 4 marbles can be drawn?

54. *Marble Samples* In Exercise 53, if the bag contains 3 yellow, 4 white, and 8 blue marbles, how many samples of 2 can be drawn in which both marbles are blue?

55. *Convention Delegation Choices* A city council is composed of 5 liberals and 4 conservatives. Three members are to be selected randomly as delegates to an urban convention.

(a) How many delegations are possible?
(b) How many delegations could have all liberals?
(c) How many delegations could have 2 liberals and 1 conservative?
(d) If 1 member of the council serves as mayor, how many delegations are possible that include the mayor?

56. *Delegation Choices* Seven workers decide to send a delegation of 2 to their supervisor to discuss their grievances.

(a) How many different delegations are possible?
(b) If it is decided that a certain employee must be in the delegation, how many different delegations are possible?
(c) If there are 2 women and 5 men in the group, how many delegations would include at least 1 woman?

Use any or all of the methods described in this section to solve each problem. See Examples 1–9.

57. *Course Schedule Arrangement* If Dwight Johnston has 8 courses to choose from, how many ways can he arrange his schedule if he must pick 4 of them?

58. *Pineapple Samples* How many samples of 9 pineapples can be drawn from a crate of 12?

59. *Soup Ingredients* Velma specializes in making different vegetable soups with carrots, celery, beans, peas, mushrooms, and potatoes. How many different soups can she make with any 4 ingredients?

60. *Secretary/Manager Assignments* From a pool of 7 secretaries, 3 are selected to be assigned to 3 managers, 1 secretary to each manager. In how many ways can this be done?

61. *Musical Chairs Seatings* In a game of musical chairs, 13 children will sit in 12 chairs. (1 will be left out.) How many seating arrangements are possible?

62. *Plant Samples* In an experiment on plant hardiness, a researcher gathers 6 wheat plants, 3 barley plants, and 2 rye plants. She wishes to select 4 plants at random.

(a) In how many ways can this be done?
(b) In how many ways can this be done if exactly 2 wheat plants must be included?

63. *Committee Choices* In a club with 8 women and 11 men members, how many 5-member committees can be chosen that have the following?

(a) all women
(b) all men
(c) 3 women and 2 men
(d) no more than 3 men

64. *Committee Choices* From 10 names on a ballot, 4 will be elected to a political party committee. In how many ways can the committee of 4 be formed if each person will have a different responsibility?

65. *Combination Lock* A briefcase has 2 locks. The combination to each lock consists of a 3-digit number, where digits may be repeated. How many combinations are possible? (*Hint:* The word *combination* is a misnomer. Lock combinations are permutations where the arrangement of the numbers is important.)

66. ***Combination Lock*** A typical "combination" for a padlock consists of 3 numbers from 0 to 39. Find the number of "combinations" that are possible with this type of lock, if a number may be repeated.

67. ***Garage Door Openers*** The code for some garage door openers consists of 12 electrical switches that can be set to either 0 or 1 by the owner. With this type of opener, how many codes are possible? (*Source:* Promax.)

68. ***Lottery*** To win the jackpot in a lottery game, a person must pick 4 numbers from 0 to 9 in the correct order. If a number can be repeated, how many ways are there to play the game?

69. ***Keys*** In how many distinguishable ways can 4 keys be put on a circular key ring?

70. ***Sitting at a Round Table*** In how many different ways can 8 people sit at a round table? Assume that "a different way" means that at least 1 person is sitting next to someone different.

Prove each statement for positive integers n and r, with $r \leq n$. (Hint: Use the definitions of permutations and combinations.)

71. $P(n, n-1) = P(n, n)$

72. $P(n, 1) = n$

73. $P(n, 0) = 1$

74. $P(n, n) = n!$

75. $\binom{n}{n} = 1$

76. $\binom{n}{0} = 1$

77. $\binom{0}{0} = 1$

78. $\binom{n}{n-1} = n$

79. $\binom{n}{n-r} = \binom{n}{r}$

80. Explain why the restriction $r \leq n$ is needed in the formula for $P(n, r)$.

7.7 Basics of Probability

- Basic Concepts
- Complements and Venn Diagrams
- Odds
- Union of Two Events
- Binomial Probability

Basic Concepts Consider an experiment that has one or more possible **outcomes,** all of which are equally likely to occur. For example, the experiment of tossing a fair coin has 2 equally likely possible outcomes: landing heads up (H) or landing tails up (T). Also, the experiment of rolling a fair die has 6 equally likely outcomes: landing so the face that is up shows 1, 2, 3, 4, 5, or 6 dots.

The set S of all possible outcomes of a given experiment is the **sample space** for the experiment. (In this text, all sample spaces are finite.) The table gives several examples.

Experiment	Sample Space
Toss a coin.	$S = \{H, T\}$
Roll a die.	$S = \{1, 2, 3, 4, 5, 6\}$
Toss two coins.	$S = \{(H, H), (H, T), (T, H), (T, T)\}$
Answer a true/false question.	$S = \{\text{true}, \text{false}\}$

Use set notation (Section R.1)

Any subset of the sample space is an **event.** In the experiment with the die, for example, "the number showing is a 3" is an event, say E_1, such that $E_1 = \{3\}$. "The number showing is greater than 3" is also an event, say E_2, such that $E_2 = \{4, 5, 6\}$. To represent the number of outcomes that belong to event E, the notation $n(E)$ is used. Then $n(E_1) = 1$ and $n(E_2) = 3$.

The notation $P(E)$ is used for the *probability* of an event E. If the outcomes in the sample space for an experiment are equally likely, then the probability of event E occurring is found as follows.

Probability of Event E

In a sample space with equally likely outcomes, the **probability** of an event E, written $P(E)$, is the ratio of the number of outcomes in sample space S that belong to event E, $n(E)$, to the total number of outcomes in sample space S, $n(S)$.

$$P(E) = \frac{n(E)}{n(S)}$$

To use this definition to find the probability of the event E_1 in the die experiment, start with the sample space, $S = \{1, 2, 3, 4, 5, 6\}$, and the desired event, $E_1 = \{3\}$. Use $n(E_1) = 1$ and $n(S) = 6$.

$$P(E_1) = \frac{n(E_1)}{n(S)} = \frac{1}{6}$$

EXAMPLE 1 **Finding Probabilities of Events**

A single die is rolled. Write each event in set notation and give the probability of the event.

(a) E_3: the number showing is even

(b) E_4: the number showing is greater than 4

(c) E_5: the number showing is less than 7

(d) E_6: the number showing is 7

SOLUTION

(a) Since $E_3 = \{2, 4, 6\}$, $n(E_3) = 3$. As given earlier, $n(S) = 6$.

$$P(E_3) = \frac{3}{6} = \frac{1}{2}$$

(b) Again $n(S) = 6$. Event $E_4 = \{5, 6\}$, with $n(E_4) = 2$.

$$P(E_4) = \frac{2}{6} = \frac{1}{3}$$

(c) $E_5 = \{1, 2, 3, 4, 5, 6\}$ and $P(E_5) = \frac{6}{6} = 1$.

(d) $E_6 = \emptyset$ and $P(E_6) = \frac{0}{6} = 0$.

✓ *Now Try Exercises 7 and 9.*

In **Example 1(c)**, $E_5 = S$. Therefore, the event E_5 is certain to occur every time the experiment is performed. An event that is certain to occur always has probability 1. In **Example 1(d)**, $E_6 = \emptyset$ and $P(E_6) = 0$. The probability of an impossible event, such as E_6, is always 0, since none of the outcomes in the sample space satisfies the event. ***For any event E, $P(E)$ is between 0 and 1, inclusive.***

The properties of probability are summarized as follows.

Properties of Probability

Let E and F represent events.

1. $0 \le P(E) \le 1$
2. $P(\text{a certain event}) = 1$
3. $P(\text{an impossible event}) = 0$
4. $P(E') = 1 - P(E)$
5. $P(E \text{ or } F) = P(E \cup F) = P(E) + P(F) - P(E \cap F)$

CAUTION ***When finding the probability of a union, remember to subtract the probability of the intersection from the sum of the probabilities of the individual events.***

Binomial Probability Many probability experiments consist of a repeated number of independent trials (n) with only two possible outcomes. Consider the example of tossing a coin 5 times and observing the number of tails. In this experiment there are $n = 5$ independent trials, or coin tosses, and there are two possible outcomes, head or tail, for each trial. It is common to consider "obtaining a tail" as a success because it is the outcome of interest, so "obtaining a head" would be considered a failure.

When trials such as coin tosses are independent, the probability of a success (tail) and the probability of a failure (head) remain constant. That is, $P(\text{tail}) = \frac{1}{2}$ and $P(\text{head}) = \frac{1}{2}$ for each coin toss. When a probability experiment satisfies these conditions, it is considered a *binomial experiment.*

Binomial Probability

If a probability experiment consists of n independent trials with two possible outcomes for each trial, and the probabilities remain constant for each trial, then it is a **binomial experiment.** Let p represent the probability of a success, and let $q = 1 - p$ represent the probability of a failure. Then the probability of obtaining exactly r successes in n trials is found as follows.

$$P(r \text{ successes in } n \text{ trials}) = \binom{n}{r} p^r q^{n-r}$$

Suppose that we want to determine the probability of getting exactly 3 tails in 5 coin tosses. Here $n = 5$, $r = 3$, $p = P(\text{tail}) = \frac{1}{2}$, and $q = P(\text{head}) = 1 - \frac{1}{2} = \frac{1}{2}$.

$$\begin{aligned}
P(3 \text{ tails in 5 coin tosses}) &= \binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{5-3} && \text{Use the binomial probability formula.} \\
&= \frac{5!}{3!\,2!}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 && \text{Apply the combinations formula and subtract.} \\
&= 10\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 && \text{Evaluate factorials and divide.} \\
&= 0.3125 && \text{Apply the exponents and multiply.}
\end{aligned}$$

EXAMPLE 6 Finding Probabilities in a Binomial Experiment

An experiment consists of rolling a die 10 times and observing the number of 3s.

(a) Find the probability of getting exactly 4 threes.

(b) Find the probability that the result is not a 3 in exactly 9 of the rolls.

ALGEBRAIC SOLUTION

(a) There are $n = 10$ independent trials with $p = P(3) = \frac{1}{6}$ and $q = 1 - \frac{1}{6} = \frac{5}{6}$.

$$P\text{ (4 threes in 10 rolls)} = \binom{10}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^{10-4}$$

$$= 210\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^6$$

$$\approx 0.054$$

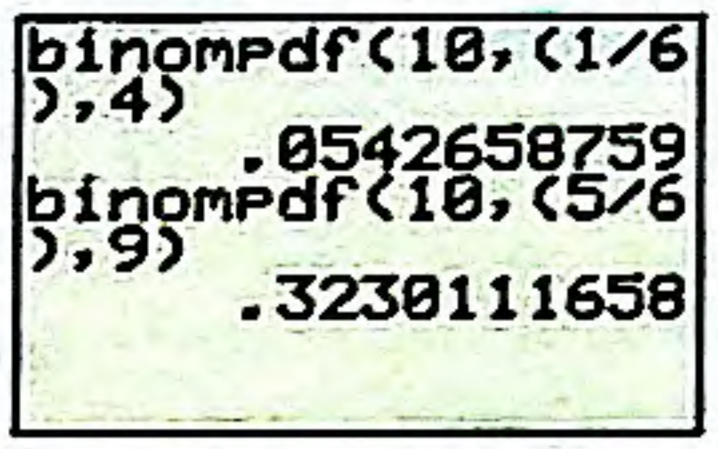
(b) Here $n = 10$, $p = P(\text{not a } 3) = \frac{5}{6}$, and $q = \frac{1}{6}$.

$$P\text{(9 non-threes in 10 rolls)} = \binom{10}{9}\left(\frac{5}{6}\right)^9\left(\frac{1}{6}\right)^1$$

$$\approx 0.323$$

GRAPHING CALCULATOR SOLUTION

Graphing calculators, such as the TI-83/84 Plus, that have statistical distribution functions give binomial probabilities. **Figure 19** shows the results for parts (a) and (b). The numbers in parentheses separated by commas represent n, p, and r, respectively.



```
binompdf(10,(1/6
),4)
        .0542658759
binompdf(10,(5/6
),9)
        .3230111658
```

Figure 19

✓ *Now Try Exercise 35.*

7.7 Exercises

Write a sample space with equally likely outcomes for each experiment.

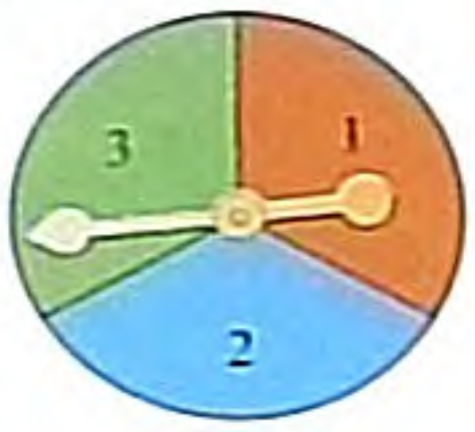
1. A two-headed coin is tossed once.

2. Two ordinary coins are tossed.

3. Three ordinary coins are tossed.

4. Slips of paper marked with the numbers 1, 2, 3, and 4 are placed in a box. A slip is drawn and set aside, its number is recorded, and then a second slip is drawn.

5. The spinner shown here is spun twice.



6. A die is rolled and then a coin is tossed.

Write each event in set notation and give the probability of the event. See Example 1.

7. Refer to Exercise 1.

(a) The result of the toss is heads. **(b)** The result of the toss is tails.

8. Refer to Exercise 2.

(a) Both coins show the same face. **(b)** At least one coin is a head.

9. Refer to Exercise 3.

(a) All three coins show the same face. (b) At least two coins are tails.

10. Refer to Exercise 4.

(a) Both slips are marked with even numbers.
(b) Both slips are marked with odd numbers.
(c) Both slips are marked with the same number.
(d) One slip is marked with an odd number, the other with an even number.

11. Refer to Exercise 5.

(a) The result is a repeated number. (b) The second number is 1 or 3.
(c) The first number is even and the second number is odd.

12. A student gives the probability of an event in a problem as $\frac{6}{5}$. Explain why this answer must be incorrect.

13. *Concept Check* If the probability of an event is 0.857, what is the probability that the event will not occur?

14. *Concept Check* Associate each probability in parts (a)–(g) with one of the statements in choices A–F. Choices may be used more than once.

(a) $P(E) = -0.1$ (b) $P(E) = 0.01$ (c) $P(E) = 1$ (d) $P(E) = 2$
(e) $P(E) = 0.99$ (f) $P(E) = 0$ (g) $P(E) = 0.5$

A. The event is certain to occur. B. The event cannot occur.
C. The event is very likely to occur. D. The event is very unlikely to occur.
E. The event is just as likely to occur as not to occur.
F. The event is impossible.

Work each problem. See Examples 1–6.

15. *Batting Average* A baseball player with a batting average of .300 comes to bat. What are the odds in favor of the ball player getting a hit?

16. *Small Business Loan* The probability that a bank with assets greater than or equal to $30 billion will make a loan to a small business is 0.002. What are the odds against such a bank making a small business loan? (*Source: The Wall Street Journal* analysis of *CA1 Reports* filed with federal banking authorities.)

17. *Drawing a Card* A card is drawn at random from a standard deck of 52 cards. Find the probabilities in parts (a)–(d).

(a) The card is a spade. (b) The card is not a spade.
(c) The card is a spade or a heart. (d) The card is a spade or a face card.
(e) What are the odds in favor of drawing a spade?

18. *Dice Rolls* Two dice are rolled. Find the probabilities in parts (a)–(d).

(a) The sum of the dots is at least 10.
(b) The sum of the dots is less than 10.
(c) The sum of the dots is either 7 or at least 10.
(d) The sum of the dots is 2, or the dice both show the same number.
(e) What are the odds against rolling a 7?

19. *Origins of Foreign-Born Population* The numbers (in thousands) of foreign-born people who were living in the United States in 2008, according to region of birth, are given in the table. Find the probability that a foreign-born U.S. resident in 2008 satisfied the following.

(a) born in Asia (b) not born in Europe
(c) born in Africa or Europe
(d) What are the odds that a randomly selected foreign-born U.S. resident was born in Latin America?

Region	Number (in thousands)
Africa	1436
Asia	10,356
Europe	4969
Latin America	20,150
Other	1050

Source: U.S. Census Bureau

20. *U.S. Population by Region* The U.S. resident population by region (in millions) for selected years is given in the table. Find the probability that a U.S. resident selected at random satisfies the following.

Region	1995	2000	2009
Northeast	51.4	53.6	55.3
Midwest	61.8	64.4	66.8
South	91.8	100.2	113.3
West	57.7	63.2	71.6

Source: U.S. Census Bureau.

(a) lived in the West in 2000
(b) lived in the Midwest in 1995
(c) lived in the Northeast or Midwest in 2000
(d) lived in the South or West in 2009
(e) What are the odds that a randomly selected U.S. resident in 2009 was not from the South?

21. *State Lottery* One game in a state lottery requires you to pick 1 heart, 1 club, 1 diamond, and 1 spade, in that order, from the 13 cards in each suit. What is the probability of getting all four picks correct and winning $5000?

22. *State Lottery* If three of the four selections in **Exercise 21** are correct, the player wins $200. Find the probability of this outcome.

23. *Male Life Table* The table is an abbreviated version of the 2006 **period life table** used by the Office of the Chief Actuary of the Social Security Administration. (The actual table includes every age, not just every tenth age.) Theoretically, this table follows a group of 100,000 males at birth and gives the number still alive at each age.

Exact Age	Number of Lives	Exact Age	Number of Lives
0	100,000	60	85,026
10	99,067	70	71,586
20	98,519	80	47,073
30	97,079	90	15,051
40	95,431	100	657
50	92,041	110	1

Source: Office of the Actuary, Social Security Administration.

(a) What is the probability that a 40-year-old man will live 30 more years?
(b) What is the probability that a 40-year-old man will not live 30 more years?
(c) Consider a group of five 40-year-old men. What is the probability that exactly three of them survive to age 70? (*Hint:* The longevities of the individual men can be considered as independent trials.)
(d) Consider two 40-year-old men. What is the probability that at least one of them survives to age 70? (*Hint:* The complement of *at least one* is *none*.)

24. *Opinion Survey* The management of a firm wishes to survey the opinions of its workers, classified as follows for the purpose of an interview:

30% have worked for the company 5 or more years,
28% are female,
65% contribute to a voluntary retirement plan, and 50% of the female workers contribute to the retirement plan.

Find each probability if a worker is selected at random.

(a) A male worker is selected.
(b) A worker is selected who has worked for the company less than 5 yr.
(c) A worker is selected who contributes to the retirement plan or is female.

25. *Growth in Stock Value* A financial analyst has determined the possibilities (and their probabilities) for the growth in value of a certain stock during the next year. (Assume these are the only possibilities.) See the table. For instance, the probability of a 5% growth is 0.15. If you invest $10,000 in the stock, what is the probability that the stock will be worth at least $11,400 by the end of the year?

Percent Growth	Probability
5	0.15
8	0.20
10	0.35
14	0.20
18	0.10

26. *Growth in Stock Value* Refer to Exercise 25. Suppose the percents and probabilities in the table are estimates of annual growth during the next 3 yr. What is the probability that an investment of $10,000 will grow in value to *at least* $15,000 during the next 3 yr? (*Hint:* Use the formula for (annual) compound interest discussed in Section 4.2.)

Insufficient Sleep The table gives the results of a 2008 survey of Americans aged 18–24 in which the respondents were asked, "During the past 30 days, for about how many days have you felt that you did not get enough sleep?"

Number of Days	0	1–13	14–29	30
Percent (as a decimal)	0.23	0.45	0.20	0.12

Source: U.S. Centers for Disease Control and Prevention.

Using the percents as probabilities, find the probability that, out of 10 respondents in the 18–24 age group selected at random, the following were true.

27. Exactly 4 did not get enough sleep on 1–13 days.

28. Exactly 5 got enough sleep every night.

29. Fewer than 2 did not get enough sleep on 14 or more days.

30. No more than 3 did not get enough sleep on 1–29 days.

College Applications The table gives the results of a survey of 219,864 first-year students from the class of 2013 at 297 of the nation's four-year colleges and universities.

Number of Colleges Applied to	1	2 or 3	4–6	7 or more
Percent (as a decimal)	0.14	0.24	0.39	0.23

Source: Higher Education Research Institute, UCLA, 2009.

Using the percents as probabilities, find the probability of each event for a randomly selected student.

31. The student applied to fewer than 4 colleges.

32. The student applied to at least 2 colleges.

33. The student applied to more than 3 colleges.

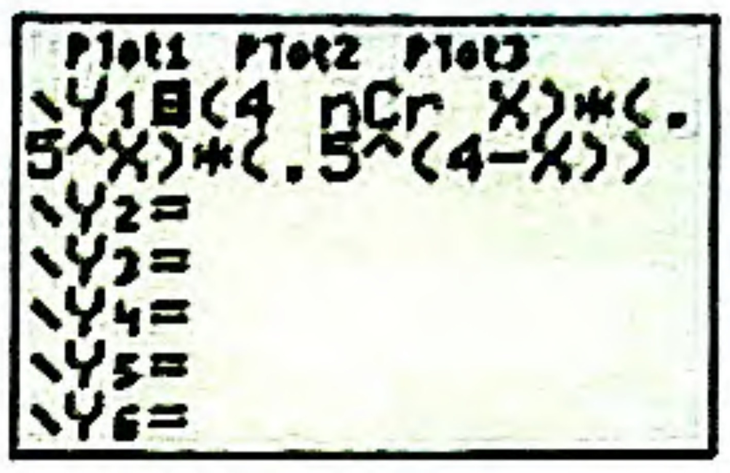
34. The student applied to no colleges.

Solve each problem.

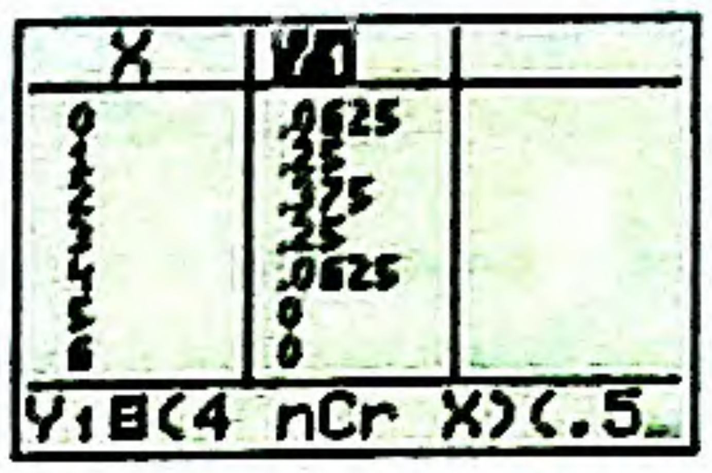
35. *Color-Blind Males* The probability that a male will be color-blind is 0.042. Find the probabilities that in a group of 53 men, the following are true.

(a) Exactly 5 are color-blind
(b) No more than 5 are color-blind.
(c) None are color-blind
(d) At least 1 is color-blind

36. The screens illustrate how the TABLE feature of a graphing calculator can be used to find the probabilities of having 0, 1, 2, 3, or 4 girls in a family of 4 children. (Note that 0 appears for X = 5 and X = 6 because these events are impossible.)



```
Plot1 Plot2 Plot3
\Y1■(4 nCr X)*(.
5^X)*(.5^(4-X))
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```



```
  X    |Y1    |
-------+------+------
  0    |.0625 |
  1    |.25   |
  2    |.375  |
  3    |.25   |
  4    |.0625 |
  5    |0     |
  6    |0     |
Y1■(4 nCr X)(.5
```

Use this approach for the following.

(a) Find the probabilities of having 0, 1, 2, or 3 boys in a family of 3 children.

(b) Find the probabilities of having 0, 1, 2, 3, 4, 5, or 6 girls in a family of 6 children.

37. *(Modeling) Spread of Disease* What will happen when an infectious disease is introduced into a family? Suppose a family has I infected members and S members who are not infected but are susceptible to contracting the disease. The probability P of exactly k people not contracting the disease during a 1-week period can be calculated by the formula

$$P = \binom{S}{k} q^k (1 - q)^{S-k},$$

where $q = (1 - p)^I$, and p is the probability that a susceptible person contracts the disease from an infected person. For example, if $p = 0.5$, then there is a 50% chance that a susceptible person exposed to 1 infected person for 1 week will contract the disease. (*Source:* Hoppensteadt, F. and C. Peskin, *Mathematics in Medicine and the Life Sciences,* Springer-Verlag.)

(a) Compute the probability P of 3 family members not becoming infected within 1 week if there are currently 2 infected and 4 susceptible members. Assume that $p = 0.1$. (*Hint:* To use the formula, first determine the values of k, I, S, and q.)

(b) A highly infectious disease can have $p = 0.5$. Repeat part (a) with this value of p.

(c) Determine the probability that everyone will become sick in a large family if, initially, $I = 1$, $S = 9$, and $p = 0.5$. Discuss the results.

38. *(Modeling) Spread of Disease* (Refer to Exercise 37.) Suppose that in a family $I = 2$ and $S = 4$. If the probability P is 0.25 of there being $k = 2$ uninfected members after 1 week, estimate graphically the possible values of p. (*Hint:* Write P as a function of p.)

Chapter 7 Test Prep

Key Terms

7.1 finite sequence
infinite sequence
terms of a sequence
general term (nth term)
convergent sequence
divergent sequence
recursive definition
Fibonacci sequence
series
summation notation
finite series
infinite series
index of summation

7.2 arithmetic sequence (arithmetic progression)
common difference
arithmetic series

7.3 geometric sequence (geometric progression)
common ratio
geometric series
annuity
future value of an annuity

7.4 Pascal's triangle
factorial notation
binomial coefficient
binomial theorem (general binomial expansion)

7.6 tree diagram
independent events
permutation
combination

7.7 outcome
sample space
event
probability
complement
Venn diagram
odds
mutually exclusive events
binomial experiment

Concepts

Examples

Binomial Theorem

For any positive integer n and any complex numbers x and y, $(x+y)^n$ is expanded as follows.

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots + \binom{n}{r}x^{n-r}y^r + \cdots + \binom{n}{n-1}xy^{n-1} + y^n$$

$$\begin{aligned}(2m+3)^4 &= (2m)^4 + \frac{4!}{1!3!}(2m)^3(3) + \frac{4!}{2!2!}(2m)^2(3)^2 \\ &\quad + \frac{4!}{3!1!}(2m)(3)^3 + 3^4 \\ &= 2^4m^4 + 4(2)^3m^3(3) + 6(2)^2m^2(9) \\ &\quad + 4(2m)(27) + 81 \\ &= 16m^4 + 12(8)m^3 + 54(4)m^2 + 216m + 81 \\ &= 16m^4 + 96m^3 + 216m^2 + 216m + 81\end{aligned}$$

kth Term of the Binomial Expansion of $(x+y)^n$

$$\binom{n}{k-1}x^{n-(k-1)}y^{k-1} \quad (\text{where } n \geq k-1)$$

The eighth term of $(a-2b)^{10}$ is

$$\begin{aligned}\binom{10}{7}a^3(-2b)^7 &= \frac{10!}{7!3!}a^3(-2)^7b^7 \\ &= 120(-128)a^3b^7 \\ &= -15{,}360a^3b^7.\end{aligned}$$

7.5 Mathematical Induction

Principle of Mathematical Induction

Let S_n be a statement concerning the positive integer n. Suppose that

1. S_1 is true;
2. for any positive integer k, $k \leq n$, if S_k is true, then S_{k+1} is also true.

Then S_n is true for every positive integer value of n.

See Examples 1 and 2 in Section 7.5.

Example 3 in Section 7.5 illustrates the generalized principle of mathematical induction.

7.6 Counting Theory

Fundamental Principle of Counting

If n independent events occur, with m_1 ways for event 1 to occur, m_2 ways for event 2 to occur, … and m_n ways for event n to occur, then there are

$$m_1 \cdot m_2 \cdot \cdots \cdot m_n$$

different ways for all n events to occur.

If there are 2 ways to choose a pair of socks and 5 ways to choose a pair of shoes, then there are

$$2 \cdot 5 = 10$$

ways to choose socks and shoes.

Permutations Formula

If $P(n,r)$ denotes the number of permutations of n elements taken r at a time, with $r \leq n$, then the following holds.

$$P(n,r) = \frac{n!}{(n-r)!}$$

How many ways are there to arrange the letters of the word *triangle* using 5 letters at a time?

Because this is an arrangement, use the permutations formula with $n = 8$ and $r = 5$.

$$P(8,5) = \frac{8!}{(8-5)!} = \frac{8!}{3!} = 6720$$

Concepts	Examples
Combinations Formula The number of combinations of n elements taken r at a time, with $r \leq n$, is determined as follows. $$C(n, r) = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}, \quad \text{or} \quad \frac{n!}{r!(n-r)!}$$	How many committees of 4 senators can be formed from a group of 9 senators? Since the arrangement of senators does not matter, this is a combinations problem with $n = 9$ and $r = 4$. $$C(9, 4) = \binom{9}{4} = \frac{9!}{4!\,5!} = 126$$
7.7 Basics of Probability	
Probability of an Event E In a sample space S with equally likely outcomes, the probability of an event E is determined as follows. $$P(E) = \frac{n(E)}{n(S)}$$	A number is chosen at random from $S = \{1, 2, 3, 4, 5, 6\}$. What is the probability that the number is less than 3? The event is $E = \{1, 2\}$, $n(S) = 6$, and $n(E) = 2$. $$P(E) = \frac{2}{6} = \frac{1}{3}$$
Properties of Probability Let E and F represent events. 1. $0 \leq P(E) \leq 1$ 2. $P(\text{a certain event}) = 1$ 3. $P(\text{an impossible event}) = 0$ 4. $P(E') = 1 - P(E)$ 5. $P(E \text{ or } F) = P(E \cup F)$ $= P(E) + P(F) - P(E \cap F)$	What is the probability that the number is 3 or more? This event is E'. $$P(E') = 1 - \frac{1}{3} = \frac{2}{3}$$
Binomial Probability In a binomial experiment, let p represent the probability of a success, and let $q = 1 - p$ represent the probability of a failure. Then the probability of obtaining exactly r successes in n trials is found as follows. $$P(r \text{ successes in } n \text{ trials}) = \binom{n}{r} p^r q^{n-r}$$	An experiment consists of rolling a die 8 times. Find the probability that exactly 5 rolls result in a 2. Here, we have $n = 8$, $r = 5$, $p = \frac{1}{6}$, and $q = \frac{5}{6}$. $$P(5 \text{ twos in 8 rolls}) = \binom{8}{5}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^{8-5}$$ $$= 56\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^3 \approx 0.004$$

Chapter 7 Review Exercises

Write the first five terms of each sequence. State whether the sequence is arithmetic, geometric, *or* neither.

1. $a_n = \dfrac{n}{n+1}$

2. $a_n = (-2)^n$

3. $a_n = 2(n + 3)$

4. $a_n = n(n + 1)$

5. $a_1 = 5$, $a_n = a_{n-1} - 3$, if $n \geq 2$

6. $a_1 = 1$, $a_2 = 3$, $a_n = a_{n-2} + a_{n-1}$, if $n \geq 3$

7. *Concept Check* Write an arithmetic sequence that consists of five terms, with first term 4, having the sum of the five terms equal to 25.

In Exercises 8–11, write the first five terms of the sequence described.

8. arithmetic, $a_2 = 10$, $d = -2$

9. arithmetic, $a_3 = \pi$, $a_4 = 1$

10. geometric, $a_1 = 6$, $r = 2$

11. geometric, $a_1 = -5$, $a_2 = -1$

12. An arithmetic sequence has $a_5 = -3$ and $a_{15} = 17$. Find a_1 and a_n.

13. A geometric sequence has $a_1 = -8$ and $a_7 = -\frac{1}{8}$. Find a_4 and a_n.

Find a_8 for each arithmetic sequence.

14. $a_1 = 6, d = 2$

15. $a_1 = 6x - 9, a_2 = 5x + 1$

Find S_{12} for each arithmetic sequence.

16. $a_1 = 2, d = 3$

17. $a_2 = 6, d = 10$

Find a_5 for each geometric sequence.

18. $a_1 = -2, r = 3$

19. $a_3 = 4, r = \dfrac{1}{5}$

Find S_4 for each geometric sequence.

20. $a_1 = 3, r = 2$

21. $a_1 = -1, r = 3$

22. $\dfrac{3}{4}, -\dfrac{1}{2}, \dfrac{1}{3}, \ldots$

Evaluate each sum that exists.

23. $\displaystyle\sum_{i=1}^{7} (-1)^{i-1}$

24. $\displaystyle\sum_{i=1}^{5} (i^2 + i)$

25. $\displaystyle\sum_{i=1}^{4} \frac{i+1}{i}$

26. $\displaystyle\sum_{j=1}^{10} (3j - 4)$

27. $\displaystyle\sum_{j=1}^{2500} j$

28. $\displaystyle\sum_{i=1}^{5} 4 \cdot 2^i$

29. $\displaystyle\sum_{i=1}^{\infty} \left(\frac{4}{7}\right)^i$

30. $\displaystyle\sum_{i=1}^{\infty} -2\left(\frac{6}{5}\right)^i$

31. $\displaystyle\sum_{i=1}^{\infty} 2\left(-\frac{2}{3}\right)^i$

32. *Concept Check* Find an infinite geometric series having common ratio $\frac{3}{4}$ and sum 6.

Evaluate each series that converges. If the series diverges, say so.

33. $24 + 8 + \dfrac{8}{3} + \dfrac{8}{9} + \cdots$

34. $-\dfrac{3}{4} + \dfrac{1}{2} - \dfrac{1}{3} + \dfrac{2}{9} - \cdots$

35. $\dfrac{1}{12} + \dfrac{1}{6} + \dfrac{1}{3} + \dfrac{2}{3} + \cdots$

36. $0.9 + 0.09 + 0.009 + 0.0009 + \cdots$

Evaluate each sum where $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, $x_4 = 3$, $x_5 = 4$, and $x_6 = 5$

37. $\displaystyle\sum_{i=1}^{4} (x_i^2 - 6)$

38. $\displaystyle\sum_{i=1}^{6} f(x_i)\,\Delta x;\ f(x) = (x-2)^3,\ \Delta x = 0.1$

Write each sum using summation notation.

39. $4 - 1 - 6 - \cdots - 66$

40. $10 + 14 + 18 + \cdots + 86$

41. $4 + 12 + 36 + \cdots + 972$

42. $\dfrac{5}{6} + \dfrac{6}{7} + \dfrac{7}{8} + \cdots + \dfrac{12}{13}$

Use the binomial theorem to expand each expression.

43. $(x + 2y)^4$

44. $(3z - 5w)^3$

45. $\left(3\sqrt{x} - \dfrac{1}{\sqrt{x}}\right)^5$

46. $(m^3 - m^{-2})^4$

Find the indicated term or terms for each expansion

47. sixth term of $(4x - y)^8$

48. seventh term of $(m - 3n)^{14}$

49. first four terms of $(x + 2)^{12}$

50. last three terms of $(2a + 5b)^{16}$

Use mathematical induction to prove that each statement is true for every positive integer n.

51. $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$

52. $2 + 6 + 10 + 14 + \cdots + (4n - 2) = 2n^2$

53. $2 + 2^2 + 2^3 + \cdots + 2^n = 2(2^n - 1)$

54. $1^3 + 3^3 + 5^3 + \cdots + (2n - 1)^3 = n^2(2n^2 - 1)$

Evaluate each expression.

55. $P(9, 2)$

56. $P(6, 0)$

57. $\binom{8}{3}$

58. $9!$

59. $C(10, 5)$

60. $10 \cdot 9!$

Solve each problem.

61. ***Median Annual Earnings*** In 2008 the median annual earnings of a high school graduate with no college attendance was $32,136. This amount is expected to increase by about $268 per year. How much will a person earning the median amount earn until retirement if he or she joins the work force at age 18 and works until age 65? (*Source:* U.S. Bureau of Labor Statistics.)

62. ***Median Annual Earnings*** In 2008 the median annual earnings of a person with 4 yr of college was $52,624. This amount is expected to increase by about $813 per year. How much will a person earning the median amount earn until retirement if he or she joins the work force at age 22 and works until age 65? (*Source:* U.S. Bureau of Labor Statistics.)

63. ***Median Annual Earnings*** Refer to Exercises 61 and 62. How much more will a person with 4 yr of college who earns the median amount make during his or her career than a person with no college attendance who earns the median amount during his or her career? If the expenses of a 4-yr college degree are estimated at $130,000, is earning a 4-yr college degree worth it? (*Source:* U.S. Bureau of Labor Statistics.)

64. ***Wedding Plans*** Two people are planning their wedding. They can select from 2 different chapels, 4 soloists, 3 organists, and 2 ministers. How many different wedding arrangements are possible?

65. ***Couch Styles*** Bob Schiffer, who is furnishing his apartment, wants to buy a new couch. He can select from 5 different styles, each available in 3 different fabrics, with 6 color choices. How many different couches are available?

66. ***Summer Job Assignments*** Four students are to be assigned to 4 different summer jobs. Each student is qualified for all 4 jobs. In how many ways can the jobs be assigned?

67. ***Conference Delegations*** A student council consists of 6 seniors and 3 juniors. Three members are to be selected to attend a conference.

(a) How many different such delegations are possible?
(b) How many are possible if 2 seniors and 1 junior must attend?

68. ***Tournament Outcomes*** Nine football teams are competing for first-, second-, and third-place titles in a statewide tournament. In how many ways can the winners be determined?

69. ***License Plates*** How many different license plates can be formed with a letter followed by 3 digits and then 3 letters? How many such license plates have no repeats?

70. ***Racetrack Bets*** Most racetracks have "compound" bets on 2 or more horses. An *exacta* is a bet in which the first and second finishers in a race are specified in order. A *quinella* is a bet on the first 2 finishers in a race, with order not specified.

(a) In a field of 9 horses, how many different exacta bets can be placed?
(b) How many different quinella bets can be placed in a field of 9 horses?

71. ***Drawing a Marble*** A marble is drawn at random from a box containing 4 green, 5 black, and 6 white marbles. Find the following probabilities.

(a) A green marble is drawn.
(b) A marble that is not black is drawn.
(c) A blue marble is drawn.
(d) What are the odds in favor of drawing a marble that is not white?

72. ***Drawing a Card*** A card is drawn from a standard deck of 52 cards. Find the following probabilities.

(a) A black king is drawn.
(b) A face card or an ace is drawn.
(c) An ace or a diamond is drawn.
(d) A card that is not a diamond is drawn.
(e) What are the odds in favor of drawing an ace?

73. ***Master's Degrees*** There were 625,023 master's degrees awarded in the United States in 2008. The table shows the numbers of degrees awarded in several fields of study.

Field of Study	Number of Master's Degrees
Business	155,637
Education	175,880
Health professions and related clinical studies	58,120
Visual and performing arts	14,164
Other	221,222

Source. U S National Center for Education Statistics.

(a) What is the probability that a randomly selected student who earned a master's degree in 2008 earned a degree in business?
(b) What is the probability that a randomly selected student who earned a master's degree in 2008 earned a degree in either health professions and related clinical studies or the visual and performing arts?
(c) What is the probability that a randomly selected student who earned a master's degree in 2008 earned a degree that was not in education?

74. ***Defective Toaster Ovens*** A sample shipment of 5 toaster ovens is chosen. The probability of exactly 0, 1, 2, 3, 4, or 5 toaster ovens being defective is given in the table.

Number Defective	0	1	2	3	4	5
Probability	0.31	0.25	0.18	0.12	0.08	0.06

Find the probability that the given number of toaster ovens are defective.

(a) no more than 3
(b) at least 2
(c) more than 5

75. ***Rolling a Die*** A die is rolled 12 times. Find the probability (to three decimal places) that exactly 2 of the rolls result in a 5.

76. ***Tossing a Coin*** A coin is tossed 10 times. Find the probability (to three decimal places) that exactly 4 of the tosses result in a tail.

Chapter 7 Test

Write the first five terms of each sequence. State whether the sequence is arithmetic, geometric, or neither.

1. $a_n = (-1)^n(n^2 + 2)$

2. $a_n = -3\left(\frac{1}{2}\right)^n$

3. $a_1 = 2$, $a_2 = 3$, $a_n = a_{n-1} + 2a_{n-2}$, for $n \geq 3$

4. A certain arithmetic sequence has $a_1 = 1$ and $a_3 = 25$. Find a_5.

5. A certain geometric sequence has $a_1 = 81$ and $r = -\frac{2}{3}$. Find a_6.

Find the sum of the first ten terms of each series.

6. arithmetic, $a_1 = -43$, $d = 12$

7. geometric, $a_1 = 5$, $r = -2$

Evaluate each sum that exists.

8. $\sum_{i=1}^{30} (5i + 2)$

9. $\sum_{i=1}^{5} (-3 \cdot 2^i)$

10. $\sum_{i=1}^{\infty} (2^i) \cdot 4$

11. $\sum_{i=1}^{\infty} 54\left(\frac{2}{9}\right)^i$

Use the binomial theorem to expand each expression.

12. $(x + y)^6$

13. $(2x - 3y)^4$

14. Find the third term in the expansion of $(w - 2y)^6$.

Evaluate each expression.

15. $8!$

16. $C(10, 2)$

17. $\binom{7}{3}$

18. $P(11, 3)$

19. Use mathematical induction to prove that for all positive integers n,

$$1 + 7 + 13 + \cdots + (6n - 5) = n(3n - 2).$$

Solve each problem.

20. *Athletic Shoe Styles* A shoe manufacturer makes athletic shoes in 4 different styles. Each style comes in 3 different colors, and each color comes in 2 different shades. How many different types of shoes can be made?

21. *Seminar Attendees* A mortgage company has 10 loan officers: 4 women and 6 men. In how many ways can 4 of these officers be selected to attend a seminar? In how many ways can 2 women and 2 men be selected to attend the seminar?

22. *Course Schedule Arrangement* A student must select 4 courses from 15 that are offered in a semester. How many different arrangements of the 4 courses are possible?

23. *Drawing Cards* A card is drawn from a standard deck of 52 cards. Find the following probabilities.

(a) A red three is drawn.
(b) A card that is not a face card is drawn.
(c) A king or a spade is drawn.
(d) What are the odds in favor of drawing a face card?

24. *Defective Light Bulbs* A sample of 4 light bulbs is chosen. The probability of exactly 0, 1, 2, 3, or 4 light bulbs being defective is given in the table. Find the probability that at most 2 are defective.

Number Defective	0	1	2	3	4
Probability	0.19	0.43	0.30	0.07	0.01

25. ***Rolling a Die*** **Find the probability (to three decimal places) of obtaining 5 on exactly two of six rolls of a single die.**

Glossary

For a more complete discussion, see the section(s) in parentheses.

A

absolute value The absolute value of a real number is the distance between 0 and the number on the number line. (Sections R.2, 1.8)

additive inverse (negative) of a matrix If the sum of two matrices is the zero matrix, then those two matrices are additive inverses (negatives) of each other. (Section 5.7)

algebraic expression Any collection of numbers or variables joined by the basic operations of addition, subtraction, multiplication, or division (except by 0), or by the operations of raising to powers or taking roots, formed according to the rules of algebra, is an algebraic expression. (Section R.3)

annuity An annuity is a sequence of equal payments made at equal periods of time. (Section 7.3)

argument In the expression $\log_a x$, x is the argument. (Section 4.3)

arithmetic sequence (arithmetic progression) An arithmetic sequence is a sequence in which each term after the first differs from the preceding term by a fixed constant. (Section 7.2)

arithmetic series An arithmetic series is the sum of the terms of an arithmetic sequence. (Section 7.2)

asymptotes of a hyperbola The two intersecting straight lines that the branches of a hyperbola approach are the asymptotes of the hyperbola. (Section 6.3)

augmented matrix An augmented matrix is a matrix whose elements are the coefficients of the variables and the constants of a system of equations. An augmented matrix is written with a vertical bar that separates the coefficients of the variables from the constants. (Section 5.2)

average rate of change The slope of a line gives the average rate of change in y per unit of change in x, where the value of y depends on the value of x. (Section 2.4)

axis (of symmetry) of a parabola The line of symmetry for a parabola is the axis of the parabola. (Section 3.1)

B

base of an exponential The base is the number that is a repeated factor in exponential notation. In the expression a^n, a is the base. (Section R.2)

base of a logarithm In the expression $\log_a x$, a is the base. (Section 4.3)

binomial A binomial is a polynomial containing exactly two terms. (Section R.3)

binomial coefficient For nonnegative integers n and r, with $r \leq n$, the binomial coefficient is the value of $\frac{n!}{r!(n-r)!}$. These values are used in calculating the coefficients of the terms of a binomial expansion. (Section 7.4)

binomial experiment In probability, an experiment consisting of n independent trials, with two possible outcomes for each trial, and constant probabilities for each trial, is a binomial experiment. (Section 7.7)

binomial theorem (general binomial expansion) The binomial theorem is used to expand a binomial raised to a power. (Section 7.4)

boundary A line that separates a plane into two half-planes is the boundary of each half-plane. (Section 5.6)

break-even point The break-even point is the point where the revenue from selling a product is equal to the cost of producing it. (Sections 1.7, 2.4)

C

center of a circle The center of a circle is the given point that is a given distance from all points on the circle. (Section 2.2)

center of an ellipse The center of an ellipse is the midpoint of the major axis. (Section 6.2)

center of a hyperbola The center of a hyperbola is the midpoint of the transverse axis. (Section 6.3)

change in x The change in x is the horizontal difference (the difference in x-coordinates) between two points on a line. (Section 2.4)

change in y The change in y is the vertical difference (the difference in y-coordinates) between two points on a line. (Section 2.4)

circle A circle is the set of all points in a plane that lie a given distance from a given point. (Section 2.2)

closed interval A closed interval is an interval that includes both of its endpoints. (Section 1.7)

coefficient (numerical coefficient) The real number factor in a term of an algebraic expression is the coefficient of the other factors. (Section R.3)

cofactor The product of a minor of an element of a square matrix and the number $+1$ (if the sum of the row number and column number is even), or -1 (if the sum of the row number and column number is odd), is a cofactor. (Section 5.3)

collinear Two points are collinear if they lie on the same line. (Section 2.1)

column matrix A matrix with exactly one column is a column matrix. (Section 5.7)

combination A subset of items selected *without regard to order* is a combination. (Section 7.6)

combined variation Variation in which one variable depends on more than one other variable is combined variation. (Section 3.6)

common difference In an arithmetic sequence, the fixed number that is added to each term to obtain the next term is the common difference. (Section 7.2)

common logarithm A base 10 logarithm is a common logarithm. (Section 4.4)

common ratio In a geometric sequence, the fixed number by which each term is multiplied to obtain the next term is the common ratio. (Section 7.3)

complement of an event In probability, the set of all outcomes in a sample space that do *not* belong to an event E is the complement of E, written E'. (Section 7.7)